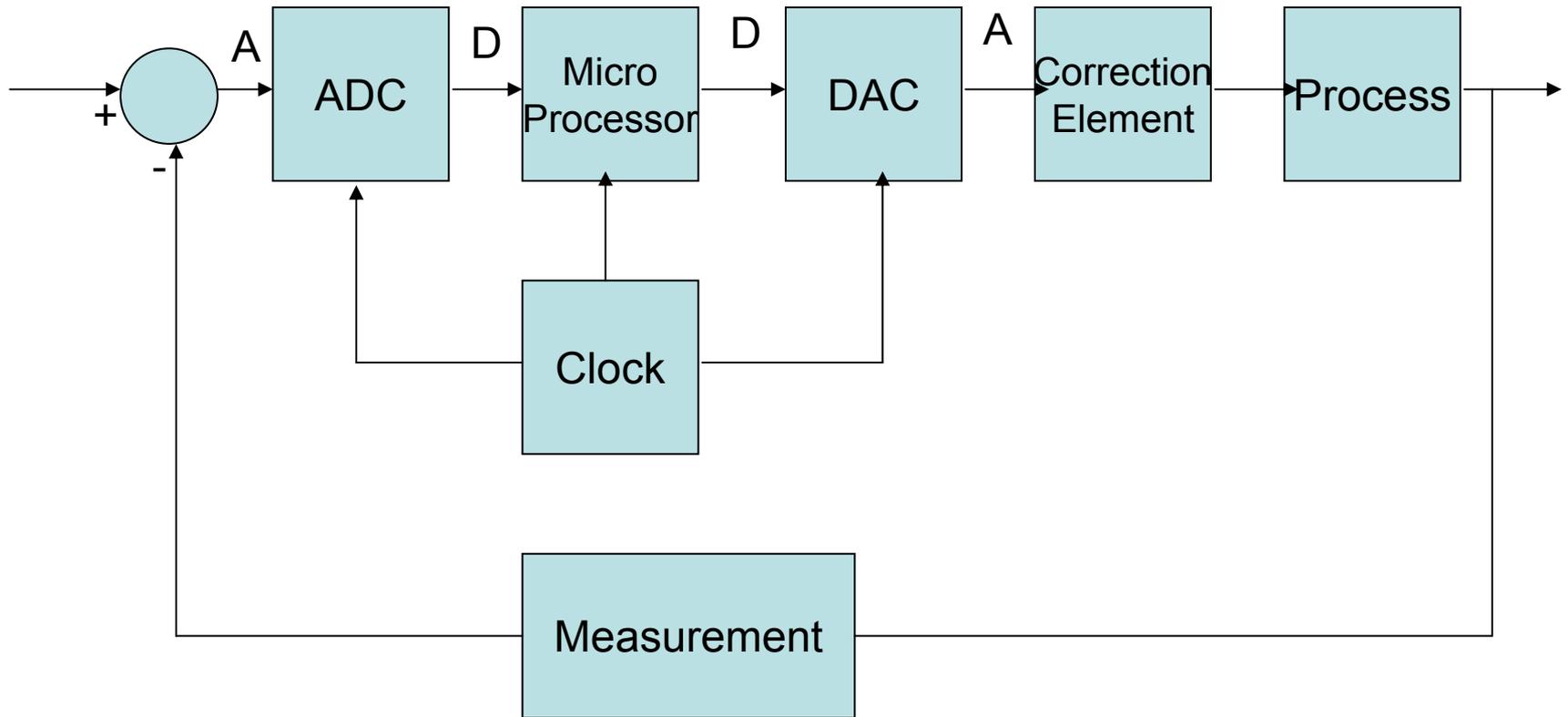
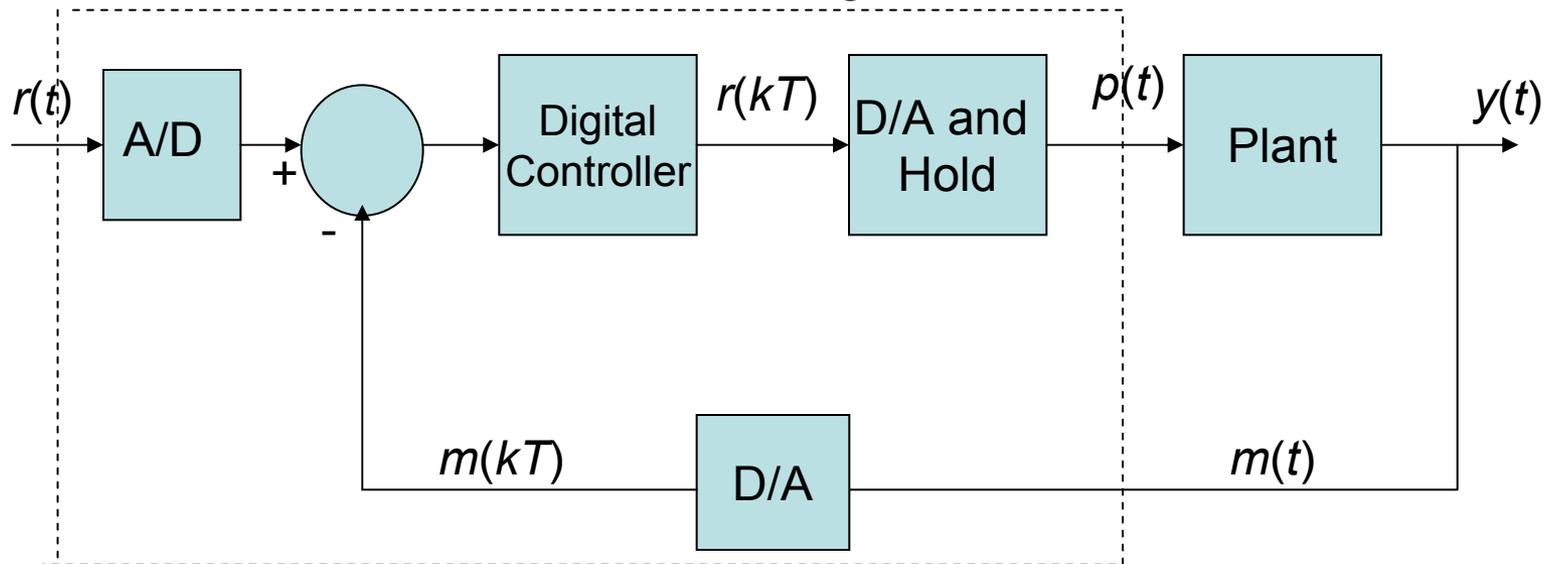
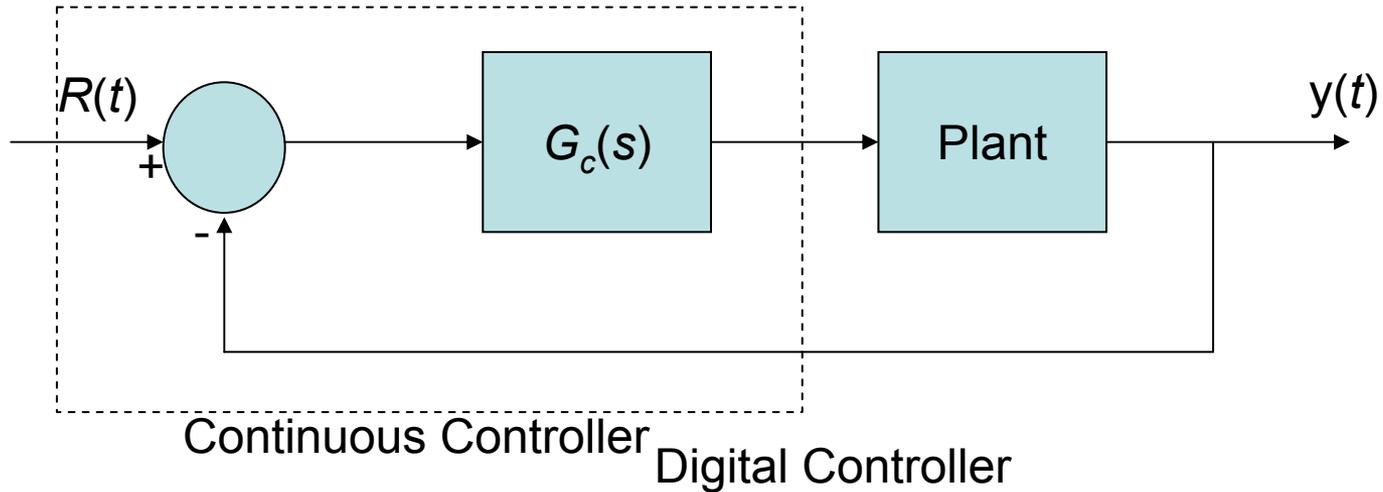


