ELG7177: MIMO Comunications

Lecture 5

Dr. Sergey Loyka

EECS, University of Ottawa

October 20, 2020

MIMO: Tx & Rx antenna arrays

- multiple Tx antennas
- multiple Rx antenna
- best Tx/Rx strategies?

MIMO Channel Model

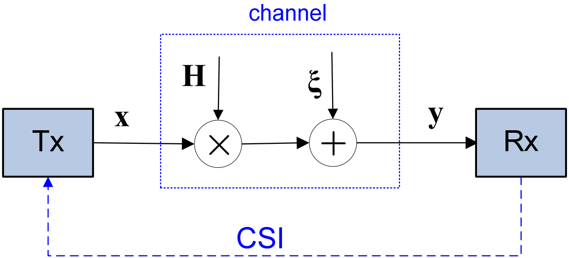
$$\mathbf{y}(t) = \mathbf{H}\mathbf{x}(t) + \boldsymbol{\xi}(t) \tag{1}$$

$\mathbf{x}(t)$ = Tx signal (vector)
$\mathbf{y}(t)$ = Rx signal (vector)
$\mathbf{H}$ = fixed channel vector; h_{ij} = channel gain from j-th Tx antenna to i-th Rx antenna
$\boldsymbol{\xi}(t)$ = Rx noise (vector)

* Compare to the SIMO/MISO models.

MIMO Channel Model

$$\mathbf{y}(t) = \mathbf{H}\mathbf{x}(t) + \boldsymbol{\xi}(t)$$



Tx/Rx Beamforming over the MIMO Channel

Tx beamforming:

$$\mathbf{x}(t) = \mathbf{w}_t \cdot x(t) \tag{2}$$

$x(t)$ = scalar Tx signal (complex amplitude, carriers the Tx data)
$\mathbf{w}_t$ = fixed Tx beamforming vector.

Rx beamforming:

$$y_r(t) = \mathbf{w}_r^+ \mathbf{y}(t) = \mathbf{w}_r^+ \mathbf{H} \mathbf{w}_t x(t) + \mathbf{w}_r^+ \boldsymbol{\xi}(t) = y_s(t) + y_n(t) \tag{3}$$

$y_s(t)$ = signal part(no noise),
$y_n(t)$ = noise part (no signal),
$\mathbf{w}_r$ = (fixed) Rx beamforming vector.

Tx/Rx Beamforming

How to choose $\mathbf{w}_t,\ \mathbf{w}_r$?

The Rx SNR γ_r (after the Rx beamformer) is

$$\gamma_r = \frac{P_s}{P_n} = \frac{\overline{|y_s|^2}}{\overline{|y_n|^2}} = \frac{|\mathbf{w}_r^+ \mathbf{H} \mathbf{w}_t|^2}{|\mathbf{w}_r|^2}\gamma_1 \tag{4}$$

where $\gamma_1 = \sigma_x^2/\sigma_0^2$ is the Rx SNR with single Tx/Rx antenna and $h = 1$.

How to maximize γ_r?

Tx/Rx Beamforming

Maximizing γ_r:

$$\gamma_r = \frac{|\mathbf{w}_r^+ \mathbf{H} \mathbf{w}_t|^2}{|\mathbf{w}_r|^2}\gamma_1 \overset{(a)}{\leq} |\mathbf{H}\mathbf{w}_t|^2\gamma_1 \overset{(b)}{\leq} \sigma_1^2(\mathbf{H})|\mathbf{w}_t|^2\gamma_1 \overset{(c)}{=} \sigma_1^2(\mathbf{H})\gamma_1 \tag{5}$$

where $\sigma_1(\mathbf{H})$ is the largest singular value of $\mathbf{H}$.

(a): how ? equality ?

(b): via the SVD properties,

$$|\mathbf{H}\mathbf{x}| \leq \sigma_1(\mathbf{H})|\mathbf{x}| \tag{6}$$

with equality iff $\mathbf{x} = \alpha \mathbf{v}_1$, where is the left singular vector of $\mathbf{H}$ corresponding to its largest singular value.

