Chapter 3
State Variable Models

The State Variables of a Dynamic System

The State Differential Equation

Signal-Flow Graph State Variables

The Transfer Function from the State Equation

Introduction

- In the previous chapter, we used Laplace transform to obtain the transfer function models representing linear, time-invariant, physical systems utilizing block diagrams to interconnect systems.

- In Chapter 3, we turn to an alternative method of system modeling using **time-domain methods**.

- In Chapter 3, we will consider physical systems described by an **nth-order ordinary differential equations**.

- Utilizing a set of variables known as **state variables**, we can obtain a set of first-order differential equations.

- The time-domain state variable model lends itself easily to computer solution and analysis.

Time-Varying Control System

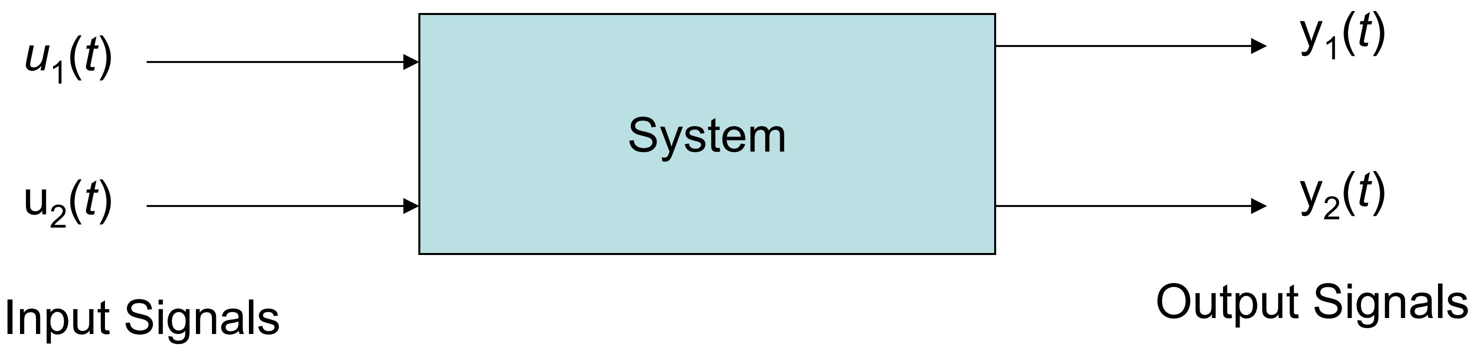
- With the ready availability of digital computers, it is convenient to consider the time-domain formulation of the equations representing control systems.
- The time-domain is the mathematical domain that incorporates the response and description of a system in terms of time t.
- The time-domain techniques can be utilized for nonlinear, time-varying, and multivariable systems (a system with several input and output signals).
- A time-varying control system is a system for which one or more of the parameters of the system may vary as a function of time.
- For example, the mass of a missile varies as a function of time as the fuel is expended during flight

Terms

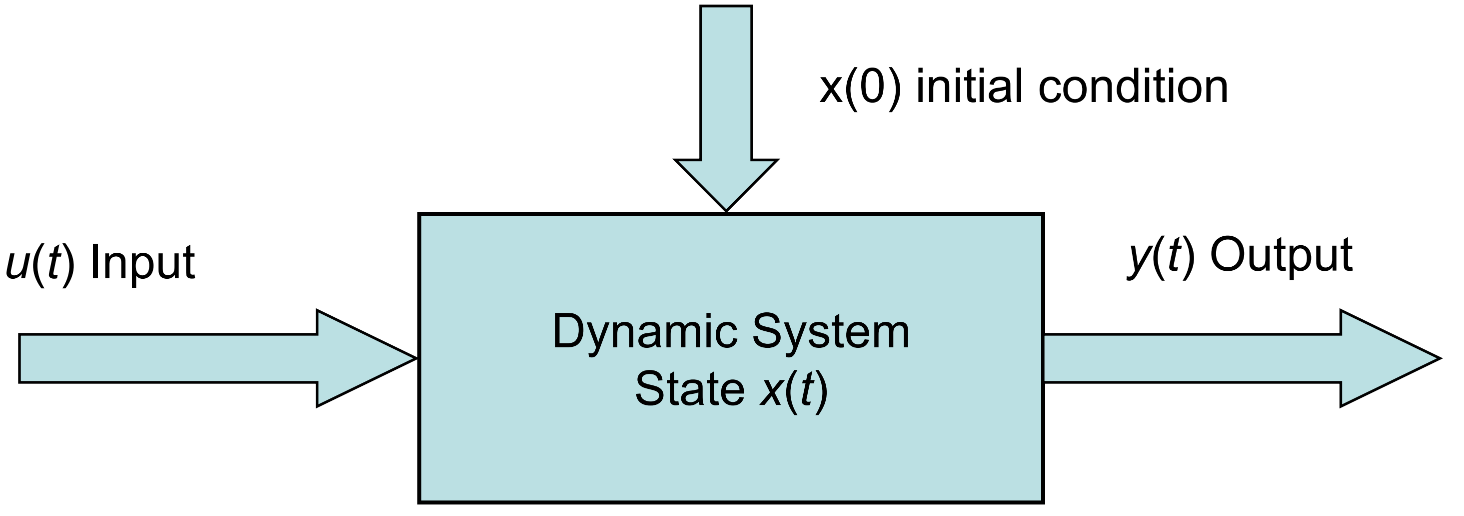
- **State:** The state of a dynamic system is the smallest set of variables (called state variables) so that the knowledge of these variables at $t = t_0$, together with the knowledge of the input for $t \geq t_0$, determines the behavior of the system for any time $t \geq t_0$.
- **State Variables:** The state variables of a dynamic system are the variables making up the smallest set of variables that determine the state of the dynamic system.
- **State Vector:** If n state variables are needed to describe the behavior of a given system, then the n state variables can be considered the n components of a vector x. Such vector is called a state vector.
- **State Space:** The n-dimensional space whose coordinates axes consist of the x_1 axis, x_2 axis, .., x_n axis, where x_1, x_2, .., x_n are state variables, is called a state space.
- **State-Space Equations:** In state-space analysis, we are concerned with three types of variables that are involved in the modeling of dynamic system: input variables, output variables, and state variables.

The State Variables of a Dynamic System

- The state of a system is a set of variables such that the knowledge of these variables and the input functions will, with the equations describing the dynamics, provide the future state and output of the system.
- For a dynamic system, the state of a system is described in terms of a set of state variables.