Digital Control System



A: Analog
D: Digital

Continuous Controller and Digital Control

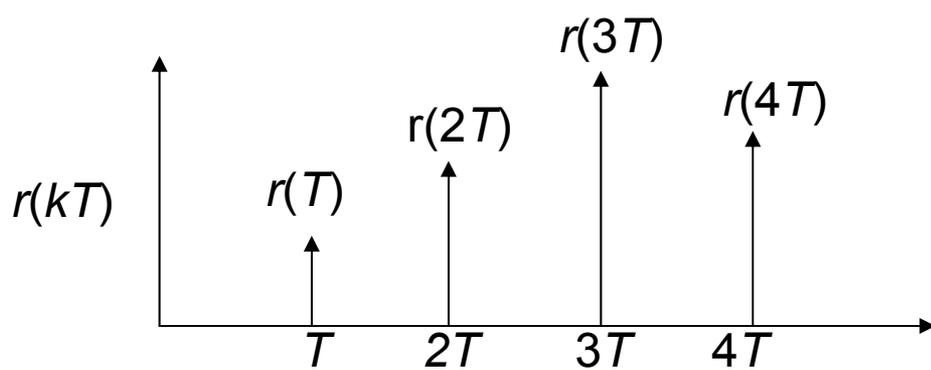
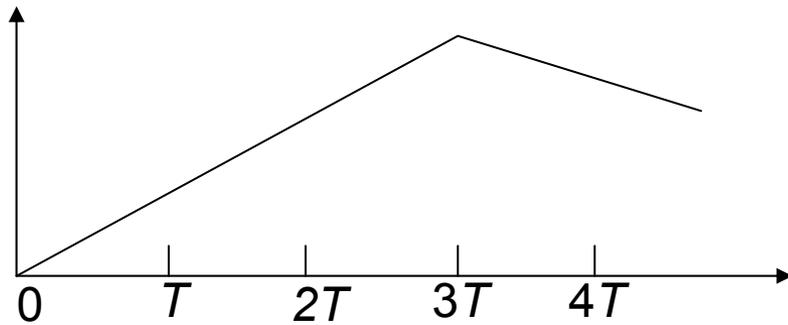
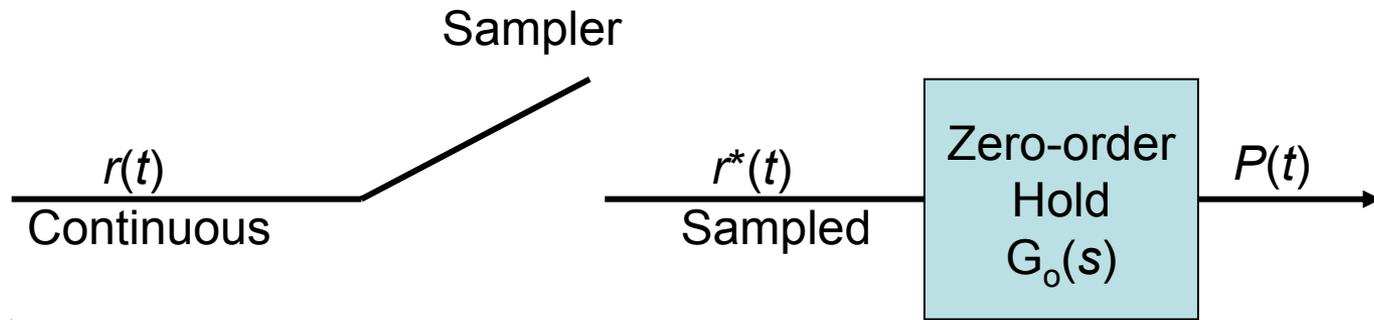


Applications of Automatic Computer Controlled Systems

- Machine Tools
- Metal Working Processes
- Chemical Processes
- Aircraft Control
- Automobile Traffic Control
- Automobile Air-Fuel Ratio
- Digital Control Improves Sensitivity to Signal Noise.