(c): $|\mathbf{w}_t| = 1$, to satisfy power constraint.

Tx/Rx Beamforming

Hence, γ_r is maximized by

$$\mathbf{w}_t = \mathbf{v}_1(\mathbf{H}),\ \mathbf{w}_r = \mathbf{u}_1(\mathbf{H}) \tag{7}$$

where $\mathbf{u}_1(\mathbf{H})$ is the left singular vector of $\mathbf{H}$ corresponding to its largest singular value $\sigma_1(\mathbf{H})$.

The maximum Rx SNR is

$$\gamma_{r,\max} = \max_{\mathbf{w}_t,\mathbf{w}_r} \gamma_r = \sigma_1^2(\mathbf{H})\gamma_1 \tag{8}$$

Singular Value Decomposition (SVD)[1,2]

Definition of singular value σ_i and its left/right singular vector $\mathbf{v}_i/\mathbf{u}_i$ of $\mathbf{H}$:

$$\mathbf{H}\mathbf{v}_i = \sigma_i \mathbf{u}_i,\ \mathbf{u}_i^+ \mathbf{H} = \sigma_i \mathbf{v}_i^+ \tag{9}$$

Applies to any matrix (not only square),

$$\mathbf{H} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^+ = \sum_i \sigma_i \mathbf{u}_i \mathbf{v}_i^+ \tag{10}$$

$\mathbf{U}$ = unitary matrix of left singular vectors of $\mathbf{H}$,
$\mathbf{V}$ = likewise for its right singular vectors,
$\boldsymbol{\Sigma}$ = diagonal matrix of its singular values,
$\mathbf{u}_i, \mathbf{v}_i$ = i-th column of $\mathbf{U}, \mathbf{V}$,
$\sigma_i \geq 0$ = i-th diagonal entry of $\boldsymbol{\Sigma}$ = i-th singular value of $\mathbf{H}$,
ordering: $\sigma_1 \geq \sigma_2 \geq ....$

[1] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge Univ. Press, 2013

[2] https://en.wikipedia.org/wiki/Singular_value_decomposition

Eigenvalue Decomposition (EVD)[3,4]

EVD: applies to any square matrix.
Definition of eigenvalue λ_i and its eigenvector $\mathbf{u}_i$ of $\mathbf{W}$:

$$\mathbf{W}\mathbf{u}_i = \lambda_i \mathbf{u}_i \tag{11}$$

For Hermitian $\mathbf{W}$,

$$\mathbf{W} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^+ = \sum_i \lambda_i \mathbf{u}_i \mathbf{u}_i^+ \tag{12}$$

$\mathbf{U}$ = unitary matrix of eigenvectors of $\mathbf{W}$,
$\boldsymbol{\Lambda}$ = diagonal matrix of its eigenvalues,
$\mathbf{u}_i$ = i-th column of $\mathbf{U}$ = i-th eigenvector of $\mathbf{W}$,
λ_i = i-th diagonal entry of $\boldsymbol{\Lambda}$ = i-th eigenvalue of $\mathbf{W}$,

[3] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge Univ. Press, 1985
[4] https://en.wikipedia.org/wiki/Eigendecomposition_of_a_matrix

Relationship of SVD and EVD

Set $\mathbf{W} = \mathbf{H}\mathbf{H}^{+}$. Then,

$$\lambda_i(\mathbf{W}) = \sigma_i^2(\mathbf{H}),\ \mathbf{u}_i(\mathbf{W}) = \mathbf{u}_i(\mathbf{H}) \tag{13}$$

and

$$\mathbf{W} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{+},\ \mathbf{H} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{+},\ \boldsymbol{\Lambda} = \boldsymbol{\Sigma}\boldsymbol{\Sigma}^{+} \tag{14}$$

i.e. the EVD can be obtained from the SVD and vice versa:

- eigenvectors of $\mathbf{H}\mathbf{H}^{+}$ are the right singular vectors of $\mathbf{H}$
- eigenvectors of $\mathbf{H}^{+}\mathbf{H}$ are the left singular vectors of $\mathbf{H}$
- eigenvalues of $\mathbf{H}\mathbf{H}^{+}$ are the squared singular values of $\mathbf{H}$

The Capacity of Tx/Rx beamforming

Extended channel: the channel + Tx/Rx beamforming.

