ELG7177: MIMO Comunications

Lecture 3

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SIMO: Rx antenna array + beamforming

- single Tx antenna
- multiple Rx antennas
- beamforming

Channel Model

$$\mathbf{y}(t) = \mathbf{h}x(t) + \boldsymbol{\xi}(t) \tag{1}$$

$x(t)$ = Tx signal (carries Tx data)
$\mathbf{y}(t)$ = Rx signal (vector)
$\mathbf{h}$ = fixed channel vector
$\boldsymbol{\xi}(t)$ = Rx noise (vector)

Rx Beamforming

Beamformer output $y_r(t)$:

$$y_r(t) = \mathbf{w}^+\mathbf{y}(t) = \mathbf{w}^+\mathbf{h}x(t) + \mathbf{w}^+\boldsymbol{\xi}(t) = y_s(t) + y_n(t) \tag{2}$$

$y_s(t)$ = signal part(no noise),
$y_n(t)$ = noise part (no signal),
$\mathbf{w}$ = (fixed) beamforming (weight) vector.

Rx Beamforming

The output SNR γ_{out} is

$$\gamma_{out} = \frac{P_s}{P_n} = \frac{\overline{|y_s|^2}}{\overline{|y_n|^2}} = \frac{|\mathbf{w}^+\mathbf{h}|^2}{|\mathbf{w}|^2}\gamma \tag{3}$$

where $\gamma = \sigma_x^2/\sigma_0^2$ is the per-antenna SNR with $h_i = 1$.

How to maximize it?

Rx Beamforming

- Maximizing γ_{out}:

$$\gamma_{out} = \frac{|\mathbf{w}^{+}\mathbf{h}|^2}{|\mathbf{w}|^2}\gamma \leq \frac{|\mathbf{w}|^2|\mathbf{h}|^2}{|\mathbf{w}|^2}\gamma = |\mathbf{h}|^2\gamma \quad (4)$$

with equality iff

$$\mathbf{w} = \alpha\mathbf{h} \quad (5)$$

where $\alpha \neq 0$ is arbitrary scalar.

- (5): max. SNR beamformer (matched filter).
- Can choose α to normalize **w**:

$$\alpha = |\mathbf{h}| \rightarrow \mathbf{w} = \mathbf{h}/|\mathbf{h}| \quad (6)$$

so that $|\mathbf{w}| = 1$.

The Capacity of Rx beamforming

Extended channel: the channel + Rx processing.

System capacity: the extended channel capacity,

$$C = \log(1 + \gamma_{out}) = \log(1 + |\mathbf{h}|^2 \gamma) \quad (7)$$

This is the largest rate (SE) the Rx beamforming can deliver.

Can we do better ???

The Capacity of SIMO channel

No constraint on Rx processing (not nec. beambforming).

The channel capacity:

$$C = \max_{p(X)} I(X;Y) \text{ s.t. } \overline{X^2} \leq P \tag{8}$$

where $I(X;Y) =$ the MI between X (input) and Y (output),

$$I(X;Y) = H(Y) - H(Y|X) \tag{9}$$

$H(Y),\ H(Y|X) =$ unconditional and conditional entropies,

Key Quantities[1,2]

$$H(Y) = -\overline{\log p(Y)} = -\int p(y) \log p(y) dy \tag{10}$$

$$H(Y|X) = -\overline{\log p(Y|X)} = -\int p(y,x) \log p(y|x) dxdy \tag{11}$$

so that

$$I(X;Y) = \overline{\log \frac{p(X,Y)}{p(X)p(Y)}} = \int p(y,x) \log \frac{p(x,y)}{p(x)p(y)} dxdy \tag{12}$$

[1] T.M. Cover, J.A. Thomas, Elements of Information Theory, John Wiley & Sons, 2006.

[2] D. Tse, P. Viswanath, Fundamentals of Wireless Communications, Cambridge University Press, 2005

Usefull Properties

$$H(Y|X) \leq H(Y) \quad (13)$$

$$H(Y|X) = H(Y) \text{ iff } Y \perp X \quad (14)$$

$$I(X;Y) = I(Y;X) \geq 0 \quad (15)$$

$$= H(Y) - H(Y|X) \quad (16)$$

$$= H(X) - H(X|Y) \quad (17)$$

$$= H(X,Y) - H(X) - H(Y) \quad (18)$$

The Capacity of SIMO Channel

The Capacity is

$$C = \max_{p(X)} I(X;Y) \text{ s.t. } \overline{X^2} \leq P \tag{19}$$

but:

How to find the max???

The Capacity of SIMO Channel

How to find the max???