Digital Control System

- A digital computer may serve as a compensator or controller in a feedback control system. Since the computer receives data only at specific intervals, it is necessary to develop a method for describing and analyzing the performance of computer control systems.
- The computer system uses data sampled at prescribed intervals, resulting in a series of signals. These time series, called sampled data, can be transformed to the s -domain, and then to the z -domain by the relation $z = e^{zT}$.
- Assume that all numbers that enter or leave the computer has the same fixed period T , called the sampling period.
- A sampler is basically a switch that closes every T seconds for one instant of time.



$$G_0(s) = \frac{1}{s} - \frac{1}{s} e^{-sT} = \frac{(1 - e^{-sT})}{s}$$

Analog to Digital Conversion: Sampling

An input signal is converted from continuous-varying physical value (e.g. pressure in air, or frequency or wavelength of light), by some electro-mechanical device into a continuously varying electrical signal. This signal has a range of amplitude, and a range of frequencies that can present. This continuously varying electrical signal may then be converted to a sequence of digital values, called samples, by some analog to digital conversion circuit.

- There are two factors which determine the accuracy with which the digital sequence of values captures the original continuous signal: the maximum rate at which we sample, and the number of bits used in each sample. This latter value is known as the quantization level

Zero-Order Hold

- The Zero-Order Hold block samples and holds its input for the specified sample period. The block accepts one input and generates one output, both of which can be scalar or vector. If the input is a vector, all elements of the vector are held for the same sample period.
- This device provides a mechanism for discretizing one or more signals in time, or resampling the signal at a different rate. The sample rate of the Zero-Order Hold must be set to that of the slower block. For slow-to-fast transitions, use the [Unit Delay](#) block.

The z-Transform

The z-Transform is used to take discrete time domain signals into a complex-variable frequency domain. It plays a similar role to the one the Laplace transform does in the continuous time domain. The z-transform opens up new ways of solving problems and designing discrete domain applications. The z-transform converts a discrete time domain signal, which is a sequence of real numbers, into a complex frequency domain representation.

$$r^*(t) = \sum_{k=0}^{\infty} r(kT) \delta(t - kT)$$

For a signal $t \geq 0$, Using the Laplace transforms, we have

$$\mathfrak{L}\{r^*(t)\} = \sum_{k=0}^{\infty} r(kT) e^{-ksT}$$

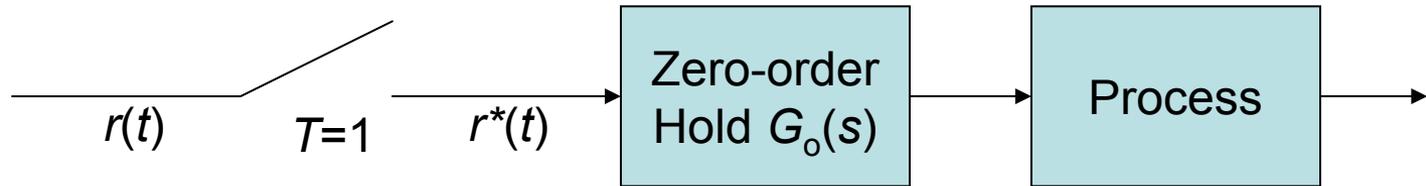
$$z = e^{sT}$$

$$Z\{r(t)\} = Z\{r^*(t)\} = \sum_{k=0}^{\infty} r(kT) z^{-k}$$

$$U(z) = \frac{z}{z-1}$$

$$Z\{f(t)\} = F(z) = \sum_{k=0}^{\infty} f(kT) z^{-k}$$

Transfer Function of Open-Loop System



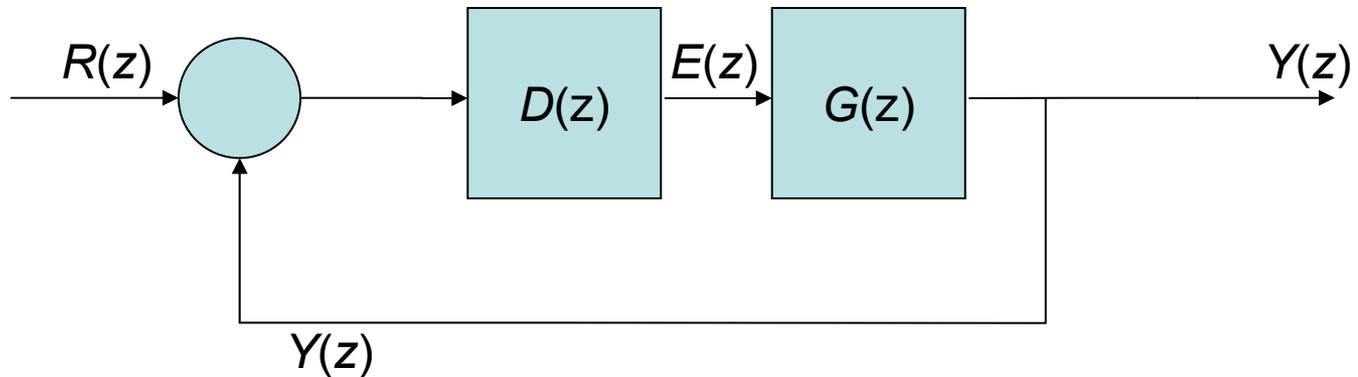
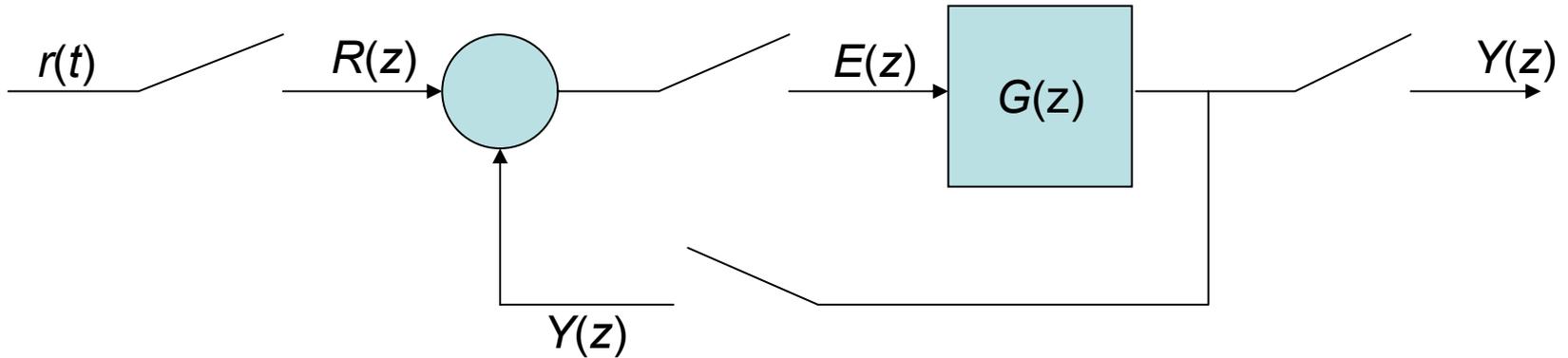
$$G_o(s) = \frac{(1 - e^{-st})}{s}; G_p(s) = \frac{1}{s(s+1)}$$