State Variables of a Dynamic System



The state variables describe the future response of a system, given the present state, the excitation inputs, and the equations describing the dynamics

The State Differential Equation

The state of a system is described by the set of first-order differential equations written in terms of the state variables (x_1, x_2, .., x_n)

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_{11}u_1 + \ldots + b_{1m}u_m$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + b_{21}u_1 + \ldots + b_{2m}u_m$$

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + b_{n1}u_1 + \ldots + b_{nm}u_m$$

$$\dot{x} = \frac{dx}{dt}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ . & . & . \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} \ldots . b_{1m} \\ \ldots\ldots\ldots\ldots \\ b_{n1} \ldots . b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ . \\ u_m \end{bmatrix}$$

A x B u

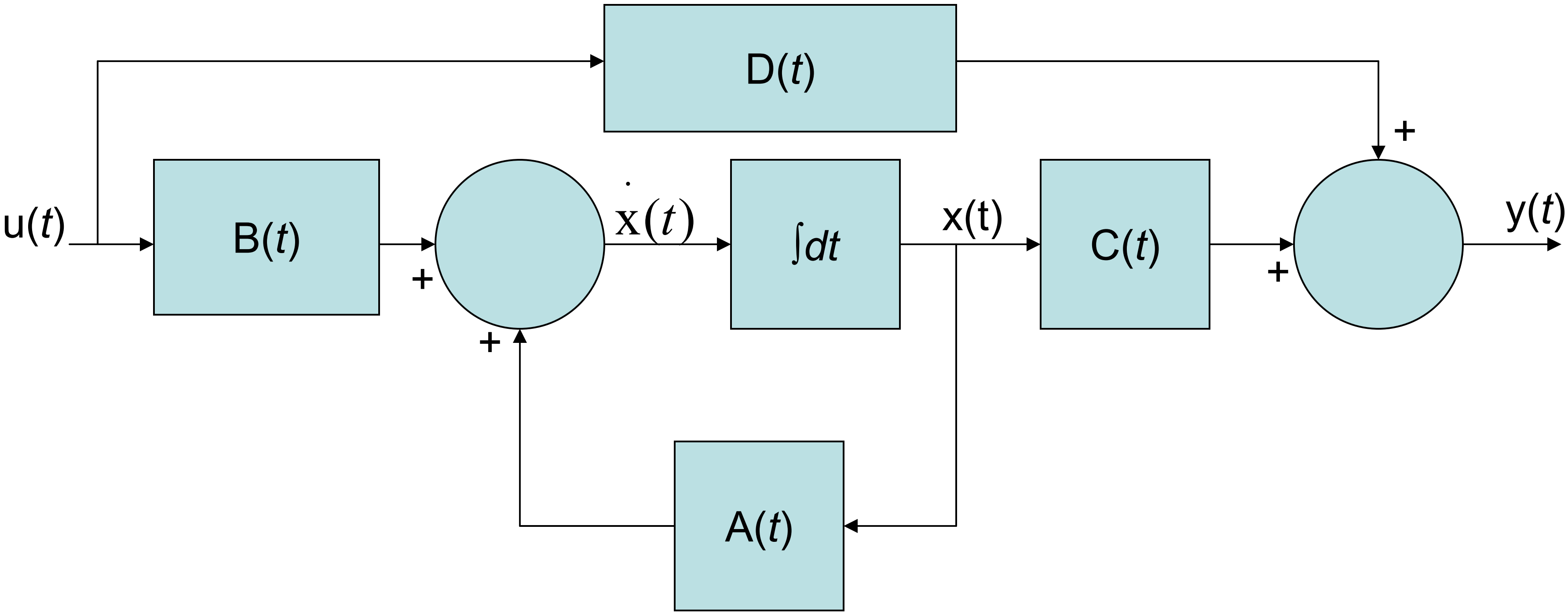
$$\dot{\mathrm{x}} = \mathrm{Ax} + \mathrm{Bu} \text{ (State differential equation)}$$

$$\mathrm{y} = \mathrm{Cx} + \mathrm{Du} \text{ (Output equation - output signals)}$$

A : State matrix; B : input matrix

C : Output matrix; D : direct transmission matrix

Block Diagram of the Linear, Continuous Time Control System



$$\dot{x}(t) = A(t)x(t) + B(t)\,u(t)$$

$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

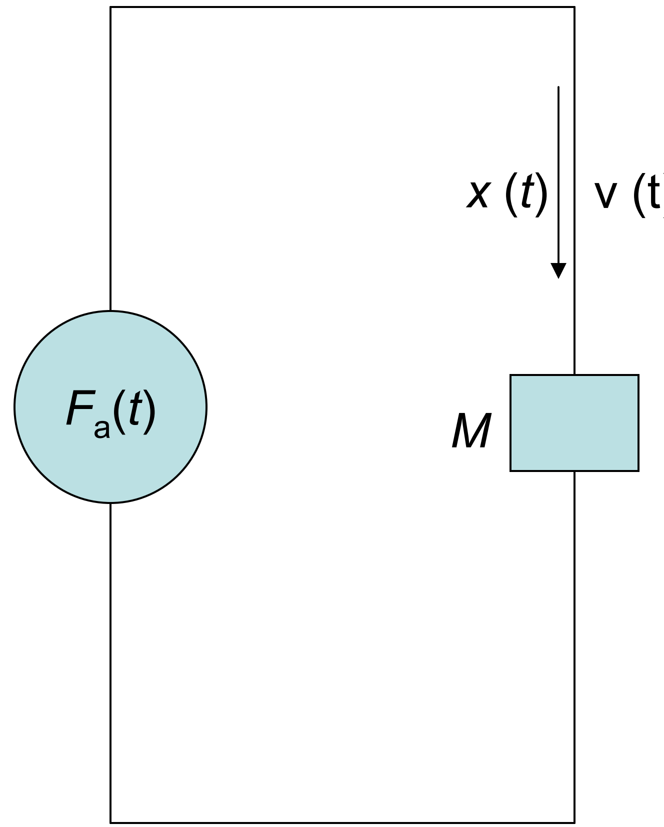
Mass Grounded, *M* (kg)

Mechanical system described by the first-order differential equation

Appied torque $T_{\mathrm{a}}(t)$ (N - m)

Linear velocity $v(t)$ (m/sec)