Key:

$$H(Y|X) = H(\Xi) = \log|\mathbf{R}_{\boldsymbol{\xi}}| + n\log(\pi e) \tag{20}$$

$$H(Y) \leq \log|\mathbf{R}_{\mathbf{y}}| + n\log(\pi e) \tag{21}$$

so that

$$I(X;Y) = H(Y) - H(\Xi) \leq \log\frac{|\mathbf{R}_{\mathbf{y}}|}{|\mathbf{R}_{\boldsymbol{\xi}}|} \tag{22}$$

where

$$\mathbf{R}_{\mathbf{y}} = \overline{\mathbf{y}\mathbf{y}^{+}},\ \mathbf{R}_{\boldsymbol{\xi}} = \overline{\boldsymbol{\xi}\boldsymbol{\xi}^{+}} \tag{23}$$

are covariance matrices of $\mathbf{y}$, $\boldsymbol{\xi}$.

The Capacity of SIMO Channel

Now observe that

$$\mathbf{R}_{\boldsymbol{\xi}} = \sigma_0^2 \mathbf{I} \tag{24}$$

since the noise is i.i.d. (isotropic), and that

$$\mathbf{R}_{\mathbf{y}} = \sigma_x^2 \mathbf{h}\mathbf{h}^+ + \sigma_0^2 \mathbf{I} \tag{25}$$

since $\mathbf{x}$ is independent of $\boldsymbol{\xi}$, so that

$$I(X;Y) \le \log \frac{|\mathbf{R}_{\mathbf{y}}|}{|\mathbf{R}_{\boldsymbol{\xi}}|} = \log |\mathbf{I} + \mathbf{h}\mathbf{h}^+ \sigma_x^2/\sigma_0^2| \tag{26}$$

$$\le \log |\mathbf{I} + \gamma \mathbf{h}\mathbf{h}^+| = \log(1 + \gamma |\mathbf{h}|^2) \tag{27}$$

where $\gamma = P/\sigma_0^2$, and the last equality is due to

$$|\mathbf{I} + \mathbf{A}\mathbf{B}| = |\mathbf{I} + \mathbf{B}\mathbf{A}| \tag{28}$$

The Capacity of SIMO Channel

Thus,

$$I(X;Y) \leq \log(1+\gamma|\mathbf{h}|^2) \tag{29}$$

and hence

$$\begin{aligned} C &= \max_{p(X)} I(X;Y) \text{ s.t. } \overline{X^2} \leq P \\ &\leq \log(1+\gamma|\mathbf{h}|^2) \end{aligned} \tag{30}$$

But the UB is attained by the Rx beamforming!

The Capacity of SIMO Channel

Therefore, the SIMO channel capacity is

$$C = \log(1 + \gamma|\mathbf{h}|^2) \tag{31}$$

and

- **the Rx beamforming is optimal**.

i.e. an optimal receiver is the max. SNR beamformer or the matched (to the channel) filter.

The optimal receiver: transforms the SIMO channel into equivalent (information-lossless) SISO channel.

An Example: Free-Space Propagation

Consider a free-space prop. channel in the far-field,

$$\mathbf{h} = a[1, ..., 1]^+ \tag{32}$$

where a = average path loss, so that

$$\mathbf{w}^* = [1, ..., 1]^+ \tag{33}$$

and

$$C = \log(1 + n\gamma_r) \tag{34}$$

where $\gamma_r = |a|^2\gamma$ is the Rx SNR with 1 antenna, $P_r = |a|^2 P$ is the Rx power with 1 antenna.

Notice that (i) $G_r = n$ is the Rx antenna (beamforming) gain, and (ii) (33) corresponds to a phased array beam at the broadside direction.

An Example: Free-Space Propagation

Therefore, the classical phased array is optimal in free space (but not necessarily otherwise) in the information-theoretic sense, i.e. to maximize the transmission rate [bit/s] or spectral efficiency [bit/s/Hz].

An Example: Free-Space Propagation

Consider now a more general case of free-space propagation:

$$\mathbf{h} = a[e^{j\phi_1}, ..., e^{j\phi_n}]^T \quad (35)$$

where ϕ_k is the phase shift between the Tx and k-th Rx antenna.

1. $\mathbf{w}^* = ?$
2. how can you interpret it?

Evaluate $y_s = \mathbf{w}^+ \mathbf{h} x$ explicitly in this case and make conclusion. What is precisely the optimal beamformer doing? Why?

Beamforming & Antenna Pattern

Consider an antenna (e.g. Rx beamformer) located in free space. Assume incoming wave = plane wave (far field in free space), no noise,

$$y_r = \mathbf{w}^+ \mathbf{y} = \sum_{i=1}^{n} w_i^* y_i = y_r(\theta) \tag{36}$$

θ = angle of arrival (AoA) of the plane wave.