$$\frac{Y(s)}{R^*(s)} = G_o(s)G_p(s) = G(s) = \frac{1 - e^{-st}}{s^2(s+1)}$$

Expanding into partial fraction : $G(s) = (1 - e^{-st}) \left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1} \right)$

$$G(z) = \frac{0.3678z + 0.2644}{z^2 - 1.3678z + 0.3678}$$

Closed-Loop Feedback Sampled-Data Systems



$$\frac{Y(z)}{R(z)} = T(z) = \frac{G(z)}{1 + G(z)} = \frac{G(z)D(z)}{1 + G(z)D(z)}$$

Now Let us Continue with the Closed-Loop System for the Same Problem

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1+G(z)} = \frac{0.3678z + 0.2644}{z^2 - z + 0.6322}$$

Assume an a unit step input : $R(z) = \frac{z}{z-1}$

$$Y(z) = \frac{z(0.3678z + 0.2644)}{(z-1)(z^2 - z + 0.6322)} = \frac{0.3678z^2 + 0.2644z}{z^3 - 2z^2 + 1.6322z - 0.6322}$$

$$Y(z) = 0.3678z^{-1} + z^{-2} + 1.4z^{-3} + 1.4z^{-4} + 1.147z^{-5}$$

Stability Analysis in the z-Plane

A linear **continuous** feedback control system is stable if all poles of the closed-loop transfer function $T(s)$ lie in the left half of the s-plane.

In the left-hand s-plane, $\sigma < 0$; therefore, the related magnitude of z varies between 0 and 1. Accordingly the imaginary axis of the s-plane corresponds to the unit circle in the z-plane, and the inside of the **unit circle** corresponds to the left half of the s-plane.

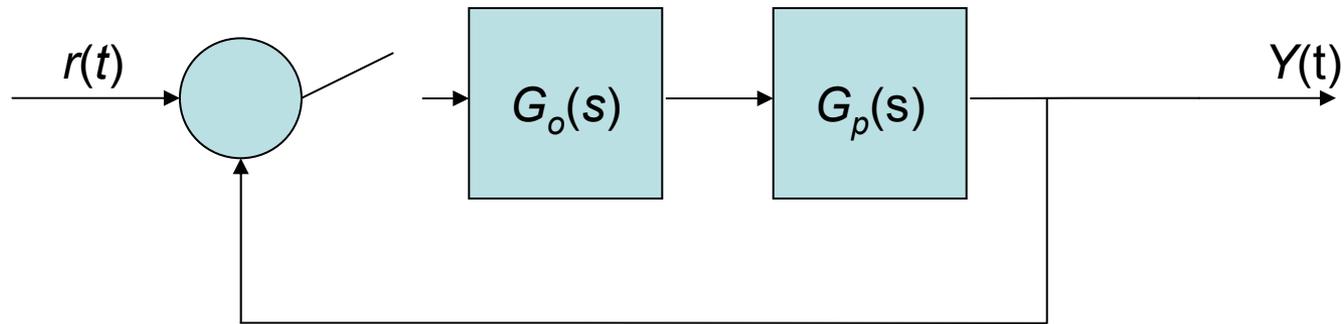
A sampled system is stable if all the poles of the closed-loop transfer function $T(z)$ lie within the unit circle of the z-plane.

$$z = e^{sT} = e^{(\sigma + j\omega)T}$$

$$|z| = e^{\sigma T}$$

$$\angle z = \omega T$$

Example 13.5: Stability of a closed-loop system



$$G_p(s) = \frac{K}{s(s+1)}; G(z) = \frac{K(0.3678z + 0.2644)}{z^2 - 1.3678z + 0.3678} = \frac{K(az + b)}{z^2 - (1+a)z + a}$$

The poles of the closed-loop transfer function $t(z)$ are the roots of the equation

$$[1 + G(z)] = 0 : z^2 - (1+a)z + a + Kaz + Kb = 0$$

$$K = 1; \quad z^2 - z + 0.6322 = (z - 0.5 + j0.6182)(z - 0.5 - j0.6182) = 0$$

The system is stable because the roots lie within the unit circle, When $K = 10$

$$z^2 + 2.310z + 3.012 = (z + 1.115 + j1.295)(z + 1.115 - j1.295) \text{ (unstable)}$$