Linear position $x(t)$ (m)



$$F_a(t) = M\frac{dv}{dt} = M\frac{d^2x(t)}{dt^2}$$

$$v(t) = \frac{1}{M}\int_{t_0}^{t} F_a(t)dt$$

Mechanical Example: Mass-Spring Damper

A set of state variables sufficient to describe this system includes the position and the velocity of the mass, therefore, we will define a set of state variables as (x_1, x_2)

$$x_1(t) = y(t)$$

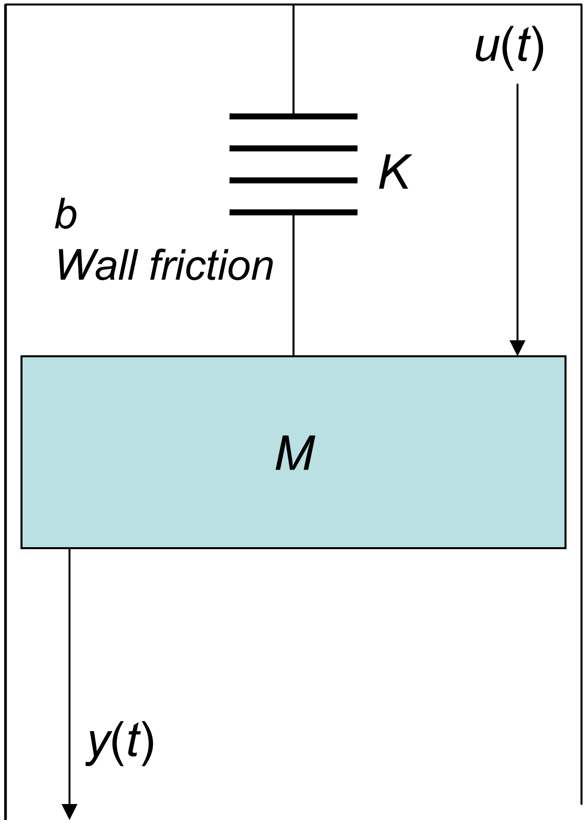
$$x_2(t) = \frac{dy(t)}{dt}$$

$$M\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = u(t)$$

$$M\frac{dx_2}{dt} + bx_2 + kx_1 = u(t)$$

$$\frac{dx_1}{dt} = x_2;$$

$$\frac{dx_2}{dt} = -\frac{b}{m}x_2 - \frac{k}{M}x_1 + \frac{1}{M}u$$



k : Spring constant

Example 1: Consider the previous mechanical system. Assume that the system is linear. The external force *u*(*t*) is the input to the system, and the displacement *y*(*t*) of the mass is the output. The displacement y(*t*) is measured from the equilibrium position in the absence of the external force. This system is a single-input-single-output system.

$$m\ddot{y} + b\dot{y} + ky = u$$

This is a second order system. It means it involves two integrators.

Let us define two variables: $x_1(t)$ and $x_2(t)$

$$x_1(t) = y(t); x_2(t) = \dot{y}(t); \text{ then } \dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

The output equation is: $y = x_1$

In a vector matrix form, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{\text{k}}{\text{m}} & -\frac{\text{b}}{\text{m}} \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \text{ (State Equation)}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{(Output Equation)}$$

The state equation and the output equation are in the standard form:

$$\dot{\text{x}} = \text{Ax} + \text{B}u; \text{y} = \text{Cx} + D\text{u}$$

$$\text{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, \text{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \text{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$$

Electrical and Mechanical Counterparts

Energy	Mechanical	Electrical
Kinetic	**Mass / Inertia** 0.5 mv^2 / 0.5 $j\omega^2$	**Inductor** 0.5 Li^2
Potential	**Gravity:** mgh **Spring:** 0.5 kx^2	**Capacitor** 0.5 Cv^2
Dissipative	**Damper / Friction** 0.5 Bv^2	Resistor Ri^2

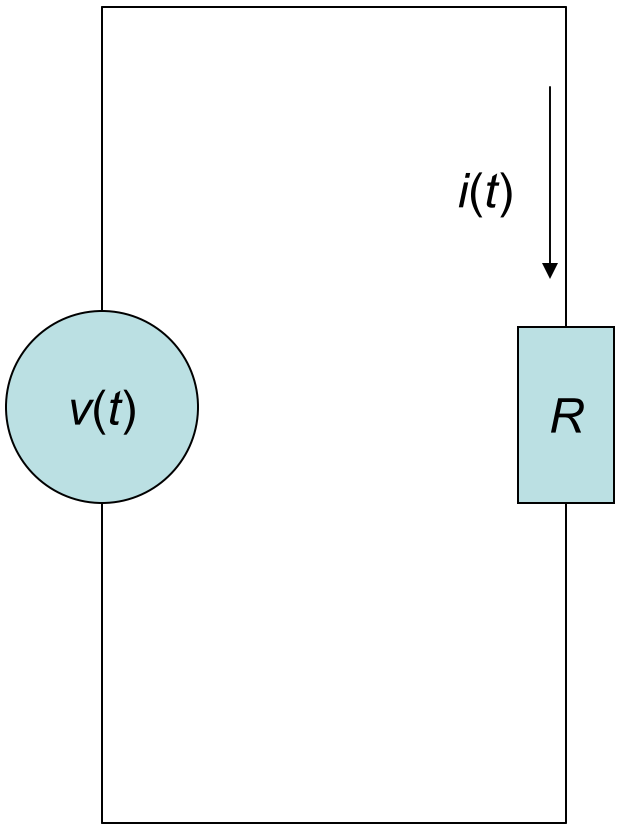
Resistance, *R* (ohm)

Appied voltage $v(t)$

Current $i(t)$

$v(t) = Ri(t)$

$i(t) = \frac{1}{R} v(t)$



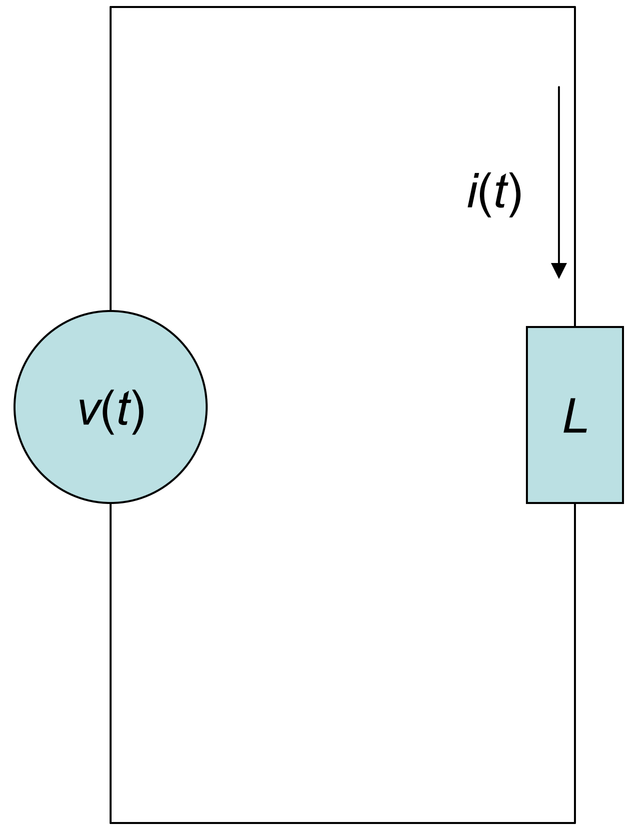
Inductance, *L* (H)