The antenna pattern is

$$F(\theta) = \frac{|y_r(\theta)|}{\max_\theta |y_r(\theta)|} \tag{37}$$

Equivalently, $F^2(\theta)$ = power pattern.

Uniform Linear Array (ULA)

Consider a uniform linear array (ULA) of isotropic elements.

Assume $\mathbf{w} = [1, .., 1]^T$, so that

$$y_r = \mathbf{w}^+\mathbf{y} = \sum_{i=1}^{n} y_i = a\sum_{i=1}^{n} e^{j\Delta\phi(i-1)} \tag{38}$$

$y_i = ae^{j\phi_i} = e^{j\Delta\phi(i-1)}$,
$a =$ wave (complex) amplitude at element 1,
$\phi_i = \Delta\phi(i-1) =$ phase at element i,
$\Delta\phi = 2\pi d\lambda^{-1}\sin\theta =$ phase difference between two adjacent elements,
$\lambda =$ wavelength, $d =$ element spacing.

Uniform Linear Array (ULA)

The ULA pattern is

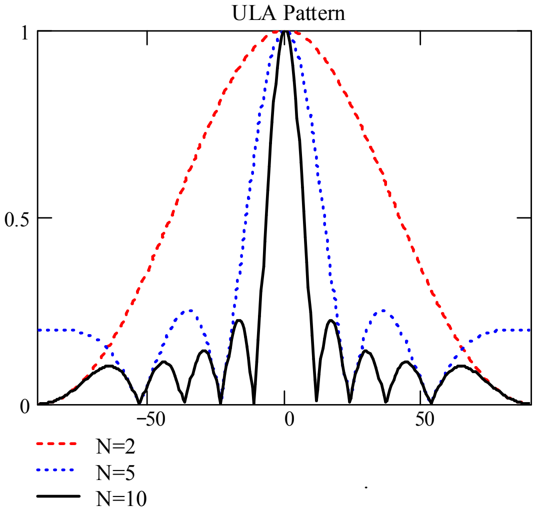
$$F(\theta) = \left| \frac{\sin(n\Delta\phi/2)}{n\sin(\Delta\phi/2)} \right| \tag{39}$$

Q1: derive this expression based on (38). Hint: use the geometric series summation formula.

Q2: plot $F(\theta)$ for $n = 1, 2, 10$, $d = \lambda/2, \lambda, 2\lambda$ and $-90^o \leq \theta \leq 90^o$. What is the impact of d? n ?

Observe that the ULA of isotropic elements has non-isotropic pattern!

ULA Pattern



Observe that the ULA of isotropic elements has non-isotropic pattern!

ULA Pattern

- main beam = area of large $F(\theta)$ (close to 1)
- secondary beams = side lobes = smaller $F(\theta)$
- nulls: $F(\theta) = 0$ or, in practice, $F(\theta) \approx 0$

Beam Steering

Moving the main beam to a desired direction θ_0, via

$$w_i = e^{j(i-1)2\pi d\lambda^{-1}\sin\theta_0} \tag{40}$$

so that the pattern is

$$F(\theta) = \left|\frac{\sin(n\psi)}{n\sin\psi}\right| = \left|\frac{\sin(n\pi d\lambda^{-1}(\sin\theta - \sin\theta_0))}{n\sin(\pi d\lambda^{-1}(\sin\theta - \sin\theta_0))}\right|, \tag{41}$$

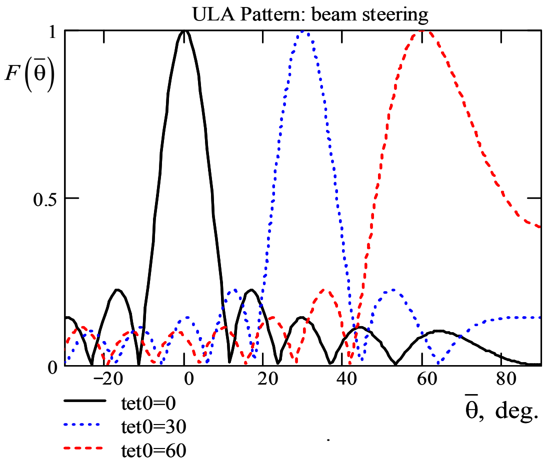
where $\psi = \pi d\lambda^{-1}(\sin\theta - \sin\theta_0)$.

Q1: plot the pattern for various values of $n,\ d, \theta_0$ and observe their impact.

