MODULE 11

INTRODUCTION OF FILTER[1] NETWORKS

1. Ref.: Dorf and Svobada, 5th Edition, chapter 16.

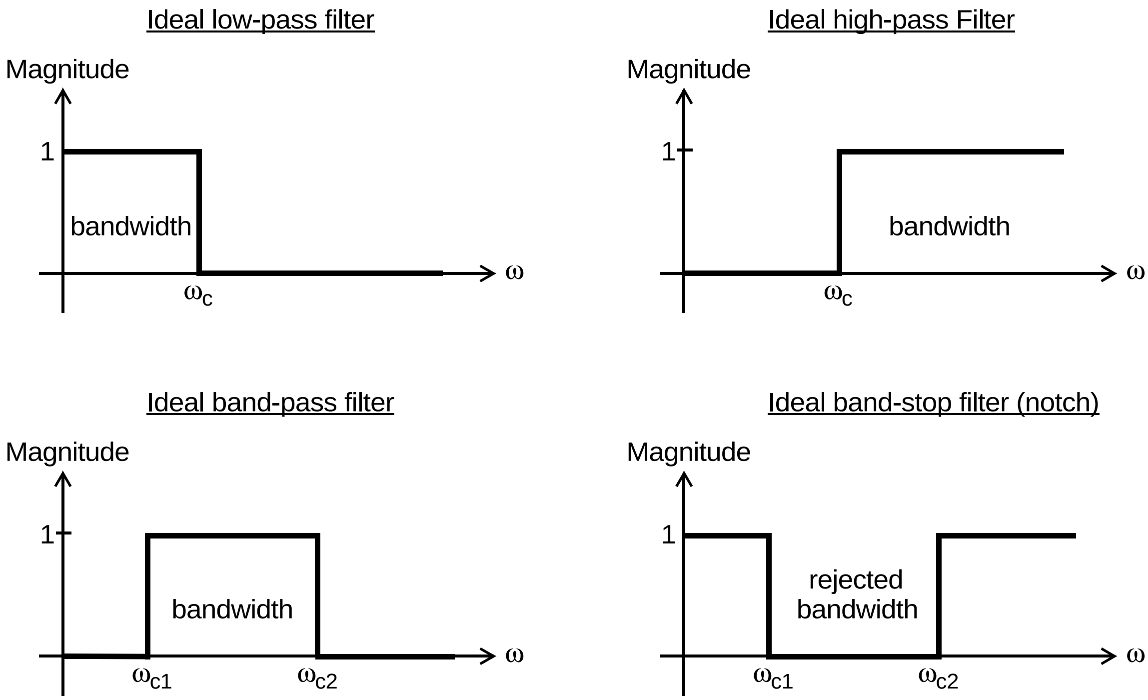
In the previous modules, we have studied analysis methods to solve electrical circuits with different configurations. Here, we shall briefly examine the behaviour of a particular type of circuit: ***filters***.

We know that an electrical signal, in general, consists of the superposition of several simple periodic signals but with different frequencies called harmonics. The filters enable us to eliminate, or more precisely attenuate some of the components which are oscillating at specific frequencies while others, oscillating at other frequencies, are amplified. This enables us to extract one part of a complex signal. For example, the equalizer in a sound system, a radio receiver or a television all have filters that extract information in the range of frequencies desired.

The analysis of filter circuits includes a study of their behaviour in terms of the frequency of signals. That is why filters are also called frequency selective circuits.

When studying a filter, we are interested in its ***frequency response,*** that is, similar to a transfer function, the range of frequencies that the filter amplifies and the range of frequencies that it attenuates. There are four main categories of filters as shown below.

FIGURE 11.1 Four main types of filters.



11.1 Bode diagram

The Bode diagram is a very useful tool to study filters. It enables to visualise graphically the impact of the filter on each of the frequency components of the signal. Thus, the frequencies with a high magnitude in the Bode diagram are amplified by the filter while the others are attenuated. The graphic representation of the phase enables to visualise the difference in phase for each of the frequency components of the signal crossing the filter.

For a given circuit or system, the Bode diagram always comprises two graphic parts:

1) The magnitude graph, $|H(j\omega)|$.

2) The phase graph, $\angle H(j\omega)$.

The Bode diagram mainly uses the notion of transfer function studied previously. $H(j\omega)$ is equivalent to the Laplace transform expressing the relationship between the input and the output of the system where $s = j\omega$. Thus, a given transfer function is associated with a system: $H(s) = H(j\omega)$.

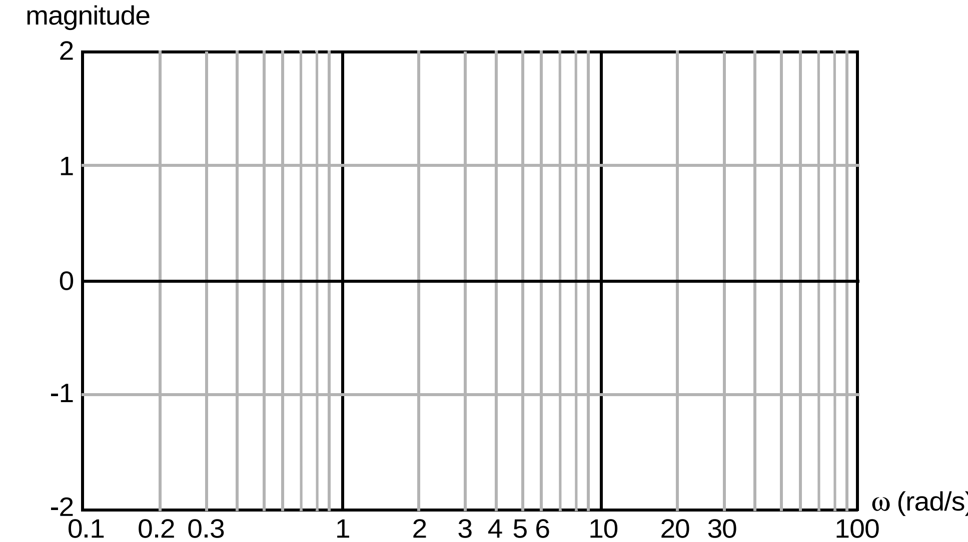
For example, for a circuit with input $V_i(s)$ and output $V_o(s)$, its transfer function is defined by:

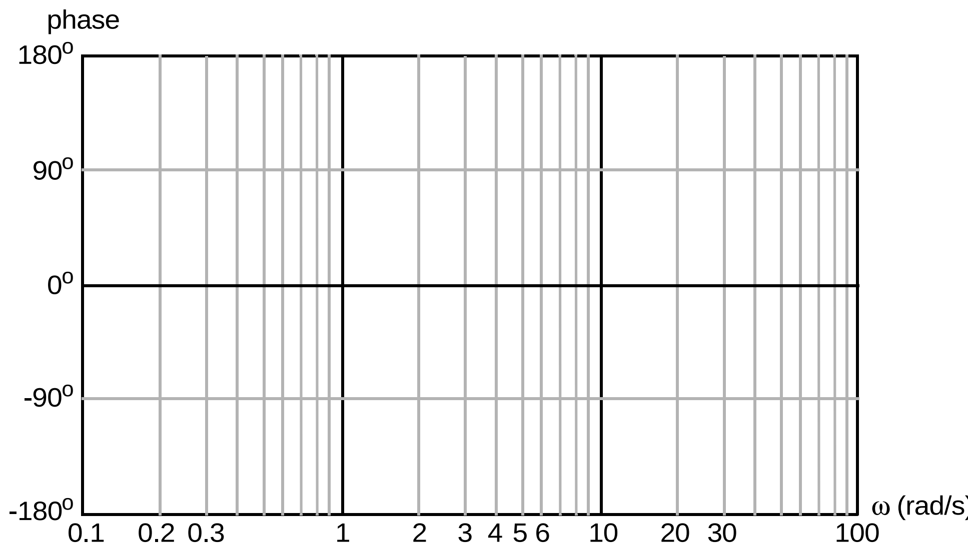
$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{sC}{LCs^2 + RCs + 1} = \frac{j\omega C}{(1 - LC\omega^2) + j\omega RC} \quad \textbf{(11.1)}$$

The function $H(j\omega)$ is a function of complex numbers. That is why the Bode diagram has two parts: magnitude and phase.

The Bode diagram can be drawn on semi-logarithmic paper. That is, the horizontal axis corresponds to the angular frequency, ω, which is logarithmic, while the vertical axis corresponds to the magnitude or the phase, and is linear. This allows to cover a larger range of frequencies. The figure below shows the scales of the Bode diagram.

FIGURE 11.2 Semi-logarithmic scales of the Bode diagram.





The graph of the magnitude can be traced either in the absolute value of $|H(s)|$ or in decibels (A_{dB}), the second approach is more common.

$$A_{dB} = 20\log|A| \quad \textbf{(11.2)}$$

It is advantageous to use decibels because the product of the elements of the transfer function, $((s+a)(s+b))/(s+c)$ then becomes simple additions and substractions. This simplifies greatly the drawing, the manipulation and the interpretation of the diagrams.

To draw the Bode diagram in magnitude and phase, we just have to vary the frequency "ω" in the transfer function and assess the magnitude $|H(s)|$ and the phase $\angle H(s)$ for each value of "ω". We then place the pair of points obtained for each "ω" on the graphs of magnitude and of phase.

Typically, the Bode diagram of a first order circuit and of a second order circuit look as shown below.

FIGURE 11.3 Bode diagram of a low-pass filter of the first order.

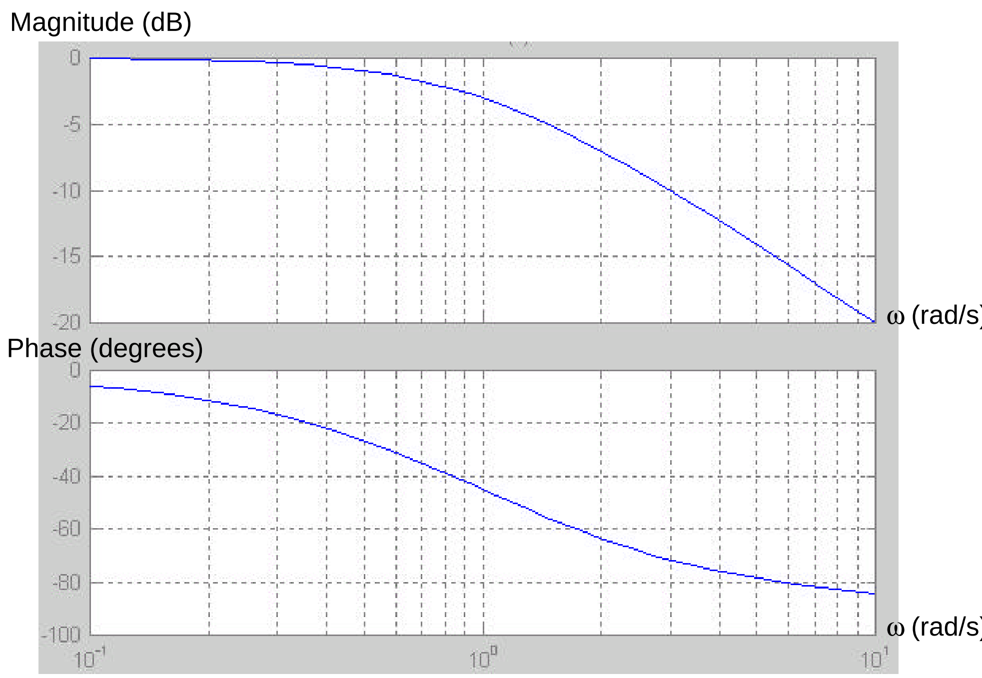
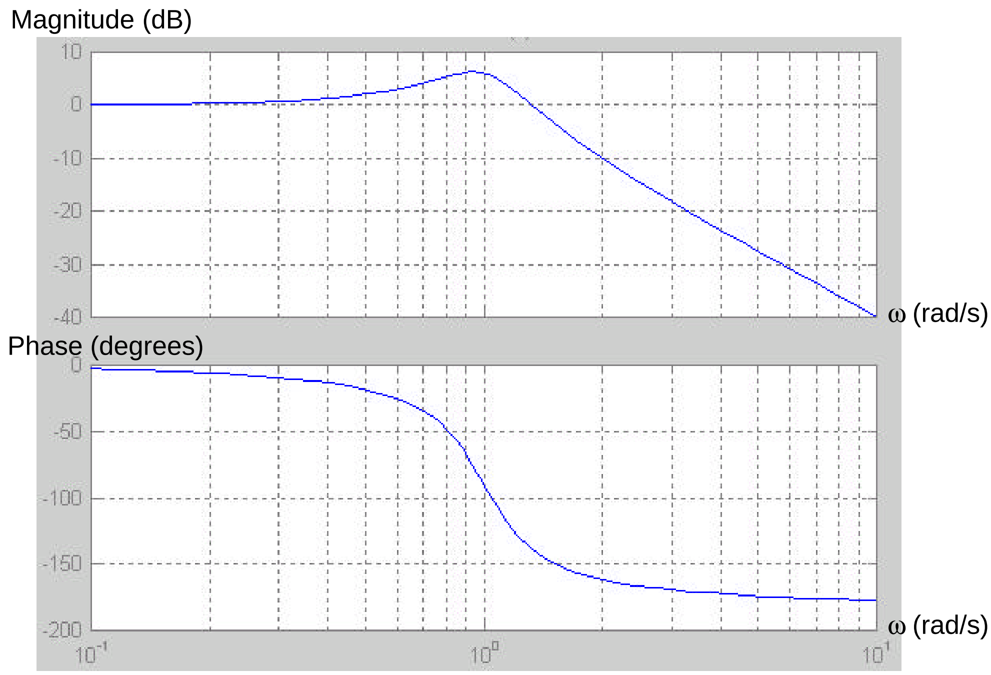


