Lagrange Equations

Use kinetic and potential energy to solve for motion!

References

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System Modeling: The Lagrange Equations (Robert A. Paz: Klipsch School of Electrical and Computer Engineering)

Electromechanical Systems, Electric Machines, and Applied Mechatronics by Sergy E. Lyshevski, CRC, 1999.

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We use Newton's laws to describe the motions of objects. It works well if the objects are undergoing constant acceleration but they can become extremely difficult with varying accelerations.

For such problems, we will find it easier to express the solutions with the concepts of kinetic energy.

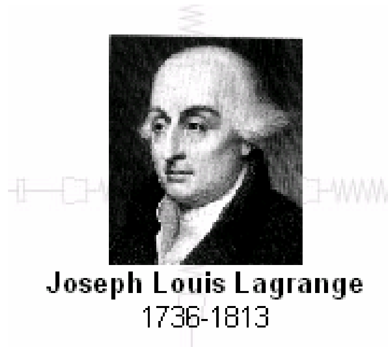
Modeling of Dynamic Systems

Modeling of dynamic systems may be done in several ways:

- Use the standard equation of motion (Newton’s Law) for mechanical systems.

- Use circuits theorems (Ohm’s law and Kirchhoff’s laws: KCL and KVL).

- Today’s approach utilizes the notation of energy to model the dynamic system (**Lagrange** model).



- Joseph-Louise Lagrange: 1736-1813.
- Born in Italy and lived in Berlin and Paris.
- Studied to be a lawyer.
- Contemporary of Euler, Bernoulli, D'Alembert, Laplace, and Newton.
- He was interested in math.
- Contribution:
  - Calculus of variations.
  - Calculus of probabilities.
  - Integration of differential equations
  - Number theory.

Equations of Motion: Lagrange Equations

- There are different methods to derive the dynamic equations of a dynamic system. As final result, all of them provide sets of equivalent equations, but their mathematical description differs with respect to their eligibility for computation and their ability to give insights into the underlying mechanical problem.

- Lagrangian method, depends on energy balances. The resulting equations can be calculated in closed form and allow an appropriate system analysis for most system applications.

- **Why Lagrange:**
  - Scalar not vector.
  - Eliminate solving for constraint forces (what holds the system together)
  - Avoid finding acceleration.
  - Uses extensively in robotics and many other fields.
  - Newton’s Law is good for simple systems but what about real systems?

Mathematical Modeling and System Dynamics

Newtonian Mechanics: Translational Motion

- The equations of motion of mechanical systems can be found using Newton's second law of motion. **F** is the vector sum of all forces applied to the body; **a** is the vector of acceleration of the body with respect to an inertial reference frame; and *m* is the mass of the body.
- To apply Newton's law, the free-body diagram (FBD) in the coordinate system used should be studied.

$$\sum \mathrm{F} = m\mathrm{a}$$

Newton approach requires that we find accelerations in all three directions, equate $\mathrm{F} = \mathrm{ma}$, solve for the constraint forces and then eliminate these to reduce the problem to "characteristic size".

Translational Motion in Electromechanical Systems

- Consideration of friction is essential for understanding the operation of electromechanical systems.

- Friction is a very complex nonlinear phenomenon and is very difficult to model friction.

- The classical Coulomb friction is a retarding frictional force (for translational motion) or torque (for rotational motion) that changes its sign with the reversal of the direction of motion, and the amplitude of the frictional force or torque are constant.

- Viscous friction is a retarding force or torque that is a linear function of linear or angular velocity.

Newtonian Mechanics: Translational Motion

- For one-dimensional rotational systems, Newton's second law of motion is expressed as the following equation. *M* is the sum of all moments about the center of mass of a body (*N*-m); *J* is the moment of inertial about its center of mass (kg/m^2); and α is the angular acceleration of the body (rad/s^2).

$$M = j\alpha$$

Newton's Second Law

The movement of a classical material point is described by the second law of Newton:

$$m\frac{d^2r(t)}{dt^2} = F(r,t) \text{ (r is a vector indicating a position of the material point in space)}$$

$$\mathrm{r} = \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{bmatrix}$$

Vector F(r, t) represents a force field, which may be calculated by taking into account interactions with other particles, or interactions with electromagnetic waves, or gravitational fields.

The second law of Newton is an idealisation, of course, even if one was to neglect quantum and relativistic effects. There is no justification why only a second time derivative of r should appear in that equation. Indeed if energy is dissipated in the system usually first time derivatives will appear in the equation too. If a material point loses energy due to EM radiation, third time derivatives will come up.

Energy in Mechanical and Electrical Systems

- In the Lagrangian approach, energy is the key issue. Accordingly, we look at various forms of energy for electrical and mechanical systems.

- For objects in motion, we have kinetic energy K_e which is always a scalar quantity and not a vector.

- The potential energy of a mass m at a height h in a gravitational field with constant g is given in the next table. Only differences in potential energy are meaningful. For mechanical systems with springs, compressed a distance x, and a spring constant k, the potential energy is also given in the next table.