Appied voltage $v(t)$

Current $i(t)$

$$v(t) = L\frac{di(t)}{dt}$$

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v(t)dt$$



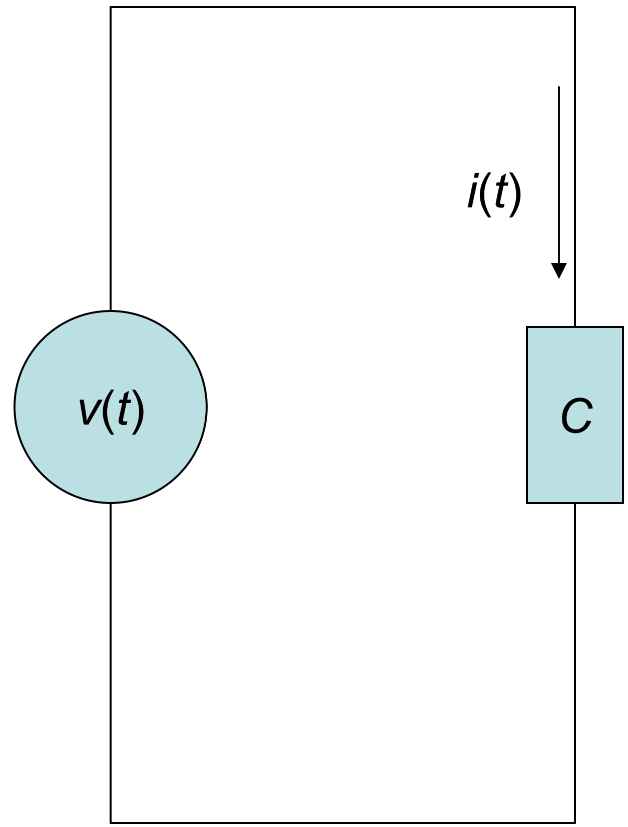
Capacitance, *C* (F)