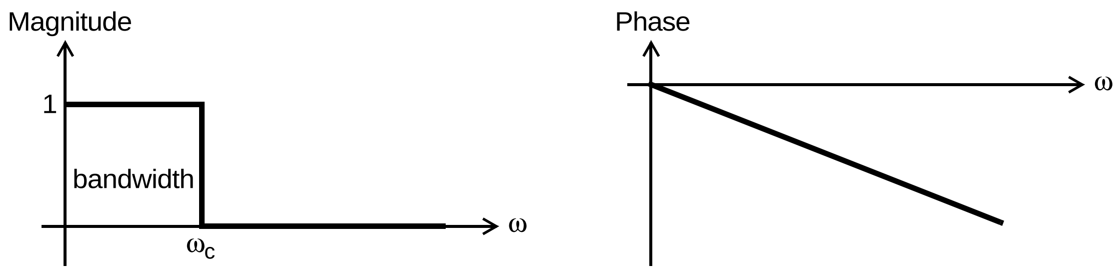
FIGURE 11.4 Bode diagram of a low-pass filter of the second order.



11.2 Low-pass filters

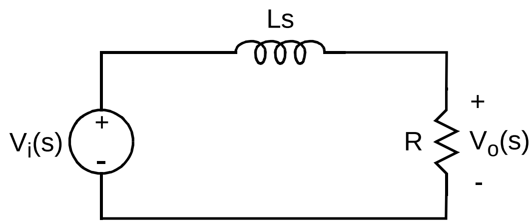
The ideal low-pass filter allows to isolate a range of frequencie with a cut-off at a specific frequency (abrupt cut-off at ω_c). Moreover, the phase should be perfectly linear as a function of the angular frequency, ω.

FIGURE 11.5 Ideal low-pass filter.



In reality, it is impossible to build a filter with such perfect characteristics. We can however build circuits whose characteristics approximate those desired. Here are a few examples:

FIGURE 11.6 *RL* circuit (1st order) with low-pass filter characteristics.



The equation of this circuit, in the Laplace domain, is obtained by the voltage division approach.

$$V_o(s) = \frac{R}{R + Ls} V_i(s) \tag{11.3}$$

The transfer function of the filter is thus:

$$F(s) = \frac{V_o(s)}{V_i(s)} = \frac{R}{R + Ls} = \frac{1}{1 + \frac{L}{R}s} = \frac{1}{1 + j\frac{\omega L}{R}} \tag{11.4}$$

The magnitude of *F(s)* in decibels is:

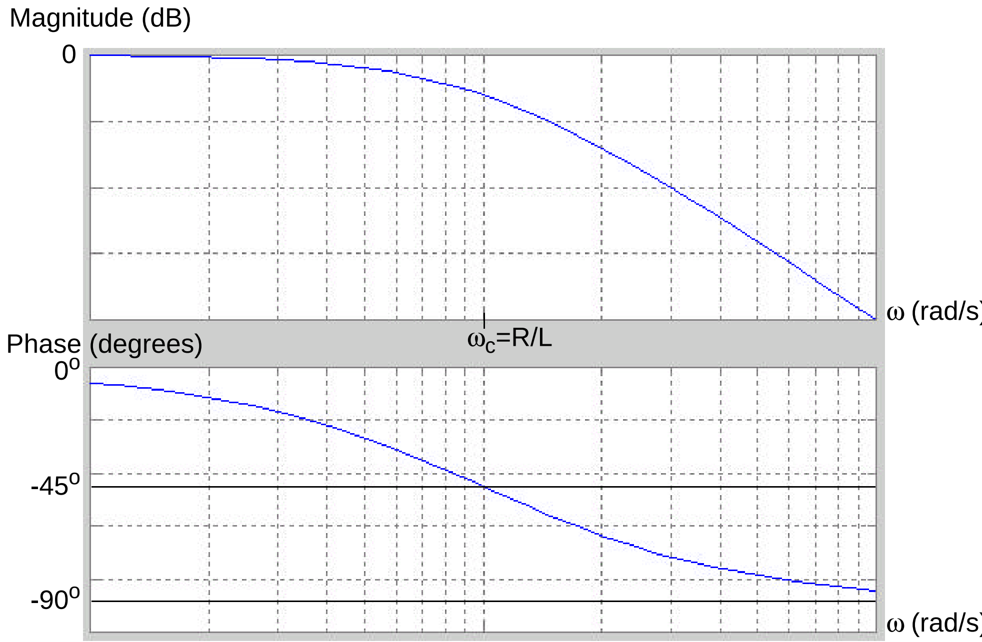
$$|F(s)|_{dB} = 20\log(1) - 20\log\left(\sqrt{1^2 + \left(\frac{\omega L}{R}\right)^2}\right) = -20\log\sqrt{1 + \left(\frac{\omega L}{R}\right)^2} \tag{11.5}$$

And the phase:

$$\angle F(s) = 0° - \mathrm{atan}\left(\frac{\frac{\omega L}{R}}{1}\right) = -\mathrm{atan}\left(\frac{\omega L}{R}\right) \tag{11.6}$$

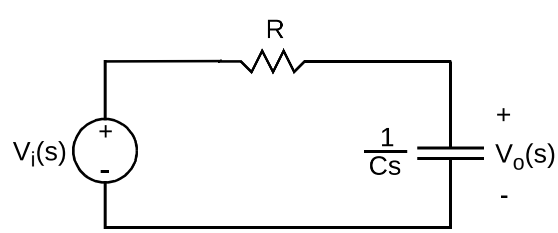
If we draw $|F(s)|_{dB}$ and $\angle F(s)$ as a function of ω, we get the Bode diagram of this circuit.

FIGURE 11.7 Bode diagram of the low-pass *RL* circuit.



An alternative approach that allows to obtain the same frequency behaviour consists in replacing the inductor by a capacitor.

FIGURE 11.8 *RC* circuit (1st order) with low-pass characteristics.