- We also have dissipated energy P in the system. For mechanical system, energy is usually dissipated in sliding friction. In electrical systems, energy is dissipated in resistors.

Electrical and Mechanical Counterparts
"Energy"

Energy	Mechanical	Electrical
Kinetic (Active) K_e	**Mass / Inertia** 0.5 mv^2 / 0.5 $j\omega^2$	**Inductor** $\frac{1}{2}Li^2 = \frac{1}{2}L\dot{q}^2$
Potential V	**Gravity:** *mgh* **Spring:** 0.5 kx^2	**Capacitor** 0.5 $Cv^2 = q^2/2C$
Dissipative P	**Damper / Friction** 0.5 Bv^2	**Resistor** $\frac{1}{2}Ri^2 = \frac{1}{2}R\dot{q}^2$

Lagrangian

The principle of Lagrange's equation is based on a quantity called "Lagrangian" which states the following: For a dynamic system in which a work of all forces is accounted for in the Lagrangian, an admissible motion between specific configurations of the system at time t1 and t2 in a natural motion if , and only if, the energy of the system remains constant.

The Lagrangian is a quantity that describes the balance between no dissipative energies.

$$L = K_e - V \ (K_e \text{ is the kinetic energy}; V \text{ is the potential energy})$$

$$K_e = \frac{1}{2} m v^2; \ V = mgh$$

$$\text{Lagrange's Equation}: \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial P}{\partial \dot{q}_i} = Q_i$$

P is power function (half rate at which energy is dissipated); Q_i are generalized external inputs (forces) acting on the system If there are three generalized coordinates, there will be three equations.

Note that the above equation is a second - order differential equation

Generalized Coordinates

- In order to introduce the Lagrange equation, it is important to first consider the *degrees of freedom* (*DOF = number of coordinates- number of constraints*) of a system. Assume a particle in a space: number of coordinates = 3 (x, y, z or r, θ, ϕ); number of constrants = 0; DOF = 3 - 0 = 3.

- These are the number of independent quantities that must be specified if the state of the system is to be uniquely defined. These are generally state variables of the system, but not all of them.

- For mechanical systems: *masses or inertias* will serve as generalized coordinates.

- For electrical systems: electrical charges may also serve as appropriate coordinates.

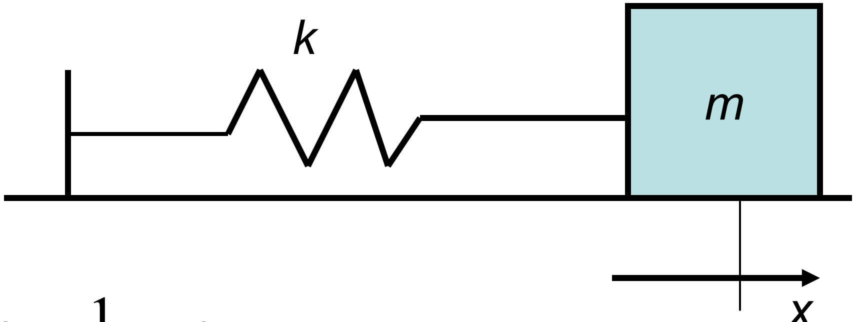
Cont..

- Use a coordinate transformation to convert between sets of generalized coordinates ($x = r \sin \theta \cos \phi$; $y = r \sin \theta \sin \phi$; $z = r \cos \theta$).
- Let a set of q_1, q_2,.., q_n of independent variables be identified, from which the position of all elements of the system can be determined. These variables are called generalized coordinates, and their time derivatives are generalized velocities. The system is said to have *n* degrees of freedom since it is characterized by the *n* generalized coordinates.
- Use the word generalized, frees us from abiding to any coordinate system so we can chose whatever parameter that is convenient to describe the dynamics of the system.

For a large class of problems, Lagrange equations can be written in standard matrix form

$$\frac{d}{dt}\begin{bmatrix} \dfrac{\partial L}{\partial \dot{q}_1} \\ \cdot \\ \cdot \\ \dfrac{\partial L}{\partial \dot{q}_n} \end{bmatrix} - \begin{bmatrix} \dfrac{\partial L}{\partial q_1} \\ \cdot \\ \cdot \\ \dfrac{\partial L}{\partial q_n} \end{bmatrix} + \begin{bmatrix} \dfrac{\partial P}{\partial \dot{q}_1} \\ \cdot \\ \cdot \\ \dfrac{\partial P}{\partial \dot{q}_n} \end{bmatrix} = \begin{bmatrix} f_1 \\ \cdot \\ \cdot \\ f_n \end{bmatrix}$$