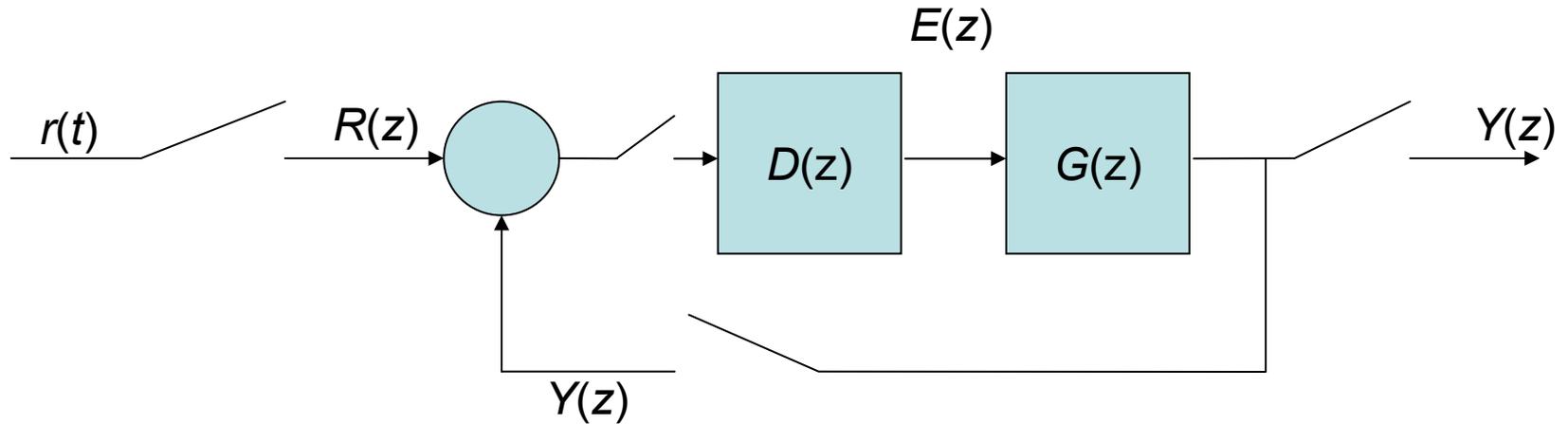
This system is stable for : $0 < K < 2.39$

Second-order sampled system is unstable for increased gain where the continuous is stable for all values of gain.

Design Procedure

- Start with continuous system.
- Add sampled-data system elements.
- Chose sample period, usually small but not too small. Use sampling period $T = 1 / 10 f_B$, where $f_B = \omega_B / 2\pi$ where ω_B is the bandwidth of the closed-loop system.
- Digitize control law.
- Check performance using discrete model or SIMULINK.

Start with a Continuous Design; $D(s)$ may be given as an existing design or by using root locus or bode design.



Add Samples Necessary for Digital Control

- Transform $D(s)$ to $D(z)$: We will obtain a discrete system with a similar behavior to the continuous one.
- Include D/A converter, usually a zero-order-device.
- Include A/D converter modeled as an ideal sampler.
- And an antialiasing filter, a low pass filter, unity gain filter with a sharp cutoff frequency.
- Chose a sample frequency ω_B based on the closed-loop bandwidth of the continuous system.

Closed-Loop System with Digital Computer Compensation

$$\frac{Y(z)}{R(z)} = T(z) = \frac{G(z)D(z)}{1 + G(z)D(z)}$$

The transfer function of the computer is $\frac{U(z)}{E(z)} = D(z)$

Consider the second order system with a zero-order hold and a plant

$$Gp(s) = \frac{1}{s(s+1)} \text{ when } T = 1; G(z) = \frac{0.3678(z+0.7189)}{(z-1)(z-0.3678)}; \text{ If we select } D(z) = \frac{k(z-0.3678)}{(z+r)}$$

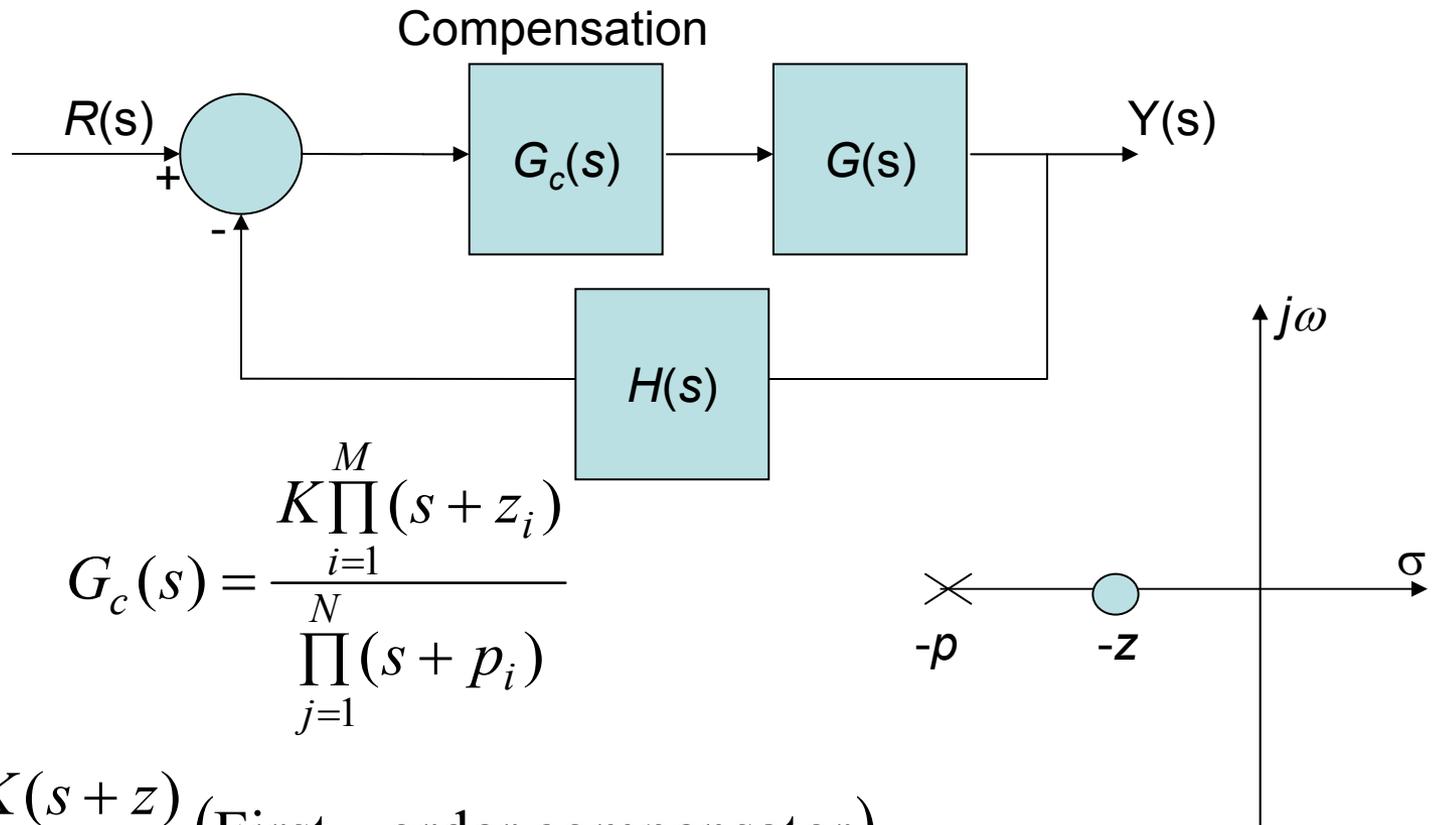
We cancel the pole of $G(z)$ at $z = 0.3678$ and have the two parameters r and K .