The equation of this circuit, in the Laplace domain, is obtained by current division.

$$V_o(s) = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} V_i(s) \tag{11.7}$$

The transfer function of this filter is thus:

$$F(s) = \frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} = \frac{1}{RCs + 1} = \frac{1}{1 + j\omega RC} \tag{11.8}$$

The magnitude of *F(s)* in decibels is:

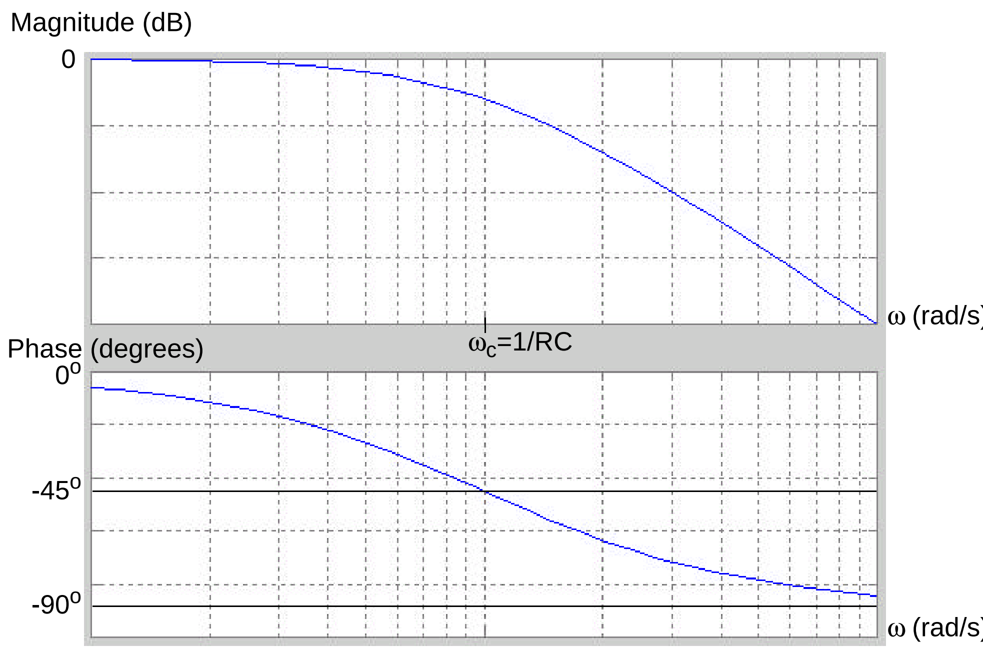
$$|F(s)|_{dB} = 20\log(1) - 20\log\left(\sqrt{1^2 + (\omega RC)^2}\right) = -20\log\sqrt{1 + (\omega RC)^2} \quad \textbf{(11.9)}$$

And the phase:

$$\angle F(s) = 0° - \operatorname{atan}\left(\frac{\omega RC}{1}\right) = -\operatorname{atan}(\omega RC) \quad \textbf{(11.10)}$$

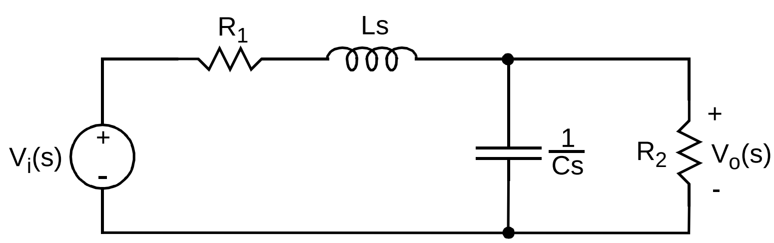
If we draw $|F(s)|_{dB}$ and $\angle F(s)$ as a function of ω, we get the Bode diagram for this circuit.

FIGURE 11.9 Bode diagram for a low-pass *RC* circuit.



In order to get a sharper cut-off at frequency, ω_c, we must increase the order of the filter, and therefore its complexity. Let us examine the case of a low-pass filter of the 2nd order.

FIGURE 11.10 Circuit of the *2nd* order with low-pass filter characteristics.



The equation of this circuit, in the Laplace domain, is obtained by voltage division, keeping in mind the equivalent impedance, Z_{eq}, of the capacitor in parallel with the resistor R_2.

$$Z_{eq} = \left(\frac{1}{R_2} + Cs\right)^{-1} = \frac{R_2}{1 + R_2 Cs} \tag{11.11}$$

$$V_o(s) = \frac{\dfrac{R_2}{1 + R_2 Cs}}{R_1 + Ls + \dfrac{R_2}{1 + R_2 Cs}} V_i(s) \tag{11.12}$$

The transfer function of the filter is:

$$F(s) = \frac{V_o(s)}{V_i(s)} = \frac{\dfrac{R_2}{1 + R_2 Cs}}{R_1 + Ls + \dfrac{R_2}{1 + R_2 Cs}} = \frac{R_2}{R_1 + R_1 R_2 Cs + Ls + R_2 CLs^2 + R_2} \tag{11.13}$$

$$F(s) = \frac{\dfrac{1}{LC}}{s^2 + \left(\dfrac{R_1}{L} + \dfrac{1}{R_2 C}\right)s + \dfrac{R_1 + R_2}{R_2 LC}} = \frac{\dfrac{R_2}{R_1 + R_2}}{\dfrac{R_2 LC}{R_1 + R_2}s^2 + \dfrac{\left(\dfrac{R_1}{L} + \dfrac{1}{R_2 C}\right)}{\dfrac{R_1 + R_2}{R_2 LC}}s + 1} \tag{11.14}$$

Replacing "*s*" by "*j*ω":

$$F(j\omega) = \frac{\dfrac{R_2}{R_1 + R_2}}{\left(1 - \dfrac{R_2 LC\omega^2}{R_1 + R_2}\right) + j\dfrac{\left(\dfrac{R_1}{L} + \dfrac{1}{R_2 C}\right)}{\dfrac{R_1 + R_2}{R_2 LC}}\omega} \tag{11.15}$$

The magnitude of *F(s)* in decibels is:

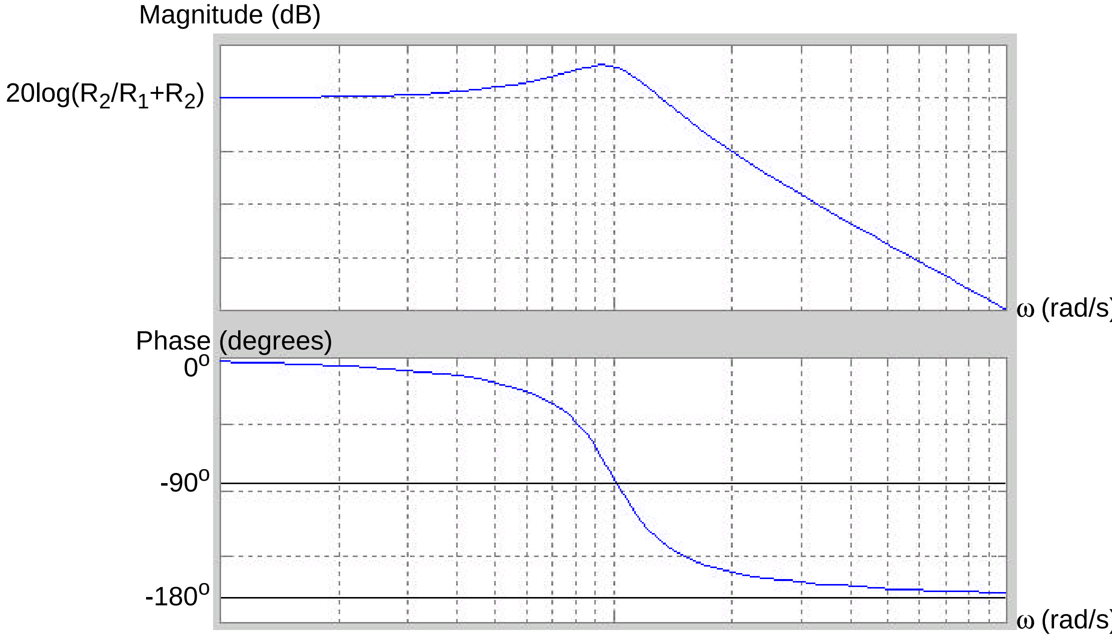
$$|F(s)|_{dB} = 20\log\left(\frac{R_2}{R_1 + R_2}\right) - 20\log\left(\sqrt{\left(1 - \frac{R_2 LC\omega^2}{R_1 + R_2}\right)^2 + \left(\frac{\left(\dfrac{R_1}{L} + \dfrac{1}{R_2 C}\right)}{\dfrac{R_1 + R_2}{R_2 LC}}\omega\right)^2}\right) \tag{11.16}$$

And the phase:

$$\angle F(s) = 0° - \text{atan}\left(\frac{\frac{\left(\frac{R_1}{L} + \frac{1}{R_2 C}\right)}{\frac{R_1 + R_2}{R_2 LC}}\omega}{1 - \frac{R_2 LC\omega^2}{R_1 + R_2}}\right) = -\text{atan}\left(\frac{\frac{\left(\frac{R_1}{L} + \frac{1}{R_2 C}\right)}{\frac{R_1 + R_2}{R_2 LC}}\omega}{1 - \frac{R_2 LC\omega^2}{R_1 + R_2}}\right) \quad \textbf{(11.17)}$$

If we draw $|F(s)|_{dB}$ and $\angle F(s)$ as a function of ω, we get the Bode diagram for this circuit.

FIGURE 11.11 Bode diagram of a low-pass second order circuit.

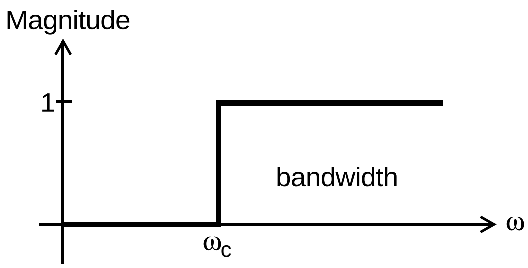


We note that the decreasing slope of the magnitude graph is sharper than that of 1st order filters studied previously, which produces a sharper cut-off of the unwanted high frequencies. However, the difference of phase produced by this filter is far larger and increases with the angular frequency.

11.3 High-pass filters

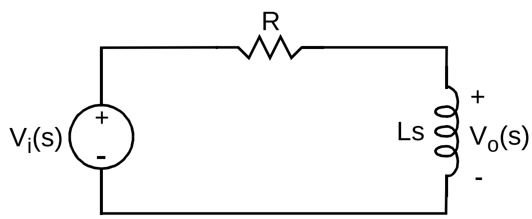
The ideal high-pass filter enables to isolate a range of frequencie with a cut-off at a specific frequency (abrupt cut-off at ω_c). Moreover, the phase must be perfectly linear as a function of the angular frequency, ω.

FIGURE 11.12 Ideal high-pass filter.



In reality, it is impossible to build a filter with such perfect characteristics. We can however build circuits whose characteristics approximate those desired. Here are a few examples.

FIGURE 11.13 *RL* circuit (1st order) with high-pass filter characteristics.



The equation of this circuit, in the Laplace domain, is obtained by voltage division.

$$V_o(s) = \frac{Ls}{R + Ls} V_i(s) \tag{11.18}$$

The transfer function of the filter is:

$$F(s) = \frac{V_o(s)}{V_i(s)} = \frac{Ls}{R + Ls} = \frac{\frac{L}{R}s}{1 + \frac{L}{R}s} = \frac{j\frac{\omega L}{R}}{1 + j\frac{\omega L}{R}} \tag{11.19}$$

The magnitude of *F(s)* in decibels is:

$$|F(s)|_{dB} = 20\log\left(\sqrt{0^2 + \left(\frac{\omega L}{R}\right)^2}\right) - 20\log\left(\sqrt{1^2 + \left(\frac{\omega L}{R}\right)^2}\right) \tag{11.20}$$

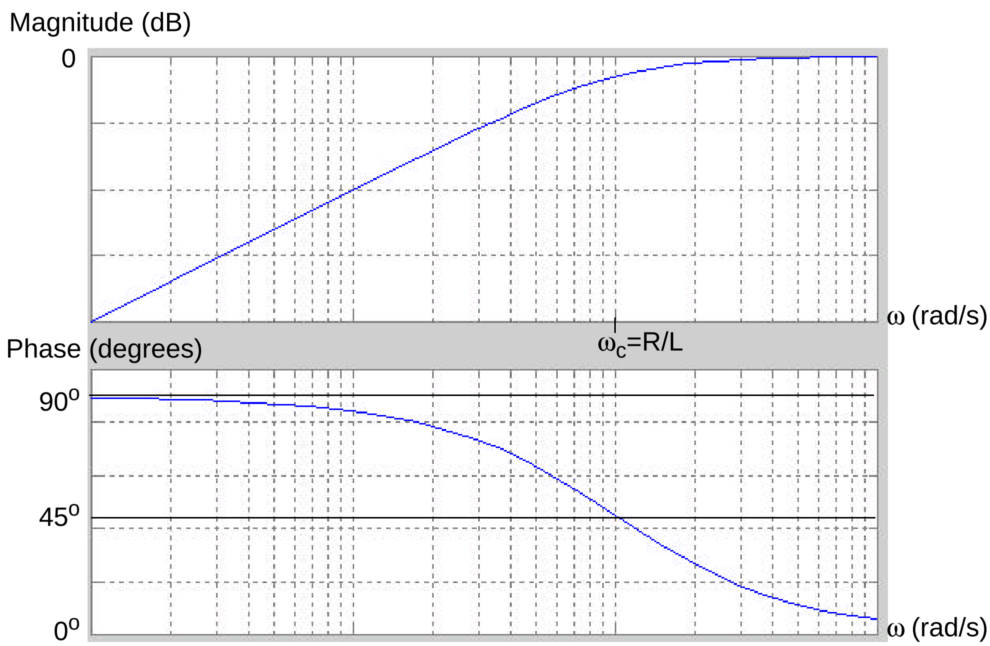
$$|F(s)|_{dB} = 20\log\left(\frac{\omega L}{R}\right) - 20\log\left(\sqrt{1 + \left(\frac{\omega L}{R}\right)^2}\right) \tag{11.21}$$

