The Capacity of Gaussian MISO Channels Under Total and Per-Antenna Power Constraints

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Introduction

- Multi-antenna (MIMO) systems/channels
- Very popular in modern wireless applications (WiFi, 4/5G, etc.)
- Capacity under the total power (TP) constraint is well-known
  - water-filling over the channel eigenmodes (MRT for MISO)
- Per-antenna (PA) constraint: practical

Recent studies under the PA constraint

- Gaussian MIMO-BC: numerical algorithm [Yu,Lan'07][1]
- Gaussian MISO channel: analytical solution [Vu'11][2]
  - beamforming, EGT
- Gaussian MIMO channel
  - numerical algorithm based on a partial analytical solution [Vu'11][3]
  - closed-form full-rank solution [Tuninetti'14][4]
- General case is an open problem

[1] W. Yu and T. Lan, Transmitter optimization for the multi-antenna downlink with per-antenna power constraint, IEEE Trans. Signal Process., June 2007

[2] M. Vu, MISO Capacity with Per-antenna power constraint, IEEE Trans. on Commun., May 2011.

[3] M. Vu, MIMO Capacity with Per-Antenna Power Constraint, IEEE Globecom, Houston, USA, 5-9 Dec., 2011.

[4] D. Tuninetti, On the capacity of the AWGN MIMO channel under per-antenna power constraints, ICC-14, Sydney, June 2014.

Joint TP + PA constraints

- Practical motivation
  - TP constraint: limited energy/power supply, battery life
  - PA constraint: power-limited amplifiers
  - Both constraints are present in real systems
- MISO channel: analytical solution for the 2x1 case [Cao et al'15][5]
- General case: open problem

This paper

- Closed-form analytical solution for the Gaussian MISO channel

[5] P. Cao et al, Optimal Transmission Rate for MISO Channels with Joint Sum and Per-antenna Power Constraints, ICC-15, London, June 2015

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Joint TP + PA constraints

This paper

- Closed-form solution for the Gaussian MISO channel
- Capacity
- Optimal signalling
  - beamforming is optimal
  - optimal power allocation: hybrid, MRT + EGT
- Bound + simple approximation

Gaussian MISO Channel

$$y = \mathbf{h}^{+}\mathbf{x} + \xi \qquad (1)$$

- $y, \mathbf{x}$ are the received and transmitted signals
- ξ is Gaussian noise
- $\mathbf{h}$ is the channel; h_i^* is i-th channel gain (between i-th Tx antenna and the Rx).
- Ordered channel gains: $|h_1| \geq |h_2| \geq ..|h_m| > 0$

Capacity of Gaussian MISO Channel

- Gaussian signaling is optimal (TP, PA or TP+PA)
- Finding capacity = finding optimal Tx covariance:

$$C = \max_{\mathbf{R} \in S_R} \ln(1 + \mathbf{h}^{+}\mathbf{R}\mathbf{h}) \quad (2)$$

$\mathbf{R} \geq 0$ is transmit covariance matrix;
S_R is the constraint set.

Capacity of Gaussian MISO Channel under TPC

- TP constraint (TPC):

$$S_R = \{\mathbf{R} : \mathbf{R} \geq 0, tr\mathbf{R} \leq P_T\} \tag{3}$$

- The capacity:

$$C_{MRT} = \ln(1 + P_T|\mathbf{h}|_2^2) \tag{4}$$

- Optimal signaling = MRT:

$$\mathbf{R}^* = P_T\mathbf{h}\mathbf{h}^+/|\mathbf{h}|_2^2 \tag{5}$$

Capacity of Gaussian MISO Channel under PAC[6]

- PA constraint (PAC):

$$S_R = \{\mathbf{R} : \mathbf{R} \geq 0, r_{ii} \leq P\} \tag{6}$$

- The capacity:

$$C_{EGT} = \ln(1 + P|\mathbf{h}|_1^2) \tag{7}$$

- Optimal signaling = EGT:

$$\mathbf{R}^* = P\mathbf{u}\mathbf{u}^+, \; u_i = e^{j\phi_i}, \; \phi_i = \arg(h_i), \tag{8}$$

$\mathbf{u}$ = beamforming vector.

[6] M. Vu, MISO Capacity with Per-antenna power constraint, IEEE Trans. on Commun., May 2011.

- The problem:

$$C = \max_{\mathbf{R}} \ln(1 + \mathbf{h}^{+}\mathbf{R}\mathbf{h}) \ \ \text{s.t.} \ \ \mathbf{R} \in S_R \tag{9}$$

- The joint constraints:

$$S_R = \{\mathbf{R} : \mathbf{R} \geq 0, \ tr\mathbf{R} \leq P_T, \ r_{ii} \leq P\} \tag{10}$$

- **Key result**: capacity and optimal signalling

MISO Channel under the TP+PA constraints

Theorem (The MISO capacity under the TPC+PAC)

Optimal signaling = beamforming:

$$\mathbf{R}^* = P^*\mathbf{u}\mathbf{u}^+, \;\; P^* = \min(P_T, mP) \tag{11}$$

$\mathbf{u}$ *is a unitary beamforming vector:*

$$u_i = a_i e^{j\phi_i}, \;\; \phi_i = \arg(h_i), \tag{12}$$

a_i *= amplitude distribution across antennas:*

$$a_i = \begin{cases} 1/\sqrt{m^*}, & i = 1..k \\ \sqrt{1 - k/m^*}|h_i|/|\mathbf{h}_{k+1}^m|_2, & i = k+1..m \end{cases} \tag{13}$$

$m^* = P^*/P$, $\mathbf{h}_{k+1}^m = [h_{k+1}...h_m]^T$, k *is the number of active PACs.*

MISO Channel under the TP+PA constraints

Theorem (cont.)