System capacity: the extended channel capacity,

$$C = \log(1 + \gamma_{r,max}) = \log(1 + \sigma_1^2(\mathbf{H})\gamma_1) \tag{15}$$

This is the largest rate (SE) the Tx/Rx beamforming can deliver.

Can we do better than that???

Special cases:

- SIMO channel: $\mathbf{H} = \mathbf{h}$, $\sigma_1(\mathbf{H})=$? $\mathbf{v}_1 =$?
- MISO channel: $\mathbf{H} = \mathbf{h}^+$, $\sigma_1(\mathbf{H})=$? $\mathbf{u}_1 =$?
- Free space: $h_{ij} = 1$ for all i, j.

The Capacity of MIMO Channel

Can we do better than Tx/Rx beamforming ???

The capacity of MIMO channel is

$$C = \max_{p(x)} I(X; Y) \text{ s.t. } \operatorname{tr} \mathbf{R_x} \leq P \tag{16}$$

X = the random Tx vector,
Y = the random Rx vector.

How to find the max???

The Capacity of MIMO Channel

How to find the max???

Key:

$$H(Y|X) = H(\Xi) = \log|\mathbf{R}_{\boldsymbol{\xi}}| + n\log(\pi e) \tag{17}$$

$$H(Y) \le \log|\mathbf{R}_{\mathbf{y}}| + n\log(\pi e) \tag{18}$$

so that

$$I(X;Y) = H(Y) - H(\Xi) \le \log\frac{|\mathbf{R}_{\mathbf{y}}|}{|\mathbf{R}_{\boldsymbol{\xi}}|} \tag{19}$$

$\mathbf{R}_{\mathbf{y}} = \overline{\mathbf{y}\mathbf{y}^{+}},\ \mathbf{R}_{\boldsymbol{\xi}} = \overline{\boldsymbol{\xi}\boldsymbol{\xi}^{+}}$ are covariance matrices of $\mathbf{y},\ \boldsymbol{\xi}$.

The UB is achieved by $X \sim CN(0, \mathbf{R}_x)$.

The Capacity of MIMO Channel

Observe that

$$\mathbf{R_y} = \mathbf{H}\mathbf{R}_x\mathbf{H}^+ + \sigma_0^2\mathbf{I} \tag{20}$$

so that

$$I(X;Y) \le \log|\mathbf{I} + \sigma_0^{-2}\mathbf{W}\mathbf{R}_x| \tag{21}$$

where $\mathbf{W} = \mathbf{H}^+\mathbf{H}$, and hence

$$C = \max_{p(x)} I(X;Y) \text{ s.t. } \operatorname{tr}\mathbf{R_x} \le P \tag{22}$$

$$\le \max_{\operatorname{tr}\mathbf{R}_x \le P} \log|\mathbf{I} + \sigma_0^{-2}\mathbf{W}\mathbf{R}_x| \tag{23}$$

Since the UB is achieved by $X \sim CN(0, \mathbf{R}_x)$, it is the capacity.

The Capacity of MIMO Channel

Thus, the capacity is

$$C = \max_{\operatorname{tr} \mathbf{R}_x \leq P} \log |\mathbf{I} + \sigma_0^{-2} \mathbf{W} \mathbf{R}_x| \tag{24}$$

and an optimal input is $X \sim CN(0, \mathbf{R}_x)$.

We will further normalize the noise power, $\sigma_0^2 = 1$.

The Capacity of MIMO Channel

Thus, the capacity is

$$C = \max_{\mathrm{tr}\,\mathbf{R}_x \leq P} \log |\mathbf{I} + \sigma_0^{-2}\mathbf{W}\mathbf{R}_x| \tag{24}$$

and an optimal input is $X \sim CN(0, \mathbf{R}_x)$.

