Chapter 5
The Performance of Feedback Control Systems

Test Input Signals

Performance of a Second-Order System

Effects of Third Pole and a Zero on the Second-Order System Response

Estimation of the Damping Ratio

The Steady-State Error of Feedback Control Systems

Performance Indices

Preview

- The ability to adjust the transient and steady-state response of a feedback control system is a beneficial outcome of the design of control systems.
- Input signals such as step and ramp are used to test the response of the control system.
- In this chapter, common time-domain specifications are introduced:
  - Transient Response and Steady-State Response
  - Percent overshoot
  - Settling time
  - Time to peak
  - Time to Rise
  - Steady-State Tracking Error.
- The concept of a performance index that represents a system's performance by a single number (or index) will be considered..

Test Input Signals

Table 2.3 is used to obtain the Laplace transform

Step	$r(t) = A, t\rangle 0$	$R(s) = A / s$
Ramp	$r(t) = At, t\rangle 0$	$R(s) = A/s^2$
Parabolic	$r(t) = At^2, t\rangle 0$	$R(s) = 2A / s^3$

See Table 5.1

Performance of a Second-Order System

$$G(s) = \frac{K}{s(s+p)}$$

$$Y(s) = \frac{G(s)}{1+G(s)}R(s) = \frac{K}{s^2+ps+K}R(s)$$

$$Y(s) = \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)} \text{[Use generalized notation in Section 2.4]}$$

$$y(t) = 1 - \frac{1}{\beta}e^{-\xi\omega_n t}\sin(\omega_n \beta t+\theta) \text{[See Laplace Transform from Table 2.3];}$$

$$\beta = \sqrt{1-\xi^2}, \theta = \cos^{-1}\xi$$

$$Y(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \text{ For a unit impulse}: R(s) = 1$$