$$D(z) = \frac{1.359(z-0.7189)}{(z+0.240)}; G(z)D(z) = \frac{0.5(z+0.7189)}{(z-1)(z+0.240)}$$

$$G_c(s) = K \frac{s+a}{s+b}; D(z) = C \frac{z-A}{z-B}; Z\{G_c(s)\} = D(z); A = e^{-aT}; B = e^{-bT}; C \frac{(1-A)}{(1-B)} = K \frac{a}{b}$$

Compensation Networks (10.3; page 557)

The compensation network, $G_c(s)$ is cascaded with the unalterable process $G(s)$ in order to provide a suitable loop transfer function $G_c(s)G(s)H(s)$.



$$G_c(s) = \frac{K(s + z)}{(s + p)} \text{ (First - order compensator)}$$

When $z < p$, the network is called a phase - lead network

Closed-Loop System with Digital Computer Compensation

There are two methods of compensator design: (1) the $G_c(s)$ -to- $D(z)$ conversion method, and (2) the root locus z-plane method.

The $G_c(s)$ -to- $D(z)$ conversion method

$$G_c(s) = K \frac{s + a}{s + b} \text{ (First - Order Compensator)}$$

$$D(z) = C \frac{z - A}{z - B} \text{ (Digital Controller)}$$

$$Z\{G_c(s)\} = D(z) \text{ (z - transform)}$$

$$A = e^{-aT}; B = e^{-bT}; C \frac{(1 - A)}{(1 - B)} = K \frac{a}{b} \text{ when } s = 0$$

Example 13.7: Design to meet a phase margin specification

$G_p(s) = \frac{1740}{s(0.25s+1)}$. We will attempt to design $G_c(s)$ so that we achieve

a phase margin of 45° with a crossover frequency $\omega_c = 125$ rad/s (Fig 10.10).

Using the Bode diagram of $G_p(s)$, we find that the phase margin is 2° (Eq 10.24).

Based on 10.4, we find that the required pole - zero ratio is $\alpha = 6.25$ (Eq 10.18).

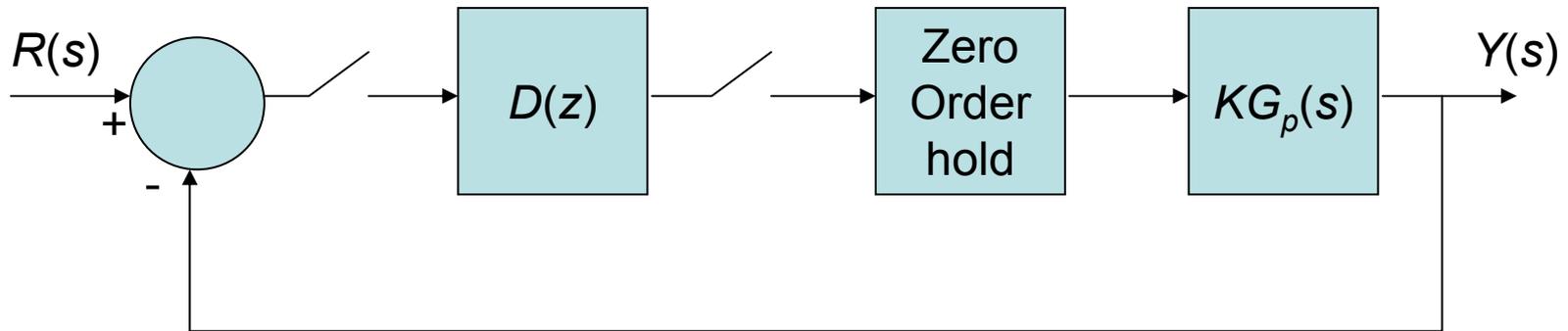
$$\omega_c = (ab)^{\frac{1}{2}} \text{ (see page 587), } a = 50 \text{ and } b = 312; G_c(s) = \frac{K(s+50)}{(s+312)}$$

We select K in order to yield $|GG_c(j\omega)| = 1$ when $\omega = \omega_c = 125$ rad/s. Then $K = 5.6$.

Now the compensator $G_c(s)$ is to be realized by $D(z)$. Set $T = 0.001$ second. We have

$$A = e^{-0.05} = 0.95, B = e^{-0.312} = 0.73, \text{ and } C = 4.85; D(z) = \frac{4.85(z-0.95)}{(z-0.73)}$$

The Root Locus of Digital Control Systems



$$\frac{Y(z)}{R(z)} = \frac{KG(z)D(z)}{1 + KG(z)D(z)}; \quad 1 + KG(z)D(z) = 0 \text{ (Characteristic equation)}$$

Plot the root locus for the characteristic equation of the sampled system as K varies.