Appied voltage $v(t)$

Current $i(t)$

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(t)dt$$

$$i(t) = C\frac{dv(t)}{dt}$$



Electrical Example: An *RLC* Circuit

$x_1 = v_C(t); x_2 = i_L(t)$

$\xi = (1/2)Li_L^2 + (1/2)Cv_c^2$

$x_1(t_0)$ and $x_2(t_0)$ is the total initial energy of the network

USE KCL at the junction

$$i_c = C\frac{dv_c}{dt} = +u(t) - i_L$$

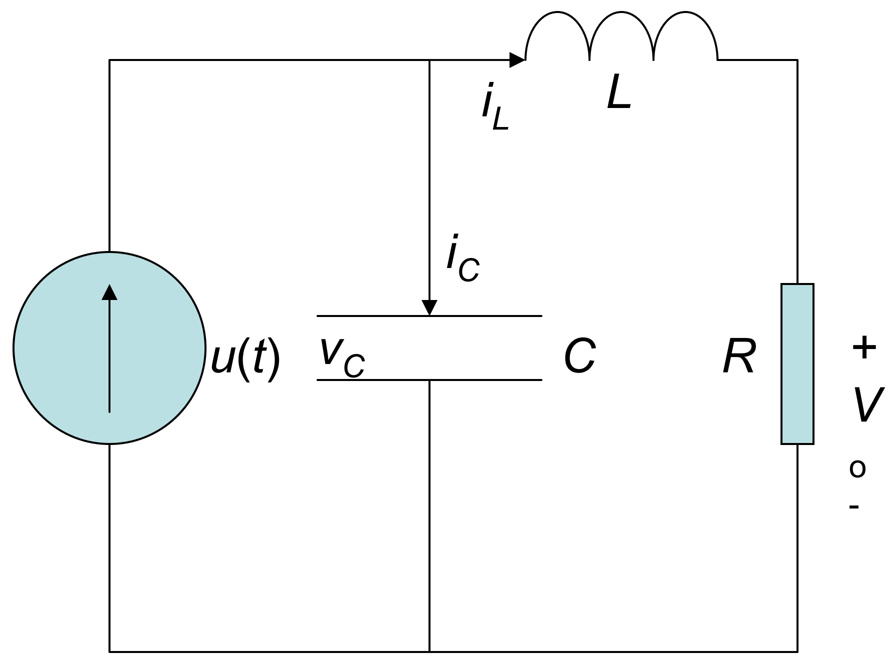
$$L\frac{di_L}{dt} = -Ri_L + v_c$$

The output of the system is represented by : $v_o = Ri_L(t)$

$$\frac{dx_1}{dt} = -\frac{1}{C}x_2 + \frac{1}{C}u(t)$$

$$\frac{dx_2}{dt} + \frac{1}{L}x_1 - \frac{R}{L}x_2$$

The output signal is then : $y_1(t) = v_o(t) = Rx_2$



Example 2: Use Equations from the *RLC* circuit

$$\dot{x} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u(t)$$

The output is

$$y = \begin{bmatrix} 0 & R \end{bmatrix} x$$

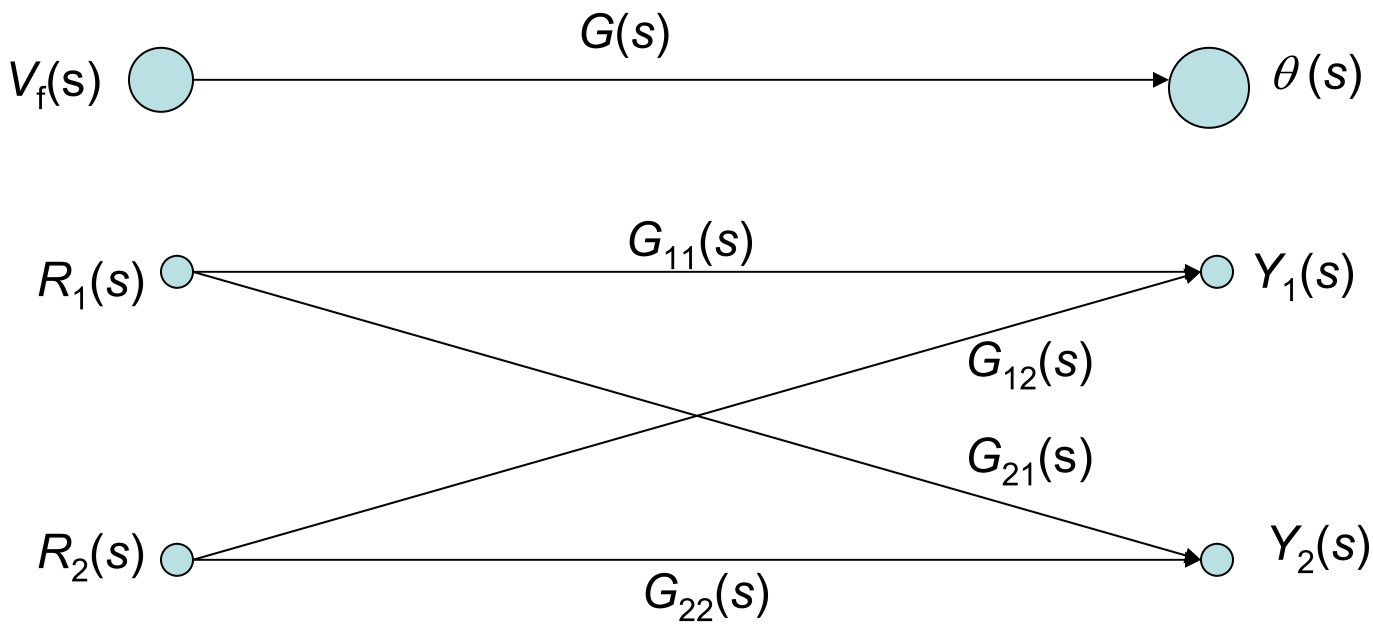
When $R = 3, L = 1, C = 1/2$, we have

$$\dot{\mathrm{x}} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 3 \end{bmatrix} x$$

Signal-Flow Graph Model

A signal-flow graph is a diagram consisting of nodes that are connected by several directed branches and is a graphical representation of a set of linear relations. Signal-flow graphs are important for feedback systems because feedback theory is concerned with the flow and processing of signals in system.



Read Examples: 2.8 - 2.11

Mason's Gain Formula for Signal Flow Graphs

In many applications, we wish to determine the relationship between an input and output variable of the signal flow diagram. The transmittance between an input node and output node is the overall gain between these two nodes.

$$P = \frac{1}{\Delta}\sum_k P_k \Delta_k$$

P_k = path gain of $\mathrm{k_{th}}$ forward path

Δ = determinant of graph

= 1 - (sum of all individual loop gain) +

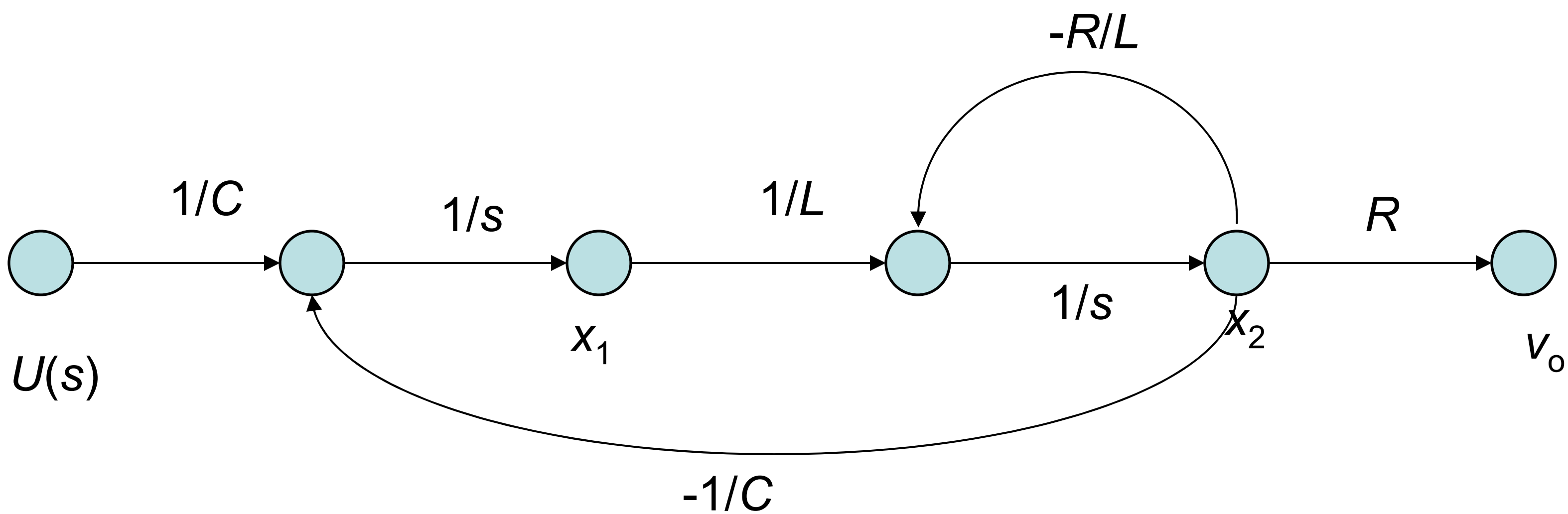
(sum of gain of all possible combinations of two nontouching loops)

- (sum of gain products of all possible combinations of these nontouching loops) + ..

$$= 1 - \sum_a L_a + \sum_{b,c} L_b L_c - \sum_{d,e,f} L_d L_c L_f$$

Δ_k = cofactor of the kth forward path determinant of the graph with the loops touching the kth forward path removed, that is, the cofactor Δ_k is obtained from Δ by removing the loops that touch path P_k.