We will further normalize the noise power, $\sigma_0^2 = 1$.

But: **How to find the max???**

The Capacity of MIMO Channel

How to find the max???

The Capacity of MIMO Channel

How to find the max???

Key: **Hadamard inequality**.

Jacques Hadamard: 8 Dec. 1865 (Versailles, France) - 17 Oct. 1963 (Paris, France).

The Capacity of MIMO Channel

The capacity is

$$
\begin{aligned}
C &= \max_{\operatorname{tr}\mathbf{R}_x \le P} \log|\mathbf{I} + \mathbf{W}\mathbf{R}_x| \\
&= \max_{\operatorname{tr}\mathbf{R}_x \le P} \log|\mathbf{I} + \mathbf{\Lambda}_W \mathbf{U}_W^+ \mathbf{R}_x \mathbf{U}_W| && (25) \\
&= \max_{\operatorname{tr}\widetilde{\mathbf{R}}_x \le P} \log|\mathbf{I} + \mathbf{\Lambda}_W \widetilde{\mathbf{R}}_x| && (26) \\
&\le \max_{\operatorname{tr}\widetilde{\mathbf{D}}_x \le P} \log|\mathbf{I} + \mathbf{\Lambda}_W \widetilde{\mathbf{D}}_x| && (27) \\
&= \max_{d_i} \sum_i \log(1 + \lambda_{wi} d_i) \text{ s.t. } \ d_i \ge 0, \ \sum_i d_i \le P && (28)
\end{aligned}
$$

$\widetilde{\mathbf{R}}_x = \mathbf{U}_W^+ \mathbf{R}_x \mathbf{U}_W$, d_i = i-th diagonal entry of $\widetilde{\mathbf{D}}_x$

The UB is achieved by $\mathbf{U}_w = \mathbf{U}_{R_x}$, so that $d_i = \lambda_i(\widetilde{\mathbf{R}}_x) = \lambda_i(\mathbf{R}_x)$

The Capacity of MIMO Channel

Thus, the capacity is

$$C = \max_{\lambda_i} \sum_i \log(1 + \lambda_{wi}\lambda_i) \text{ s.t. } \lambda_i \geq 0, \ \sum_i \lambda_i \leq P$$

and the signaling on the eigenvectors of $\mathbf{W} = \mathbf{H}^+\mathbf{H}$ (or right singular vectors of $\mathbf{H}$) is optimal,

$$\mathbf{R}^* = \mathbf{U}_W \mathbf{\Lambda}^* \mathbf{U}_W^+ = \sum_i \lambda_i^* \mathbf{u}_{wi} \mathbf{u}_{wi}^+ \tag{29}$$

where $\mathbf{\Lambda}^* = \text{diag}\{\lambda_i^*\}$, i.e. an optimal power allocation to the channel eigenmodes.

But: **How to find the max??? How to implement** (29)**???**

The Water-Filling (WF) Algorithm

The $\max_{\lambda_i}$ is given by

$$\lambda_i^* = (\mu^{-1} - \lambda_{wi}^{-1})_+ \tag{30}$$

where $(x)_+ = \max(x, 0)$ is positive part; μ is the Lagrange multiplier responsible for the total power constraint $\sum_i \lambda_i \leq P$.

μ is the (unique) solution to

$$\sum_i (\mu^{-1} - \lambda_{wi}^{-1})_+ = P \tag{31}$$

Numerically: via e.g. bisection method. Analytically: possible in some special cases.

This is the optimal power allocation among the eigenmodes and is known as "water-filling" (WF).

The MIMO Capacity via the WF

The MIMO capacity is

$$C = \sum_i \log(1 + \lambda_{wi}\lambda_i^*) = \sum_{i:\lambda_{wi}>\mu} \log(\mu^{-1}\lambda_{wi}) \tag{32}$$

so that active eigenmodes satisfy $\lambda_{wi} > \mu$.

Proof of WF: via the KKT conditions for constrained optimization (Lagrange multipliers).