$$y(t) = \frac{\omega_n}{\beta}e^{-\xi\omega_n t}\sin\omega_n \beta t$$

Standard Performance Measures

Rise Time T_r and Peak Time, T_p

$$\text{Percent overshoot P. O.} = \frac{M_{p_i} - f_v}{f_v} \times 100\%$$

M_{p_i} is the peak value of the time response,

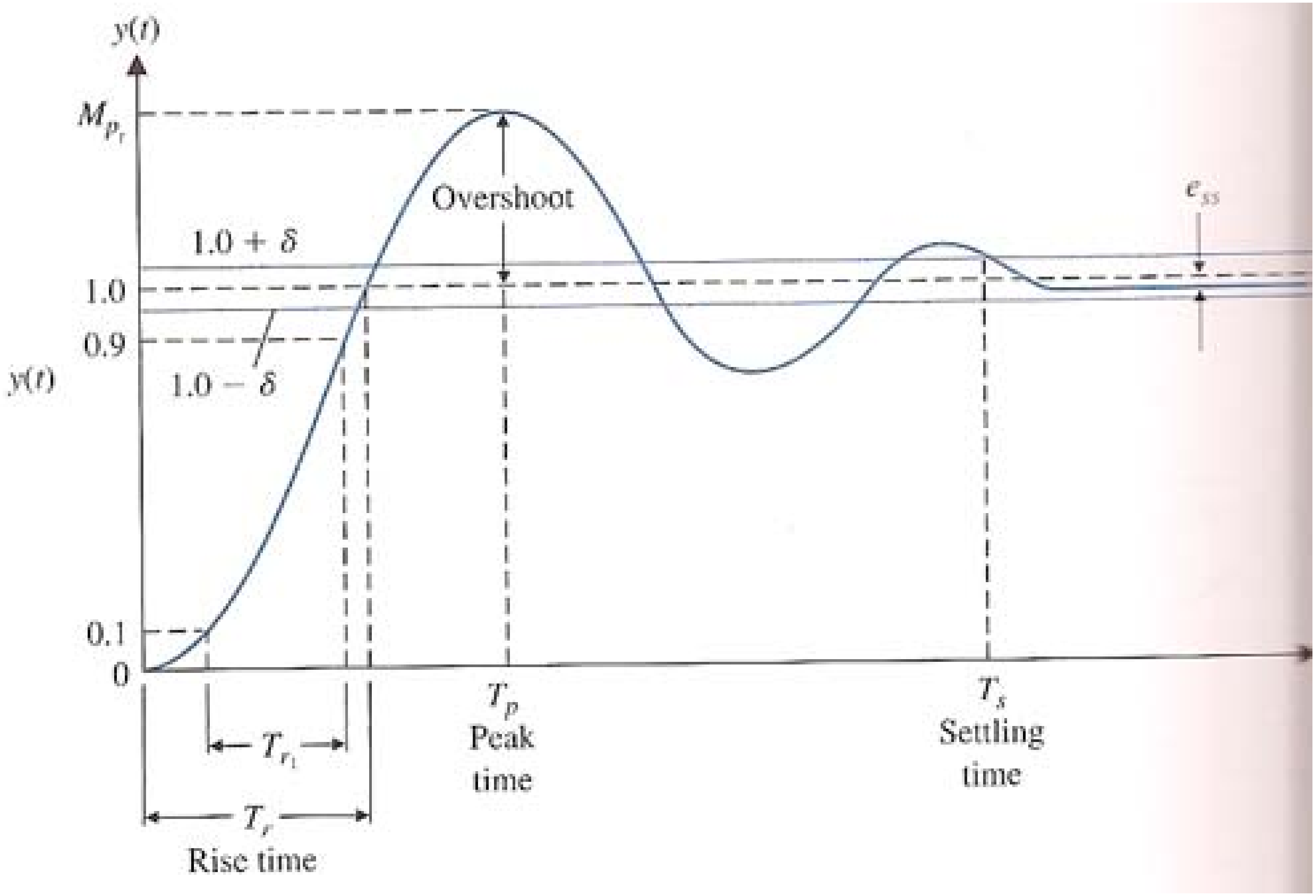
f_v is the final value of the response

$$\text{Settling time, } T_s = \frac{4}{\xi\omega_n} \text{ Peak Time, } T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}};$$

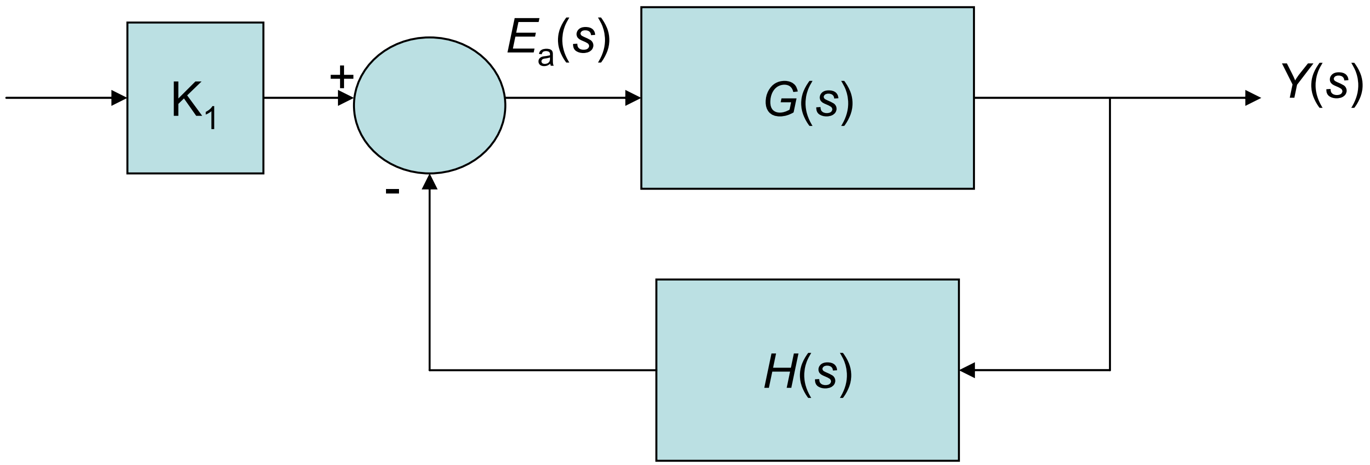
$$\text{Peak response, } M_{p_i} = 1 + e^{-\xi\pi/\sqrt{1-\xi^2}}; \text{P.O.} = 100e^{-\xi\pi/\sqrt{1-\xi^2}}$$