Signal-Flow Graph State Models



$$G(s) = \frac{V_o(s)}{U(s)} = \frac{\alpha}{s^2 + \beta s + \gamma}$$

$$\dot{x}_1 = -\frac{1}{C} x_2 + \frac{1}{C} u(t)$$

$$\dot{x}_2 = \frac{1}{L} x_1 - \frac{R}{L} x_2 ; v_o = R x_2$$

$$\frac{Vo(s)}{U(s)} = \frac{R / LCs^2}{1 + (R / Ls) + (1 / LCs^2)} ; = \frac{R / LC}{s^2 + (R / L)s + (1 / LC)}$$

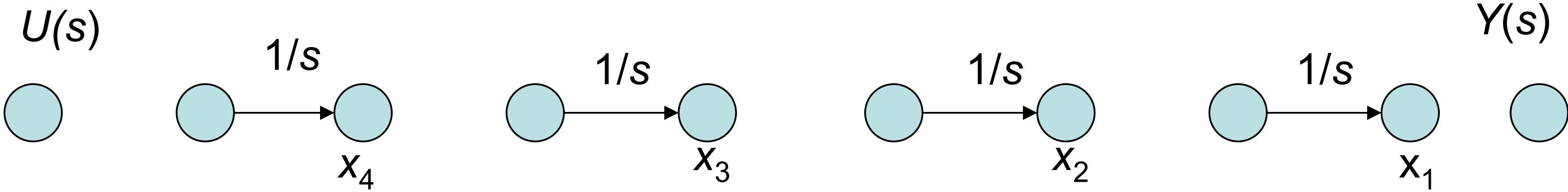
$$G(s) = \frac{Y(s)}{U(s)} = \frac{s^m + b_{m-1}s^{m-1} + \ldots. + b_1 s + b_o}{s^n + a_{n-1}s^{n-1} + \ldots. + a_1 s + a_o}$$

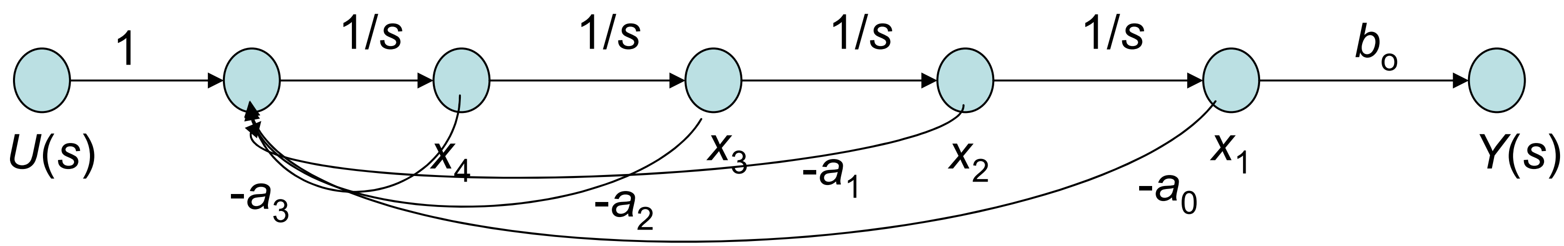
$$G(s) = \frac{Y(s)}{U(s)} = \frac{s^{-(n-m)} + b_{m-1}s^{-(n-m+1)} + \ldots. + b_1 s^{-(n-1)} + b_o s^{-n}}{1 + a_{n-1}s^{-1} + \ldots. + a_1 s^{-(n-1)} + a_o s^{-n}}$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\sum_k P_k \Delta k}{\Delta}$$

$$G(s) = \frac{\sum_k P_k}{1 - \sum_{q=1}^{N} Lq} = \frac{\text{Some of the forward-path factors}}{\text{1-sum of the feedback loop factor}}$$

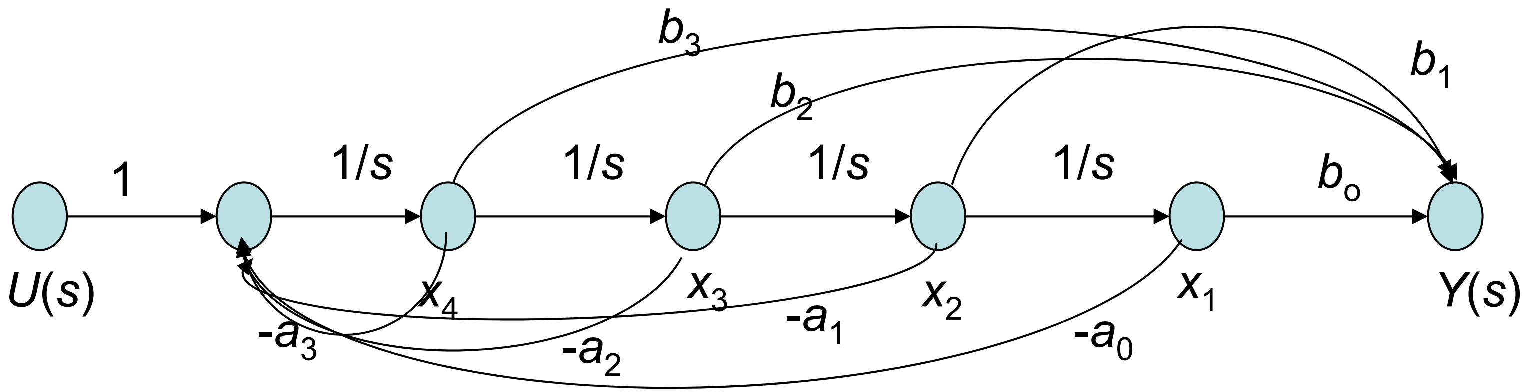
Phase Variable Format: Let us initially consider the fourth-order transfer function. Four state variables (x_1, x_2, x_3, x_4); Number of integrators equal the order of the system.





$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$= \frac{b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}$$



$$\dot{x}_1 = x_2; \dot{x}_2 = x_3; \dot{x}_3 = x_4$$

$$G(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$= \frac{b_3 s^{-1} + b_2 s^{-2} + b_1 s^{-3} + b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}$$