Q.: prove that (31) (i) always has a solution, and (ii) the solution is unique. Hint: show that the l.h.s of (31) is monotonically decreasing in μ. When $\mu = 0$? $\mu = \infty$?

WF Examples

1. Identical eigenvalues of $\mathbf{W}$: $\lambda_{wi} = \lambda_w \ \forall \ i$,

$$\lambda_i^* = \frac{P}{m}, \ \mathbf{R}^* = \frac{P}{m}\mathbf{I}, \ C = m \log\left(1 + \frac{P}{m}\lambda_w\right) \tag{33}$$

where $P = \gamma = \text{SNR}$ (with $m = 1$).

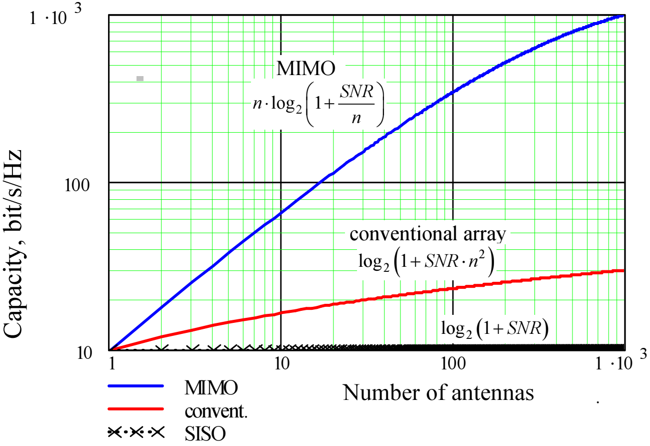
2. Rank-1 $\mathbf{W}$: $\lambda_{w1} = \lambda_w, \lambda_{w2} = ... = \lambda_{wm} = 0$,

$$\begin{aligned} &\lambda_1^* = P, \ \lambda_2^* = ... = \lambda_m^* = 0, \ \mathbf{R}^* = P\mathbf{u}_1\mathbf{u}_1^+ \\ &C = \log\left(1 + \lambda_w P\right) \end{aligned} \tag{34}$$

3. Optimal Tx structure?

WF Examples

1. Identical eigenvalues of $\mathbf{W}$: $\lambda_{wi} = \lambda_w \ \forall \ i$,



WF Properties

Q1: prove that only the strongest eigenmode is active at low SNR, while all eigenmodes are active at high SNR. Derive conditions for low/high SNR.

Q2: prove that the number of active eigenmodes increases with the SNR.

Q3: prove that stronger eigenmodes get more power, i.e. "rich get richer" or, equivalently, "capitalism is better than communism".

The Capacity of MIMO Channel

Q4: compare the MIMO channel capacity in (32) to that of the Tx-Rx beamforming in (15). Which is better (consider the most general case)? When are they equal?

Q5: consider now the MIMO channel with correlated noise,

$$\mathbf{y}(t) = \mathbf{H}\mathbf{x}(t) + \boldsymbol{\xi}(t) \tag{35}$$

where $\boldsymbol{\xi} \sim CN(0, \mathbf{R}_\xi)$, $\mathbf{R}_\xi =$ noise covariance matrix. Find its capacity. Correlated ("colored") noise can model interference.

Q6: In Q5, what happens if $\mathbf{R}_\xi$ is singular?

Optimal Covariance $\mathbf{R}^*$: Implications

- How to use $\mathbf{R}^*$ for Tx/Rx design?
- How to generate optimal input $\mathbf{x}$ with covariance $\mathbf{R}^*$?
- Insight into MIMO benefits?

Summary

- MIMO channel: Tx & Rx antenna arrays
- Tx/Rx beamforming, its capacity
- The MIMO channel capacity
- Water-filling algorithm
- Examples and special cases

Reading

- D. Tse, P. Viswanath, Fundamentals of Wireless Communications - Ch. 7.1-2, 8.1-8.3, Appendix A, B.
- J.R. Barry, E.A. Lee, D.G. Messerschmitt, Digital Communications (3rd Ed.), Kluwer, Boston, 2004. - Ch. 10.3.