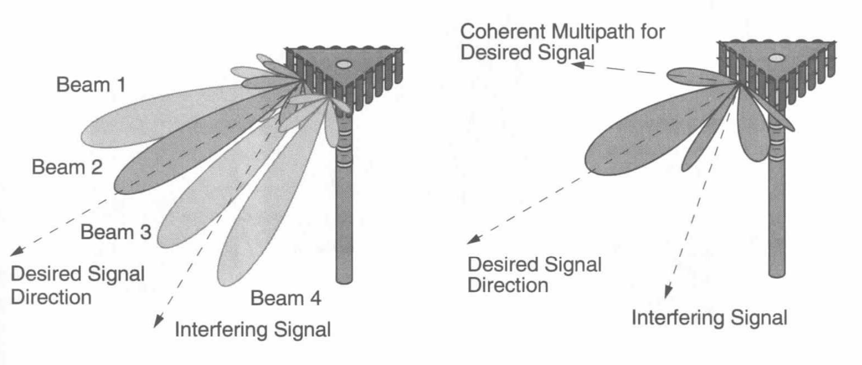
Q2: what is precisely the meaning of ψ ?

Q3: derive (41).

ULA Pattern: Beam Steering



More Practical Example



(a) Switched Beam Systems can select one of several beams to enhance receive signals. Beam 2 is selected here for the desired signal.

(b) An adaptive antenna can adjust its antenna pattern to enhance the desired signal, null or reduce interference, and collect correlated multipath power.

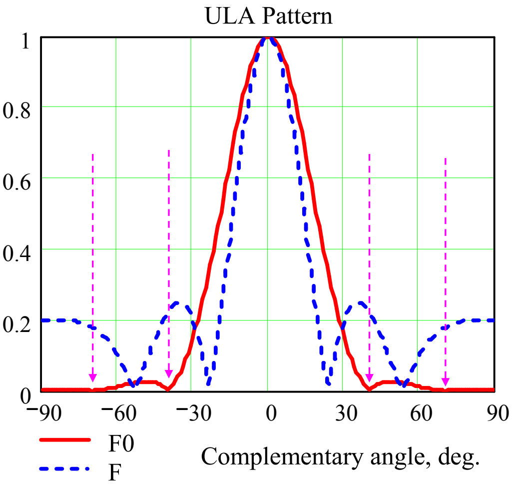
J.C. Liberti, Jr., T.S. Rappaport, Smart Antennas for Wireless Communications, Prentice Hall, 1999.

Null Steering: Forming Nulls in Given Directions

Also known as "Zero Forcing" (ZF).

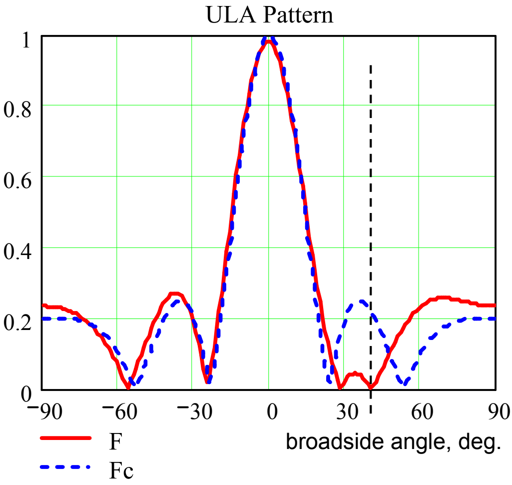
$n = 5,\ d = \lambda/2,$
$\theta_I = \pm 40^o,\ \pm 70^o.$

$$\mathbf{w} = \begin{bmatrix} 0.09 \\ 0.25 \\ 0.33 \\ 0.25 \\ 0.09 \end{bmatrix}$$



Max. SNR Beamformer

$n = 5,\ d = \lambda/2,\ \theta_I = 40^o$, SNR=10 dB, INR=10 dB.
$\text{SNIR}_{in} = 0.9,\ \text{SNIR}_{out} = 47.6,\ \text{Gain} = 52.3.$



Summary

- SIMO channel, its model
- Rx beamforming, its capacity
- the SIMO channel capacity
- Optimality of Rx beamforming
- Free-space propagation
- Antenna pattern & beam steering

Reading

- D. Tse, P. Viswanath, Fundamentals of Wireless Communications, Cambridge University Press, 2005. Appendix B, Ch. 5.1-5.3.
- J.R. Barry, E.A. Lee, D.G. Messerschmitt, Digital Communications (3rd Ed.), Kluwer, Boston, 2004. Ch. 10 and 11.
- E. Biglieri, G. Taricco, Transmission and Reception with Multiple Antennas: Theoretical Foundations, Foundations and Trends in Communications and Information Theory, v. 1, no. 2, 2004.