And the phase:

$$\angle F(s) = 90° - \operatorname{atan}\left(\frac{\frac{\omega L}{R}}{1}\right) = 90° - \operatorname{atan}\left(\frac{\omega L}{R}\right) \qquad \textbf{(11.22)}$$

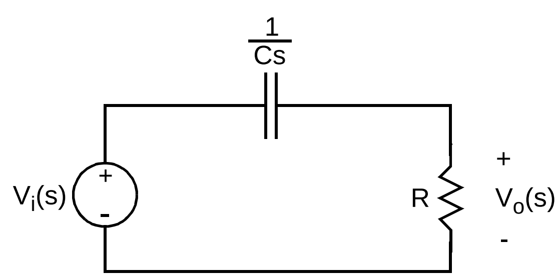
If we draw $|F(s)|_{dB}$ and $\angle F(s)$ as a function of ω, we get the Bode diagram of this circuit.

FIGURE 11.14 Bode diagram of the high-pass *RL* circuit.



As for low-pass filters, another way to get the same frequency behaviour is to replace the inductor by a capacitor.

FIGURE 11.15 *RC* circuit (1st order) with high-pass filter characteristics.



The equation of this circuit, in the Laplace domain, is obtained by voltage division.

$$V_o(s) = \frac{R}{R + \frac{1}{Cs}} V_i(s) \qquad \textbf{(11.23)}$$

The transfer function of this filter is:

$$F(s) = \frac{V_o(s)}{V_i(s)} = \frac{R}{R + \frac{1}{Cs}} = \frac{RCs}{RCs + 1} = \frac{j\omega RC}{1 + j\omega RC} \qquad \textbf{(11.24)}$$

The magnitude of *F(s)* in decibels is:

$$|F(s)|_{dB} = 20\log(\sqrt{0^2 + (\omega RC)^2}) - 20\log(\sqrt{1^2 + (\omega RC)^2}) \qquad \textbf{(11.25)}$$

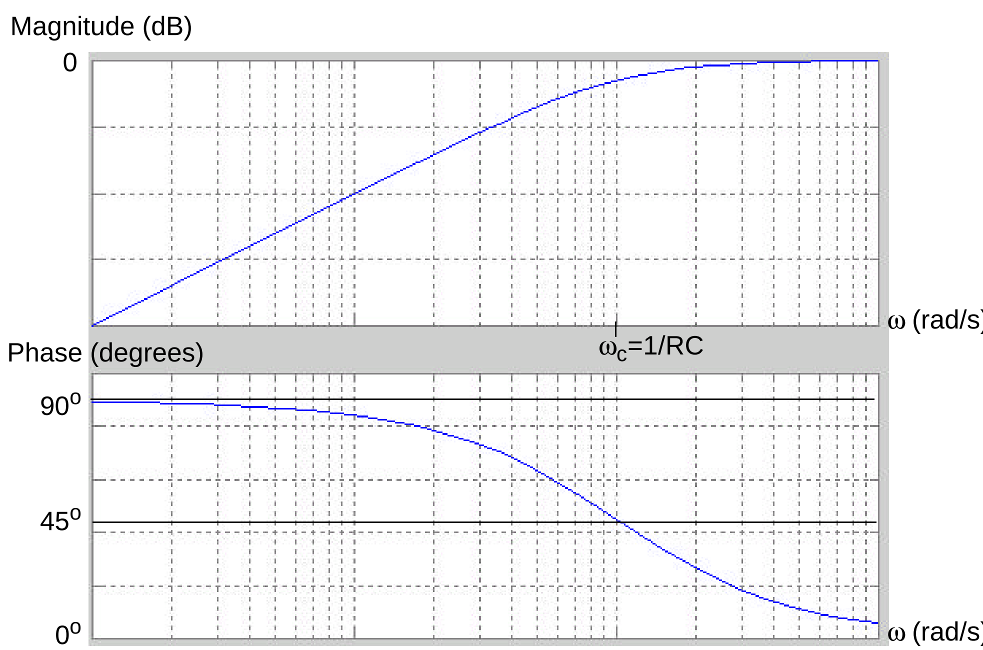
$$|F(s)|_{dB} = 20\log(\omega RC) - 20\log(\sqrt{1 + (\omega RC)^2}) \qquad \textbf{(11.26)}$$

And the phase:

$$\angle F(s) = 90° - \mathrm{atan}\left(\frac{\omega RC}{1}\right) = 90° - \mathrm{atan}(\omega RC) \qquad \textbf{(11.27)}$$

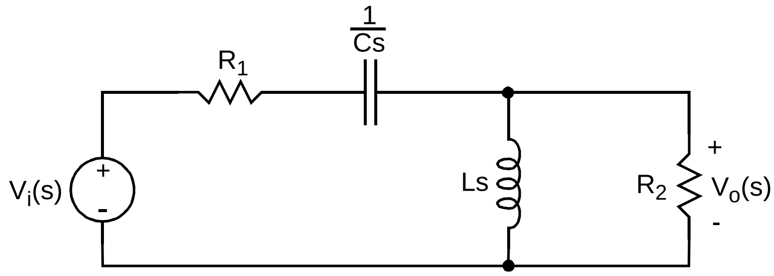
If we draw $|F(s)|_{dB}$ and $\angle F(s)$ as a function of ω, we get the Bode diagram of this circuit.

FIGURE 11.16 Bode diagram of the *RC* high-pass filter circuit.



To obtain a sharper cut-off at the frequency, ω_c, we can increase the order of the high-pass filter. Let us examine the case of a high-pass filter of the 2nd order.

FIGURE 11.17 Circuit of the *2nd* order with high-pass filter characteristics.