The capacity is $C = \ln(1+\gamma^*)$, *where* γ^* *is the optimal Rx SNR:*

$$\gamma^* = P^*(c_1|\mathbf{h}_1^k|_1 + c_2|\mathbf{h}_{k+1}^m|_2^2)^2 \tag{14}$$

- 1st term - EGT, 2nd term - MRT
- **hybrid** transmission: $\mathbf{h} = [\underbrace{h_1..h_k}_{\text{EGT}}, \underbrace{h_{k+1}..h_m}_{\text{MRT}}]$
- number of active PACs k: least solution of

$$|h_{k+1}| \le |\mathbf{h}_{k+1}^m|_2/\sqrt{m^* - k} \tag{15}$$

if $P_T < mP$ and $k = m$ otherwise.

MISO Channel under the TP+PA constraints

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Example: capacity of $\mathbf{h} = [3, 1, 0.5, 0.1]^T$

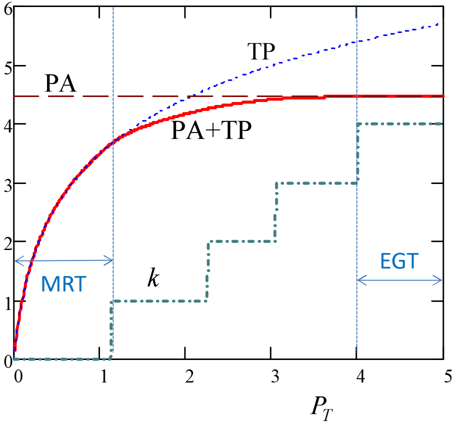

Figure: The capacity of MISO channel under the PA, TP and joint PA+TP constraints and the number of active PA constraints k vs. total power P_T; $P = 1$, $\mathbf{h} = [3, 1, 0.5, 0.1]^T$.

Example: amplitude distribution

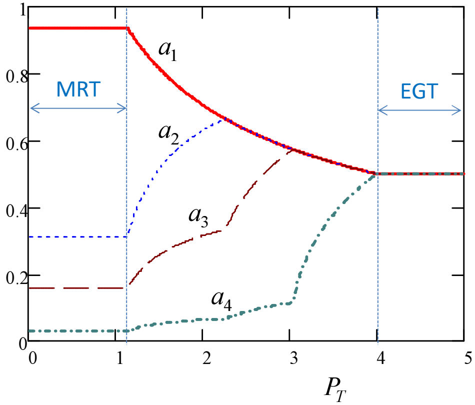

Figure: The amplitude distribution under the joint power constraints for the scenario in Fig. 1.

Optimality of MRT and EGT

Corollary

All PA constraints are inactive → MRT is optimal iff

$$|h_1| \leq |\mathbf{h}|_2 \sqrt{P/P_T} \tag{16}$$

At least 1 PA constraint is active otherwise.

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All PA constraints are active → EGT is optimal iff

$$P_T \geq mP \tag{17}$$

At least 1 PA constraint is inactive otherwise.

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Bound and simple approximation

- The TPC+PAC capacity is bounded by either one:

$$C \leq \min(C_{MRT}, C_{EGT}) \tag{18}$$

- The bound is tight in many cases:

$$C \approx \min(C_{MRT}, C_{EGT}) \tag{19}$$

where

$$C_{MRT} = \ln(1 + P_T|\mathbf{h}|_2^2), \;\; C_{EGT} = \ln(1 + P|\mathbf{h}|_1^2) \tag{20}$$

Example: $\mathbf{h} = [4, 3, 2.5, 2]^T$

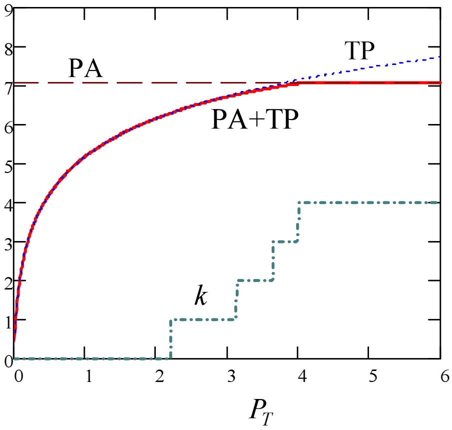

Figure: The capacity of MISO channel under the PA, TP and joint PA+TP constraints and the number of active PA constraints k vs. total power P_T; $P = 1$, $\mathbf{h} = [4, 3, 2.5, 2]^T$.

Different PA constraints

- The results can be extended to different PA constraints

$$S_R = \{\mathbf{R} : \mathbf{R} \geq 0, tr\mathbf{R} \leq P_T, r_{ii} \leq P_i\} \tag{21}$$

- All PA constraints are inactive and thus the MRT is optimal iff

$$|h_1| \leq |\mathbf{h}|_2 \sqrt{P_1/P_T} \tag{22}$$

  and at least 1 PA constraint is active otherwise.

- All PA constraints are active and hence the EGT is optimal iff

$$P_T \geq \sum_{i=1}^{m} P_i \tag{23}$$

Conclusion

- MISO channel under joint (PA+TP) constraints
  - capacity
  - optimal signaling (beamforming)
- Hybrid transmission: MRT+EGT
- Optimality of MRT or EGT
- Bound and simple approximation