$$\dot{x}_4 = -a_0 x_1 - a_1 x_2 - a_2 x_3 - a_3 x_4 + u$$

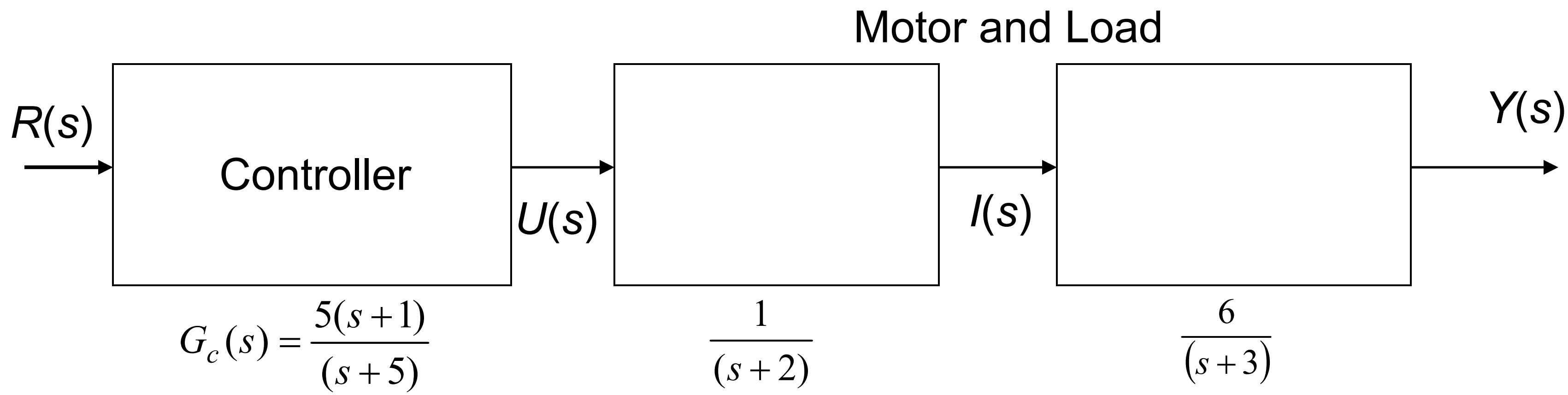
$$y(t) = b_0 x_1 + b_1 x_2 + b_2 x_3 + b_3 x_4$$

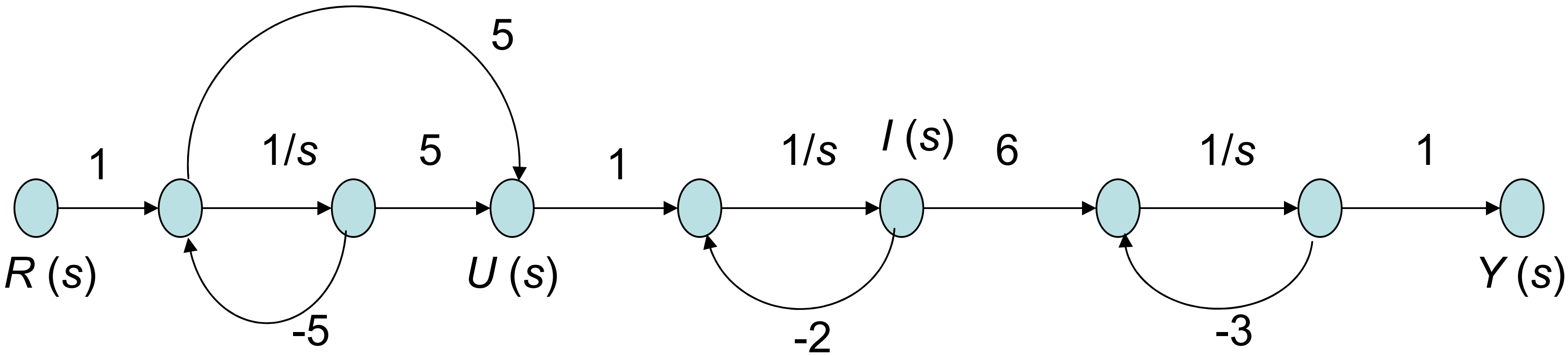
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

Read Examplev 3.1 of the textbook

$$y(t) = \mathrm{Cx} = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

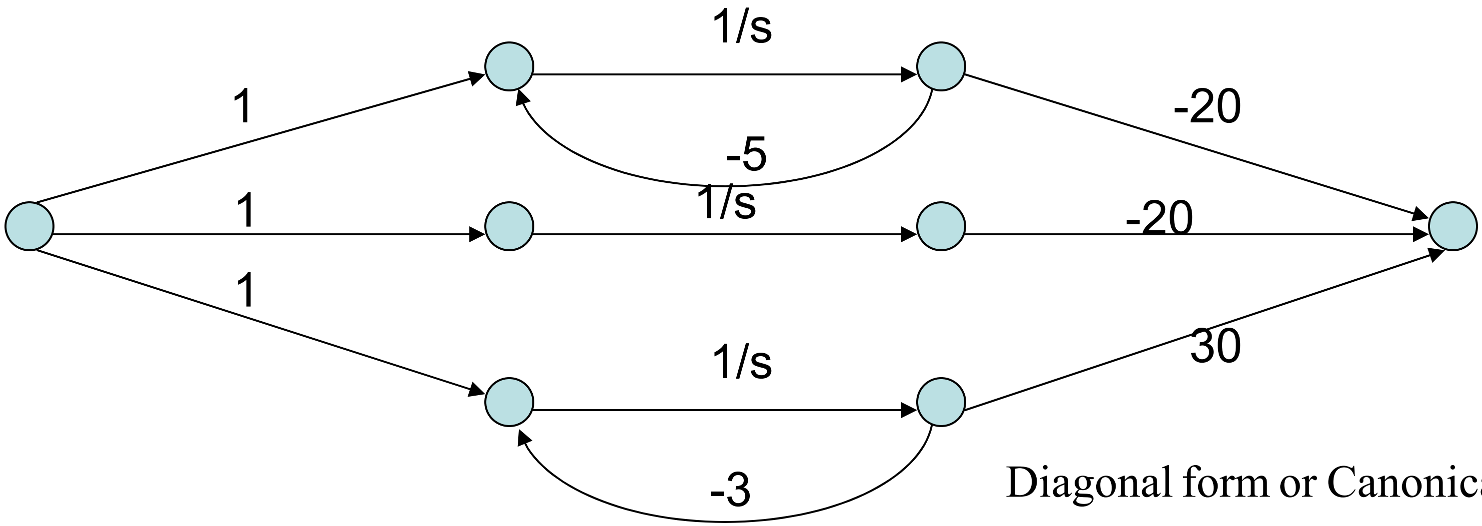
Alternative Signal-Flow Graph State Models





$$\dot{x} = \begin{bmatrix} -3 & 6 & 0 \\ 0 & -2 & -20 \\ 0 & 0 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} r(t) \qquad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

The State Variable Differential Equations



Diagonal form or Canonical form

$$\frac{Y(s)}{R(s)} = T(s) = \frac{30(s+1)}{(s+5)(s+2)(s+3)} = \frac{q(s)}{(s-s_1)(s-s_2)(s-s_3)}$$

$$\frac{Y(s)}{R(s)} = T(s) = \frac{k_1}{(s+5)} + \frac{k_2}{(s+2)} + \frac{k_3}{(s+3)}$$