The equation of this circuit, in the Laplace domain, is obtained by voltage division while keeping in mind the equivalent impedance, Z_{eq}, of the inductor in parallel with the resistor R_2.

$$Z_{eq} = \left(\frac{1}{R_2} + \frac{1}{Ls}\right)^{-1} = \frac{R_2 Ls}{R_2 + Ls} \quad (11.28)$$

$$V_o(s) = \frac{\frac{R_2 Ls}{R_2 + Ls}}{R_1 + \frac{1}{Cs} + \frac{R_2 Ls}{R_2 + Ls}} V_i(s) \quad (11.29)$$

The transfer function of the filter is:

$$F(s) = \frac{V_o(s)}{V_i(s)} = \frac{\frac{R_2 Ls}{R_2 + Ls}}{R_1 + \frac{1}{Cs} + \frac{R_2 Ls}{R_2 + Ls}} = \frac{R_2 LCs^2}{R_2 LCs^2 + R_2 + Ls + R_1 LCs^2 + R_1 R_2 Cs} \quad (11.30)$$

$$F(s) = \frac{\frac{R_2 s^2}{R_1 + R_2}}{s^2 + \frac{L + R_1 R_2 C}{(R_1 + R_2)LC}s + \frac{R_2}{(R_1 + R_2)LC}} = \frac{LCs^2}{\frac{(R_1 + R_2)LC}{R_2}s^2 + \frac{L + R_1 R_2 C}{R_2}s + 1} \quad (11.31)$$

Replacing "*s*" by "*j*ω":

$$F(j\omega) = \frac{-LC\omega^2}{\left(1 - \frac{(R_1 + R_2)LC\omega^2}{R_2}\right) + j\frac{L + R_1 R_2 C}{R_2}\omega} \quad (11.32)$$

The magnitude of *F(s)* in decibels is:

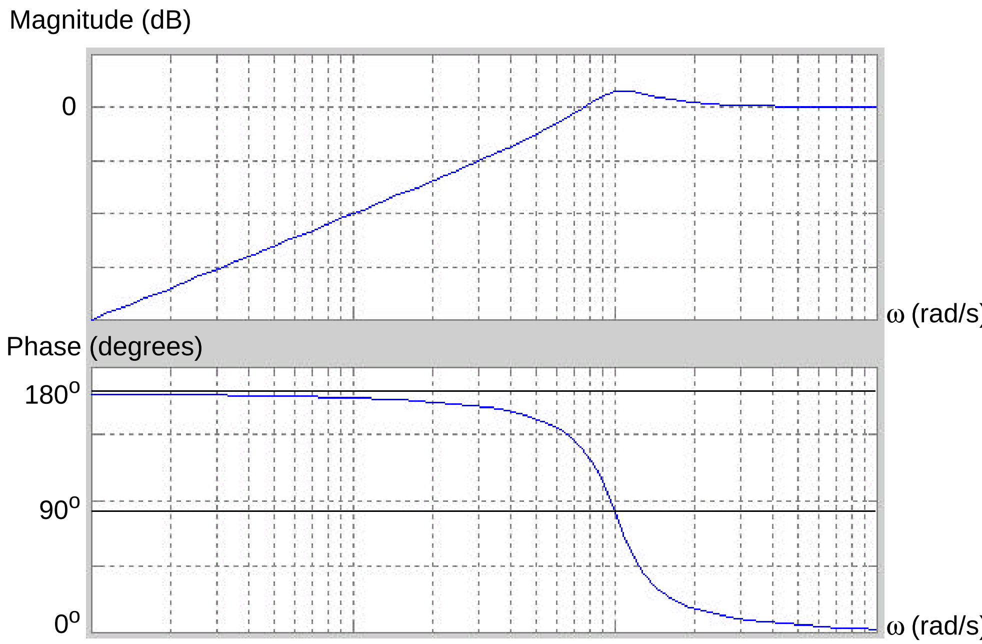
$$|F(s)|_{dB} = 20\log(LC\omega^2) - 20\log\left(\sqrt{\left(1 - \frac{(R_1 + R_2)LC\omega^2}{R_2}\right)^2 + \left(\frac{L + R_1 R_2 C}{R_2}\omega\right)^2}\right) \quad (11.33)$$

And the phase:

$$\angle F(s) = 180° - \operatorname{atan}\left(\frac{\frac{L + R_1 R_2 C}{R_2}\omega}{1 - \frac{(R_1 + R_2)LC\omega^2}{R_2}}\right) \tag{11.34}$$

If we draw $|F(s)|_{dB}$ and $\angle F(s)$ as a function of ω, we get the Bode diagram of this circuit.

FIGURE 11.18 Bode diagram of a high-pass *RLC* circuit.

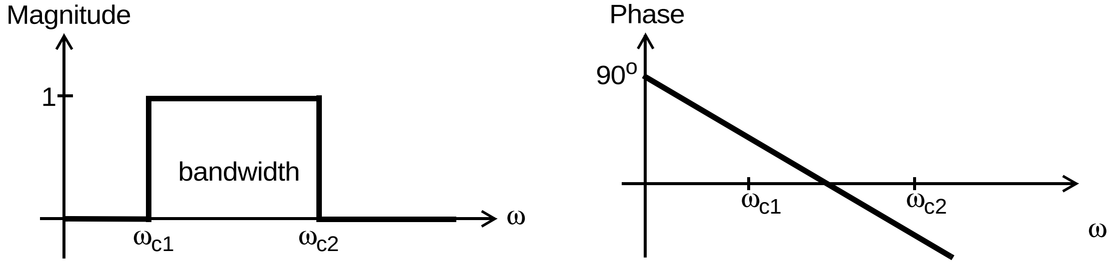


We note that the rising slope of the magnitude graph is sharper than that of 1st order filters studied previously, which makes a sharper cut-off of the unwanted low frequencies. However, the difference of phase caused by this filter is larger and increases as the angular frequency diminishes.

11.4 Band-pass filters

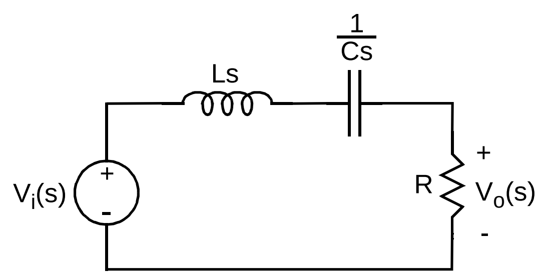
The ideal band-pass filter enables to isolate a range of frequencies between two specific cut-off points (between ω_{c1} and w_{c2}). Moreover, the phase should be perfectly linear as a function of the angular frequency, ω.

FIGURE 11.19 Ideal band-pass filter.



In reality, band-pass filters are an approximation of the ideal characteristics. Moreover, we need a circuit of at least the 2nd order to build a band-pass filter since we need 2 cut-off frequencies.

FIGURE 11.20 *RLC* circuit (2nd order) with band-pass filter characteristics.