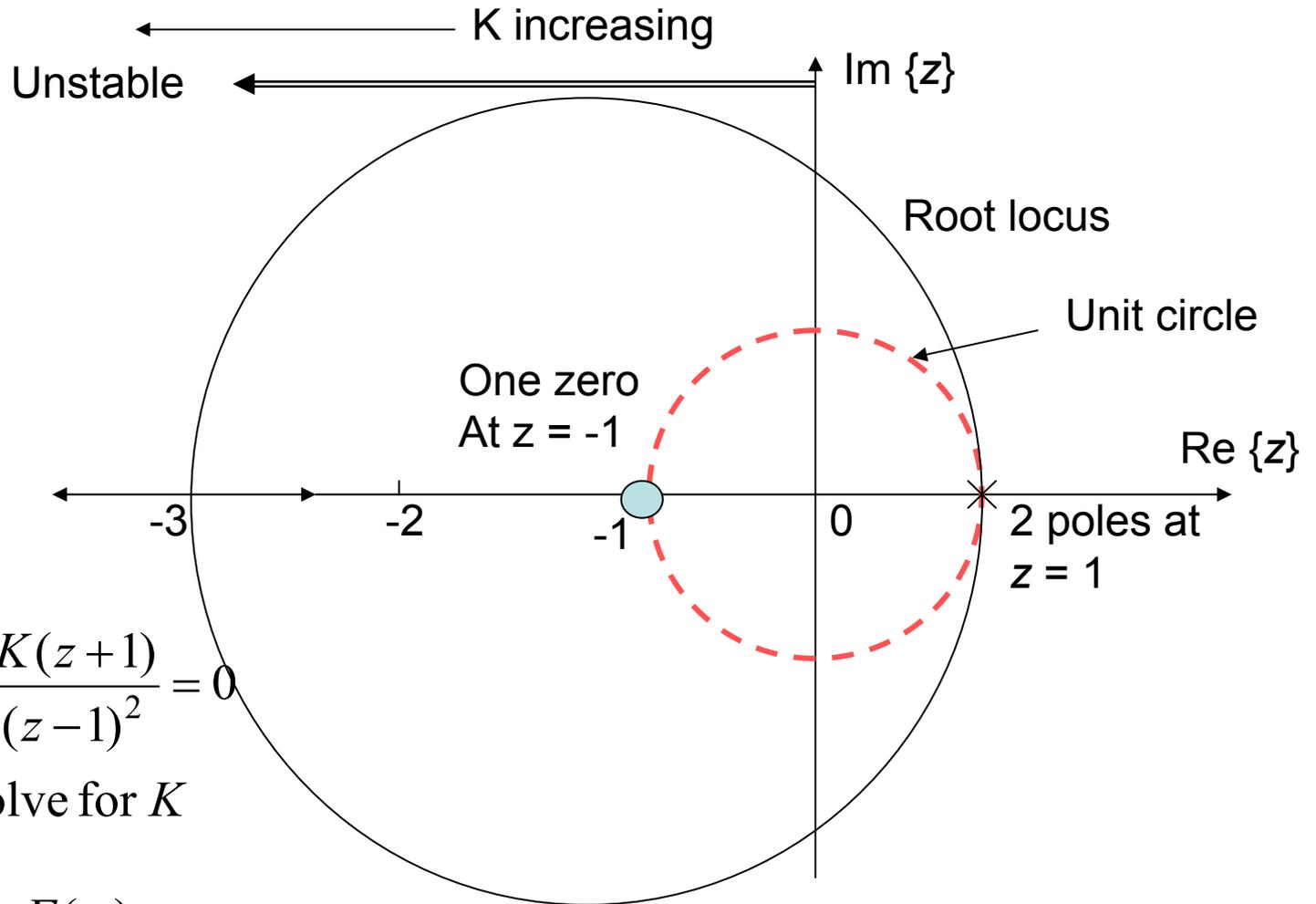
1. The root locus starts at the poles and progresses to the zeros.

2. The root locus lies on a section of the real axis to the left of an odd number of poles and zeros.

3. The root locus is symmetrical with respect to the horizontal real axis.

4. $1 + KG(z)D(z) = 0$ or $|KG(z)D(z)| = 1$ and $\angle KG(z)D(z) = 180^\circ \pm k360^\circ$

Root Locus of a Second Order System



$$1 + KG(z) = 1 + \frac{K(z+1)}{(z-1)^2} = 0$$

Let $z = \sigma$ and solve for K

$$K = -\frac{(\sigma-1)^2}{(\sigma+1)} = F(\sigma)$$

$$\frac{dF(\sigma)}{d\sigma} = 0; \sigma_1 = -3; \sigma_2 = 1$$

Design of a Digital Controller

In order to achieve a specified response utilizing a root locus method,

we will select a controller $D(z) = \frac{(z - a)}{(z - b)}$

Use $(z - a)$ to cancel one pole at $G(z)$ that lies on the positive real axis of the z - plane.

Select $(z - b)$ so that the locus of the compensated system will give a set of complex roots at a desired point within the unit circle on the z - plane.

Example 13.9: Design of a digital compensator

Let us design a compensator $D(z)$ that will result in a stable system when $G_p(s)$ is as described in Example 13.8.

With $D(z) = 1$, we have unstable system. Select $D(z) = \frac{z - a}{z - b}$

$$KG(z)D(z) = \frac{K(z+1)(z-a)}{(z-1)^2(z-b)}$$

If we select $a = 1$ and $b = 0.2$,

$$\text{we have } KG(z)D(z) = \frac{k(z+1)}{(z-1)(z-0.2)}$$

Using the equation for $F(\sigma)$, we obtain the entry point as $z = -2.56$.

The root locus is on the unit circle at $K = 0.8$.