ξ : Damping ratio

Step Response of a Control System



The Steady-State Error of Feedback Control Systems



$$H(s) = \frac{K_2}{(\tau s + 1)}; \lim_{s \to 0} H(s) = K_2$$

$$E(s) = R(s) - Y(s) = [1 - T(s)]R(s)$$

$$T(s) = \frac{K_1 G(s)}{1 + K_1 G(s)}; E(s) = \frac{1}{1 + K_1 G(s)} R(s)$$

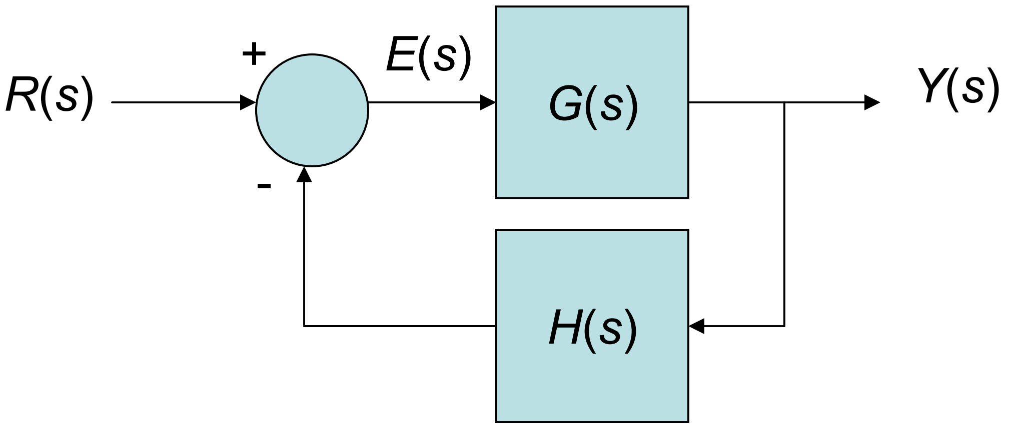
$$e_{ss} = \lim_{s \to 0} sE(s) = \frac{1}{1 + K_1 G(0)}$$

Table 5.5 Summary of Steady-State Errors

K_p: position error constant; $K_{v:}$ Velocity error constant

Number of Integrations in $G(s)$, Type Number	Step $R(s)$=A/s	Input Ramp A/s^2
0	e_{ss}=1/1+K_p	Infinite
1	e_{ss}= 0	A/K_v
2	e_{ss}= 0	0

E5.1: In order to get $e_{ss} = 0$; When the input is a step we require one integration (type 1 system). For a ramp input we require type 2 system.



See Table 5.5

Performance Indices

A performance index is a quantitative measure of the performance of a system and is chosen so that emphasis is given to the important system specifications.

A system is considered an optimum control system when the system parameters are adjusted so that index reaches an extremum value, commonly a minimum value.

A performance index, to be useful, must be a number that is always positive or zero

$$\text{ISE} = \int_0^T e^2(t)\,dt \ \text{[Integral of the square of the error]}$$

$$\text{IAE} = \int_0^T \left|e(t)\right|dt \ \text{[Integral of the absolute magnitude of the error]}$$

$$\text{ITAE} = \int_0^{\text{T}} \text{t}\left|\text{e(t)}\right|dt \ \text{[Integral of time multiplied by absolute error]}$$

The Optimum Coefficients of *T*(*s*) Based on the ITAE Criterion for a Step Input

$$s+\omega_n$$

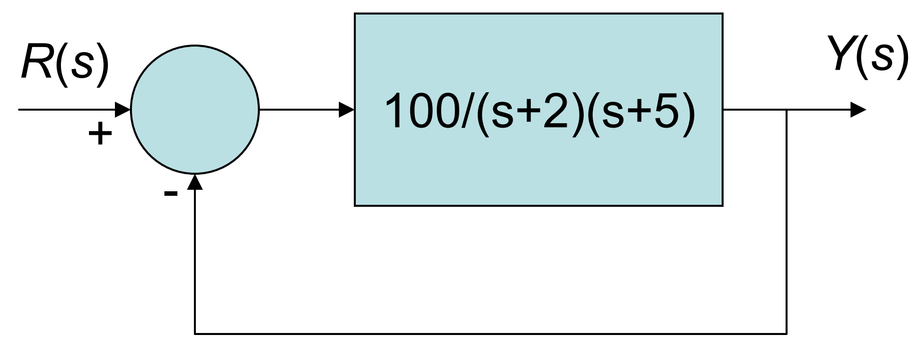
$$s^2+1.4\omega_n s+\omega_n^2$$

$$s^3+1.75\omega_n s^2+2.15\omega_n^2 s+\omega_n^3$$

The coefficients that will minimize the ITAE performance criterion for a step input have been determined for the general closed - loop transfer function