$k_1 = -20, k_2 = -10,$ and $k_3 = 30$

$$\dot{x} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} r(t);\ y(t) = \begin{bmatrix} -20 & -10 & 30 \end{bmatrix} x$$

The State Variable Differential Equations

$$\dot{x} = \begin{bmatrix} -3 & 6 & 0 \\ 0 & -2 & -5 \\ 0 & 0 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} r(t)$$

$$\frac{Y(s)}{R(s)} = T(s) = \frac{30(s+1)}{(s+5)(s+2)(s+3)} = \frac{q(s)}{(s-s_1)(s-s_2)(s-s_3)}$$

$$\frac{Y(s)}{R(s)} = T(s) = \frac{k_1}{(s+5)} + \frac{k_2}{(s+2)} + \frac{k_3}{(s+3)}$$

$$k_1 = -20, k_2 = -10, \text{and } k_3 = 30$$

$$\dot{x} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} r(t)$$

$$y(t) = \begin{bmatrix} -20 & -10 & 30 \end{bmatrix} x$$

The Transfer Function from the State Equation

Given the transfer function $G(s)$, we may obtain the state variable equations using the signal-flow graph model. Recall the two basic equations

$$\dot{\mathrm{x}} = \mathrm{A}\mathrm{x} + \mathrm{B}u$$

$$y = \mathrm{C}\mathrm{x}$$

y is the single output and u is the single input.

Take the Laplace transform

$$s\mathrm{X}(s) = \mathrm{A}\,\mathrm{X}(s) + \mathrm{B}\,U(s)$$

$$Y(s) = \mathrm{C}\mathrm{X}(s)$$

$$\left(s\mathrm{I} - \mathrm{A}\right)\mathrm{X}(s) = \mathrm{B}U(s)$$

$$\text{Since } \left[s\mathrm{I} - \mathrm{A}\right]^{-1} = \Phi(s)$$

$$\mathrm{X}(s) = \Phi(s)\,\mathrm{B}\,U(s)$$

$$Y(s) = \mathrm{C}\,\Phi(s)\,\mathrm{B}\,U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \mathrm{C}\,\Phi(s)\,\mathrm{B}$$

Exercises: E3.2 (DGD)

A robot-arm drive system for one joint can be represented by the differential equation,

$$\frac{dv(t)}{dt} = -k_1 v(t) - k_2 y(t) + k_3 i(t)$$

where $v(t)$ = velocity, $y(t)$ = position, and $i(t)$ is the control-motor current. Put the equations in state variable form and set up the matrix form for $k_1=k_2=1$

$$v = \frac{dy}{dt}$$

$$\frac{dv}{dt} = -k_1 v(t) - k_2 y(t) + k_3 i(t)$$

$$\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -k_2 & -k_1 \end{bmatrix}\begin{pmatrix} y \\ v \end{pmatrix} + \begin{bmatrix} 0 \\ k_3 \end{bmatrix} i$$

$$\text{Define } u = i, \text{ and let } k_1 = k_2 = 1$$

$$\dot{x} = Ax + Bu;\ A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ k_3 \end{bmatrix},\ x = \begin{pmatrix} y \\ v \end{pmatrix}$$

E3.3: A system can be represented by the state vector differential equation of equation (3.16) of the textbook. Find the **characteristic roots** of the system (DGD).

$$\dot{x} = Ax + Bu \qquad A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\mathrm{Det}\,(\lambda I - A) = \mathrm{Det}\left(\begin{bmatrix} \lambda & -1 \\ 1 & (\lambda + 1) \end{bmatrix}\right)$$

$$= \lambda(\lambda + 1) + 1 = \lambda^2 + \lambda + 1 = 0$$

$$\lambda_1 = -\frac{1}{2} + j\frac{\sqrt{3}}{2};\lambda_2 = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$$

E3.7: Consider the spring and mass shown in Figure 3.3 where *M* = 1 kg, *k* = 100 N/m, and *b* = 20 N/m/sec. (a) Find the state vector differential equation. (b) Find the roots of the characteristic equation for this system (DGD).

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -100x_1 - 20x_2 + u$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -100 & -20 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\text{Det}\,(\lambda \text{I} - \text{A}) = \text{Det}\begin{bmatrix} \lambda & -1 \\ 100 & \lambda + 20 \end{bmatrix} = \lambda^2 + 20\lambda + 100$$

$$= (\lambda + 10)^2 = 0; \lambda_1 = \lambda_2 = -10$$

E3.8: The manual, low-altitude hovering task above a moving land deck of a small ship is very demanding, in particular, in adverse weather and sea conditions. The hovering condition is represented by the **A** matrix (DGD)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -5 & -2 \end{bmatrix}$$

$$\mathrm{Det}(\lambda I - A) = \mathrm{Det}\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 5 & \lambda = 2 \end{bmatrix} =$$

$$\lambda\left(\lambda^2 + 2\lambda + 5)\right) = 0$$

$$\lambda_1 = 0; \lambda_2 = -1 + j2; \lambda_3 = -1 - j2$$

E3.9: See the textbook (DGD)

$$\dot{x}_1 = \frac{1}{2}x_2 - x_1$$

$$\dot{x}_2 = -x_1 - x_2$$

$$\dot{\mathrm{x}} = \begin{bmatrix} -1 & 1/2 \\ 1 & -3/2 \end{bmatrix}\mathrm{x},\ \mathrm{y} = \begin{bmatrix} 1 & -3/2 \end{bmatrix}\mathrm{x}$$

$$\mathrm{s}^2 + \frac{5}{2}s + 1 = (s+2)\left(s + \frac{1}{2}\right) = 0$$

$$\dot{\mathrm{z}} = \begin{bmatrix} -2 & 0 \\ 0 & -1/2 \end{bmatrix}z;\ \mathrm{y} = \begin{bmatrix} -0.35 & -1.79 \end{bmatrix}\mathrm{z}$$

P3.1 (DGD-ELG4152):

Apply KVL

$$v(t) = Ri(t) + L\frac{di}{dt} + +v_c$$

$$v_c = \frac{1}{C}\int i dt$$

(a) Select the state variables as $x_1 = i$ and $x_2 = v_\mathrm{c}$

(b) The state equations are :

$$\dot{x}_1 = \frac{1}{L}v - \frac{R}{L}x_1 - \frac{1}{L}x_2$$

$$\dot{x}_2 = \frac{1}{C}x_1$$

(c) $\dot{\mathrm{x}} = \begin{bmatrix} -\mathrm{R}/L & -1/L \\ 1/C & 0 \end{bmatrix}\mathrm{x} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix}u$