The equation of this circuit, in the Laplace domain, is obtained by voltage division.

$$V_o(s) = \frac{R}{R + Ls + \frac{1}{Cs}} V_i(s) \tag{11.35}$$

The transfer function of the filter is:

$$F(s) = \frac{V_o(s)}{V_i(s)} = \frac{R}{R + Ls + \frac{1}{Cs}} = \frac{RCs}{LCs^2 + RCs + 1} \tag{11.36}$$

The magnitude of *F(s)* in decibels is:

$$|F(s)|_{dB} = 20\log(\sqrt{0^2 + (RC\omega)^2}) - 20\log\sqrt{(1 - LC\omega^2)^2 + (RC\omega)^2} \tag{11.37}$$

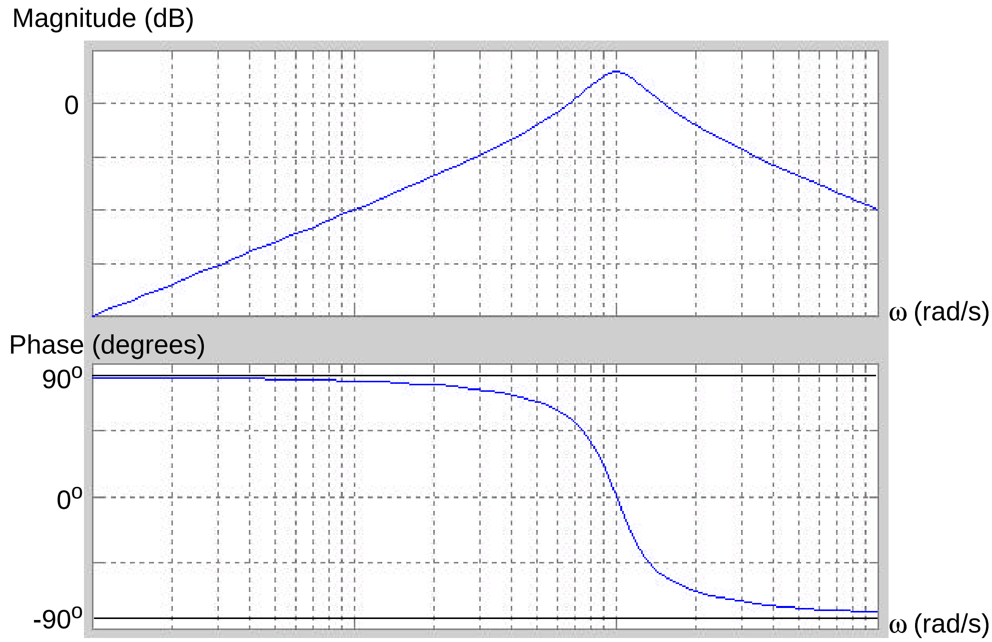
$$|F(s)|_{dB} = 20\log(RC\omega) - 20\log\sqrt{(1 - LC\omega^2)^2 + (RC\omega)^2} \tag{11.38}$$

And the phase:

$$\angle F(s) = 90° - \operatorname{atan}\left(\frac{RC\omega}{1 - LC\omega^2}\right) \tag{11.39}$$

If we draw $|F(s)|_{dB}$ and $\angle F(s)$ as a function of ω, we get the Bode diagram of this circuit.

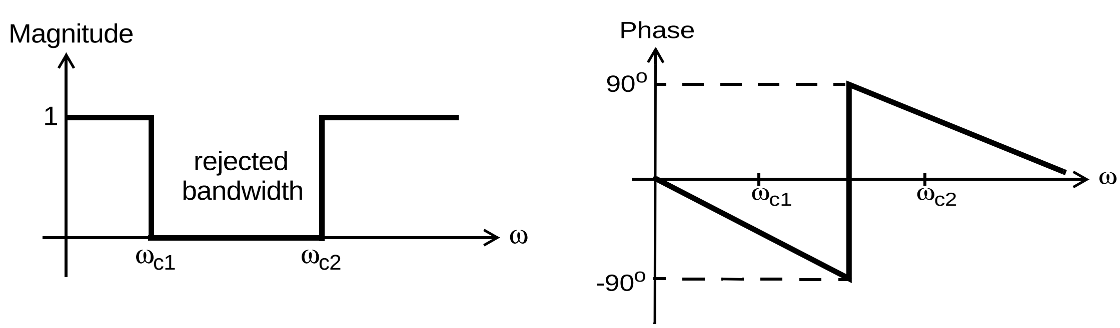
FIGURE 11.21 Bode diagram of a band-pass *RLC* circuit.



11.5 Band-stop filters (notch)

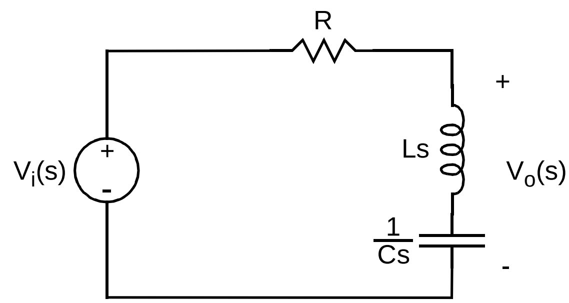
The ideal band-stop filter enables to cut a range of frequencies found between two specific cut-off points (between ω_{c1} and w_{c2}). Moreover, the phase must be perfectly linear as a function of the angular frequency, ω, except for the rejected bandwidth portion.

FIGURE 11.22 Ideal band-stop filter.



In reality, band-stop filters are an approximation of the ideal characteristics. Moreover, we need a circuit of at least the 2nd order to build a band-stop filter since we need 2 cut-off frequencies.

FIGURE 11.23 *RLC* circuit (2nd order) with band-stop filter characteristics.



The equation of this circuit, in the Laplace domain, is obtained by voltage division.

$$V_o(s) = \frac{Ls + \frac{1}{Cs}}{R + Ls + \frac{1}{Cs}} V_i(s) \tag{11.40}$$

The transfer function of the filter is:

$$F(s) = \frac{V_o(s)}{V_i(s)} = \frac{Ls + \frac{1}{Cs}}{R + Ls + \frac{1}{Cs}} = \frac{LCs^2 + 1}{LCs^2 + RCs + 1} \tag{11.41}$$

The magnitude of *F(s)* in decibels is:

$$|F(s)|_{dB} = 20\log((1 - LC\omega^2)) - 20\log\sqrt{(1 - LC\omega^2)^2 + (RC\omega)^2} \tag{11.42}$$

And the phase:

$$\angle F(s) = 0° - \operatorname{atan}\left(\frac{RC\omega}{1 - LC\omega^2}\right) \tag{11.43}$$

If we draw $|F(s)|_{dB}$ and $\angle F(s)$ as a function of ω, we get the Bode diagram of this circuit.

FIGURE 11.24 Bode diagram for a band-stop *RLC* circuit.