Problem in Observer Design: Second Order System Observer Design

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}$$

In this problem we can only observe the state $y = x_1$. The observer will provide estimate of the second state x_2. We will follow the full observer (estimates of all the states) not reduced-order observer.

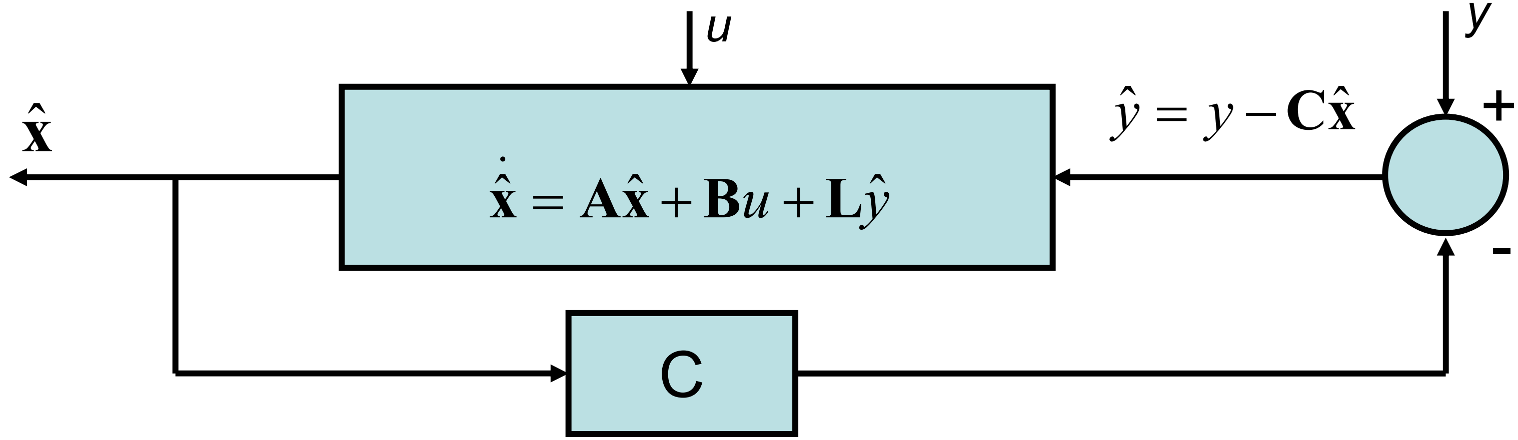
Check the system observability to guarantee the stability of the estimation error.

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \quad \text{and } \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Observability matrix $\mathbf{P}_0 = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$; $\mathbf{P}_o = 3$ (Observable)

Suppose the **desired** characteristic equation : $\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2$

We can select $\xi = 0.8$, $\omega_n = 10$ for settling time of 0.5 second



Compute the **actual** characteristic

$$\text{Det}\,(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{LC})) = \lambda^2 + (L_1 - 6)\lambda - 4(L_1 - 2) + 3(L_2 + 1)$$

Equating the coefficients in both the desired and actual equation

$$L_1 - 6 = 16 \qquad \text{and} \qquad -4(L_1 - 2) + 3\,(L_2 + 1) = 100$$

$$\mathbf{L} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} 22 \\ 59 \end{bmatrix}$$

$$\dot{\hat{\mathbf{x}}} = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \hat{\mathbf{x}} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 22 \\ 59 \end{bmatrix} (y - \hat{x}_1)$$

Ackermann's formula can be employed to place the roots of the observer characteristic equation at the desired location.

$$p(\lambda) = \lambda^n + \beta_1 \lambda^{n-1} + \ldots + \beta_{n-1}\lambda + \beta_n$$

$$\mathbf{L} = p(\mathbf{A})\, \mathbf{P}_o^{-1} [0 \ldots 1]^T$$

$$p(\mathbf{A}) = \mathbf{A}^n + \beta_1 \mathbf{A}^{n-1} + \ldots + \beta_{n-1}\mathbf{A} + \beta_n \mathbf{I}$$

Consider the previous example: $\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2$

$\xi = 0.8; \omega_n = 10$; therefore $\beta_1 = 16 \;\; \beta_2 = 100$

Compute $p(\mathbf{A}) = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}^2 + 16\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + 100\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 133 & 66 \\ -22 & 177 \end{bmatrix}$

From the previous example $\mathbf{P_o} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ $\mathbf{P}_o^{-1} = \begin{bmatrix} 1 & 0 \\ -2/3 & 1/3 \end{bmatrix}$

$$\mathbf{L} = p(\mathbf{A})\mathbf{P}_o^{-1}[0 .. 0 \;\; 1]^T = \begin{bmatrix} 133 & 66 \\ -22 & 177 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2/3 & 1/3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 22 \\ 59 \end{bmatrix}$$

Problem on Pole Placement

Consider the following system which uses the state feedback control **u = -Kx**

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u;\ \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}\ \mathrm{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The desired closed-loop poles are: $s = -2 + j4;\ s = -2 - j4;\ s = -10$

Check the controllability matrix of the system

$$\mathbf{P}_{\mathrm{c}} = \left[\mathbf{B}\ \ \mathbf{AB}\ \ \mathbf{A}^2\mathbf{B}\ \ \mathbf{A}^{n\text{-}1}\mathbf{B}\right] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix} = 3$$

The desired characteristic equation: $(s + 2 - j4)(s + 2 + j4)(s + 10) = s^3 + 14s^2 + 60s + 200$

$$\left|s\mathbf{I} - \mathbf{A} + \mathbf{BK}\right| = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\left[k_1\ \ k_2\ \ k_3\right]$$

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 + k_1 & 5 + k_2 & s + 6 + k_3 \end{bmatrix} = s^3 + (6 + k_3)s^2 + (5 + k_2)s + 1 + k_1 = s^3 + 14s^2 + 60s + 200$$

$$k_3 + 6 = 14;\ k_2 + 5 = 60;\ k_1 + 1 = 200;\ \ k_1 = 199;\ k_2 = 55;\ k_3 = 8;\ \mathbf{K} = \left[199\ \ \ 55\ \ \ 8\right]$$

Problem on State Observer

Consider the system given. Use the observed state feedback and design a full-order state observer assuming the desired poles of the observer matrix as $s_{1,2} = -10$

$$\mathbf{A} = \begin{bmatrix} 0 & 20.6 \\ 1 & 0 \end{bmatrix}; \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Check the observability matrix : $\mathbf{P}_o = 2$ (Observable)

The desired characteristic equation : $(s+10)^2 = s^2 + 20s + 100$

$$\text{Det}\,(s\mathbf{I} - \mathbf{A} + \mathbf{LC})) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 20.6 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} s & -20.6 + L_1 \\ -1 & s + L_2 \end{bmatrix}$$

$$= s^2 + L_2 s - 20.6 + L_1 = 0$$

Compare the **desired** and the **actual** characteristic equations

$$L_1 = 120.6; \quad L_2 = 20; \quad \mathbf{L} = \begin{bmatrix} 120.6 \\ 20 \end{bmatrix}$$