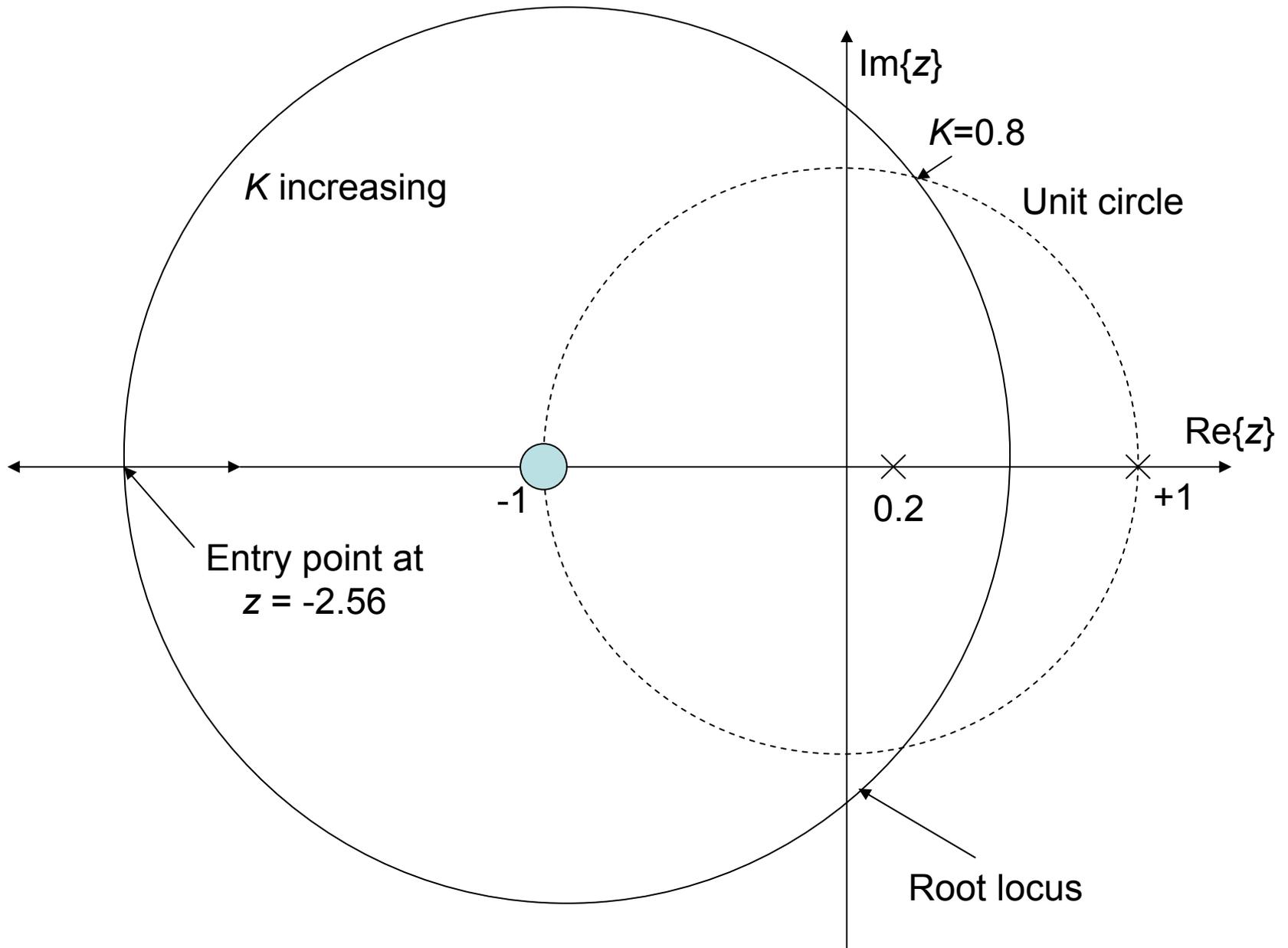
Thus the system is stable for $K < 0.8$.

If the system performance were inadequate, we would improve the root locus by selecting $a = 1$ and $b = -0.98$ so that

$$KG(z)D(z) = \frac{K(z+1)}{(z-1)(z+0.98)} \cong \frac{K}{(z-1)}$$

Then the root locus would lie on the real axis of the z - plane.

When $K = 1$, the root of the characteristic equation is at the origin.



P13.10

$$G_p(s) = \frac{1}{s(s+10)}; T = 0.1; D(z) = K$$

(a) The transfer function $G(z)D(z) = K \frac{0.0037z + 0.0026}{z^2 - 1.368z + 0.3679}$

(b) The closed-loop system characteristic equation is $1 + K \frac{0.0037z + 0.0026}{z^2 - 1.368z + 0.3679} = 0$

(c) Using root locus method, maximum value of K is 239.

(d) Using Figure 13.19 for $T/\tau = 1$ and maximum overshoot of 0.3, we find $K = 75$.

(e) When $K = 75$; $T(z) = \frac{0.2759z + 0.1982}{z^2 - 1.092z + 0.5661}$

(f) When $K = 119.5$, the poles are $z = 0.4641 \pm j0.6843$. The overshoot is 0.55.

P13.11

$$(a) G_c(s) = K \frac{s + a}{s + b}$$

By using Bode Plot, we may select $a = 0.7$, $b = 0.1$, and $K = 150$.

The compensated system overshoot and steady-state tracking error (for a ramp input) are $PO = 30\%$ and $e_{ss} < 0.01$.

$$(b) \text{ Use } G_c(s) \text{ to } D(z) \text{ method } (T = 0.1): D(z) = C \frac{z - A}{z - B} = 155.3 \frac{z - 0.9324}{z - 0.99}$$

$$A = e^{-aT}; B = e^{-bT}; C \frac{1 - A}{1 - B} = K \frac{a}{b}$$

$$A = e^{-0.007} = 0.9324; B = e^{-0.01} = 0.99; C = 155.3$$

$$(d) \text{ Use } G_c(s) \text{ to } D(z) \text{ method } (T = 0.01): D(z) = C \frac{z - A}{z - B} = 150 \frac{z - 0.993}{z - 0.999}$$

$$A = e^{-aT}; B = e^{-bT}; C \frac{1 - A}{1 - B} = K \frac{a}{b}$$

$$A = e^{-0.07} = 0.993; B = e^{-0.01} = 0.999; C = 150$$