$$T(s)=\frac{Y(s)}{R(s)}=\frac{b_0}{s^n+b_{n-1}s^{n-1}+....+b_1 s+b_0} \quad (5.47)$$

E5.2:



$$T(s)=\frac{Y(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}=\frac{100}{(s+2)(s+5)+100}=\frac{100}{s^2+2\xi\omega_n s+\omega_n^2}$$

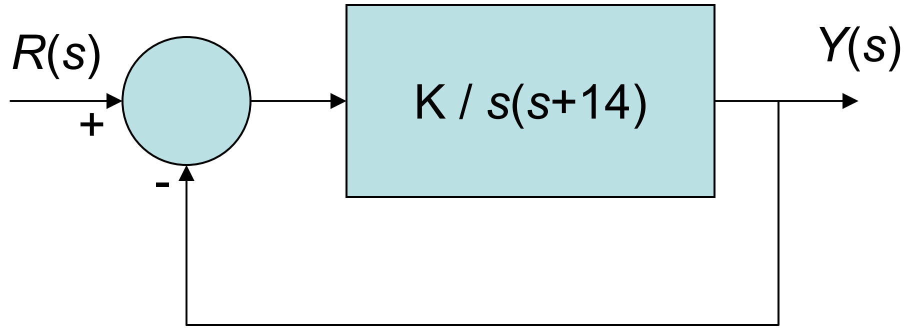
$$e_{ss}=\frac{A}{1+K_p}\,(R(s)=A/s);\;\; K_p=\lim_{s\longrightarrow 0} G(s)=\frac{100}{10}=10;\, e_{ss}=\frac{\mathrm{A}}{11}$$

The closed-loop system is a second order system with natural frequency $\omega_n=\sqrt{110}$

The damping ratio $\xi=\dfrac{7}{2\sqrt{110}}=0.334$

$$P.O.=0.909\left(100\,e^{-\pi\xi/\sqrt{1-\xi^2}}\right)=29\%\ \text{(Equation 5.16)}$$

E5.3



$$T(s) = \frac{G(s)}{1+G(s)} = \frac{K}{s(s+14)+K} = \frac{K}{s^2+14s+K}$$

Refer to Table 5.6 to find the optimum coefficients

$s^2 + 1.4\omega_n s + \omega_n^2; \quad s^2 + 14s + K$. We will get

$$\omega_n = 10; K = \omega_n^2 = 100; \ \xi = \frac{14}{2\omega_n} = 0.7$$

Use Figure 5.8 to find P.O. $\approx 5\%$

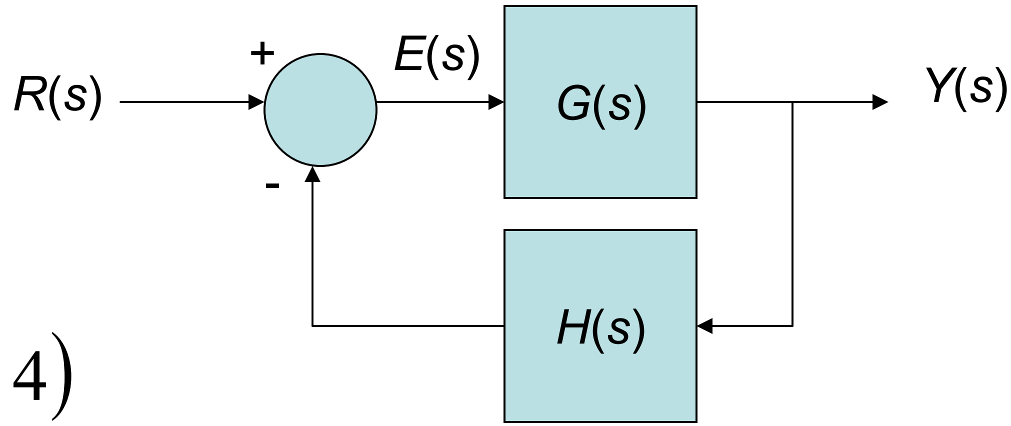
E5.8:

$$G(s) = \frac{K}{s\left(s + \sqrt{2K}\right)}; T(s) = \frac{K}{s^2 + \sqrt{2K}\, s + K}$$

$$\xi = \frac{\sqrt{2}}{2}; \omega_n = \sqrt{K}; \text{P.O.} = 100\, e^{-\pi\xi/\sqrt{1-\xi^2}} = 4.3\%; T_s = \frac{4}{\xi\omega_n} = \frac{8}{\sqrt{2K}}$$

The settling time is less than 1 second whenever $K \rangle 32$

E5.10 The system is a type 1 (Table 5.5). The error constants are



$$G(s) = \frac{10(s+4)}{s(s+1)(s+3)(s+8)}$$

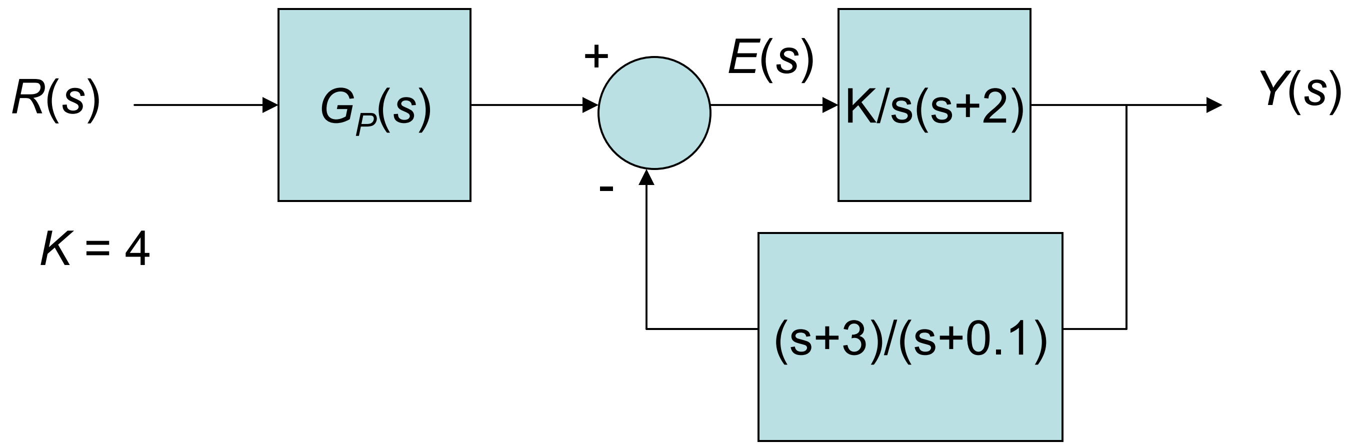
$$K_p = \infty \text{ and } K_v = 1.67 \; (Kv = \lim_{s \to 0} sG(s))$$

The steady - state error for a step input is 0

The steady - state error for a ramp is 0.6A (Equation 5.29)

A is the amplitude of the ramp input.

E5.13



$K = 4$

a) The tracking error is given by: $E(s) = [1 - T(s)]R(s)$ (Read Section 5.8)

The steady-state tracking error with $R(s) = 1/s$ is

$$e_{ss} = \lim_{s \to 0} s[1 - T(s)]R(s) = \lim_{s \to 0} [1 - T(s)] = 1 - T(0)$$

The closed-loop transfer function is

$$T(s) = \frac{K(s+0.1)}{s(s+0.1)(s+2) + K(s+3)}; T(0) = 0.033$$

$$e_{ss} = 1 - 0.033 = 0.967$$

b) Use $G_P(s) = 30$

$$\lim_{s \to 0} s[1 - T(s)G_P(s)]R(s) = 1 - \lim_{s \to 0} T(s)G_P(s) = 1 - 30T(0) = 0$$