Example of Linear Spring Mass System and Frictionless Table: **The Steps**



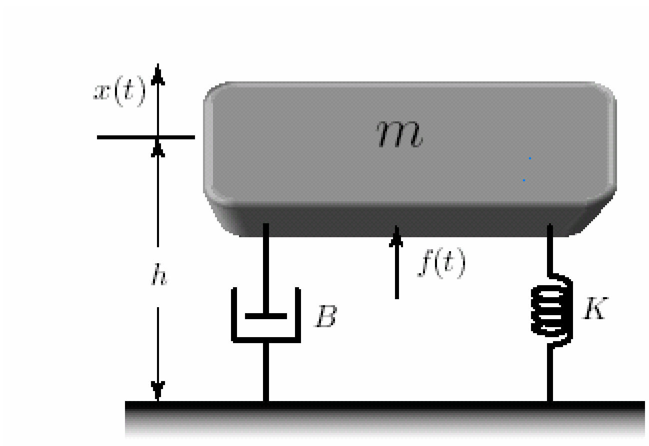
$$\text{Lagrangian}: L = K_e - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$\text{Lagrang's Equation}: \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$$\text{Do the derivatives}: \frac{\partial L}{\partial \dot{q}_i} = m\dot{x};\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = m\ddot{x};\ \frac{\partial L}{\partial q_i} = -kx$$

$$\text{Combine all together}: \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = m\ddot{x} + kx = 0$$

Mechanical Example: Mass-Spring Damper



$$Ke = \frac{1}{2}m\dot{x}^2$$

$$V = \frac{1}{2}Kx^2 + mg(h+x)$$

$$L = K_e - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}Kx^2 - mg(h+x)$$

$$P = \frac{1}{2}B\dot{x}^2$$

We have the generalized coordinate $q = x$, and thus with the applied force $Q = f$, we write the Lagrange equation :

$$f = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} + \frac{\partial P}{\partial \dot{x}}$$

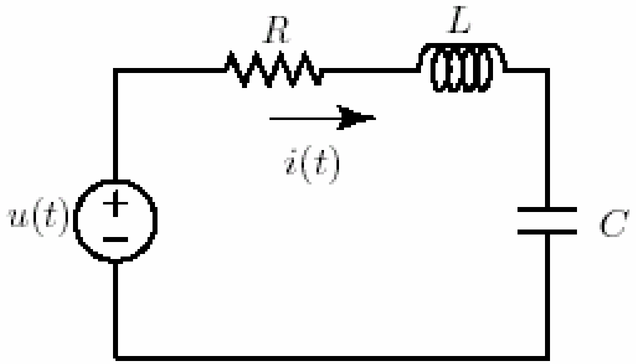
$$= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}Kx^2 - mg(h+x)\right)\right)$$

$$- \frac{\partial}{\partial x}\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}Kx^2 - mg(h+x)\right) + \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}B\dot{x}^2\right)$$

$$= \frac{d}{dt}(m\dot{x}^2) - (-Kx - mg) + (B\dot{x})$$

$$= m\ddot{x} + Kx + mg + B\dot{x}$$

Electrical Example: *RLC* Circuit



$$K_e = \frac{1}{2} L \dot{q}^2$$

$$V = \frac{1}{2C} q^2$$

$$L = K_e - V = \frac{1}{2} L \dot{q}^2 - \frac{1}{2C} q^2$$

$$P = \frac{1}{2} R \dot{q}^2$$

We have the generalized coordinate q (charge), and with the applied force $Q = u$, we have

$$u = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial P}{\partial \dot{q}}$$

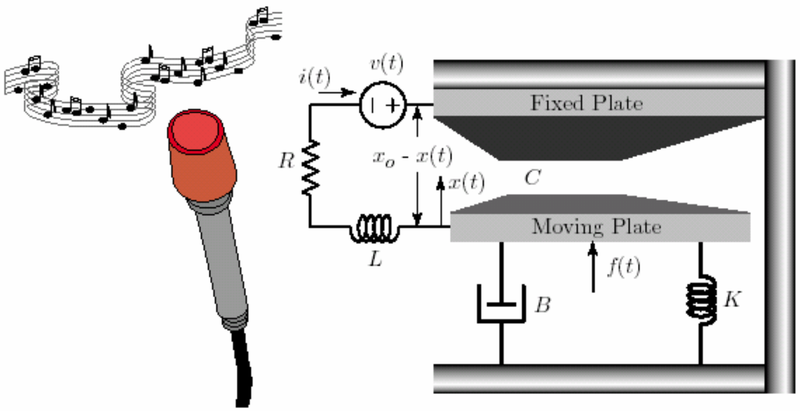
$$= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}}\left(\frac{1}{2} L \dot{q}^2 - \frac{1}{2C} q^2\right)\right) - \frac{\partial}{\partial q}\left(\frac{1}{2} L \dot{q}^2 - \frac{1}{2C} q^2\right) + \frac{\partial}{\partial \dot{q}}\left(\frac{1}{2} R \dot{q}^2\right)$$

$$= \frac{d}{dt}(L\dot{q}) + \frac{Q}{C} + R\dot{q} = L\ddot{q} + \frac{Q}{C} + R\dot{q} = L\frac{di}{dt} + v_c + Ri$$

$i = \dot{q}$ and $q = Cv_c$ for a capacitor. This is just KVL equation

Electromechanical System: Capacitor Microphone

About them see: http://www.soundonsound.com/sos/feb98/articles/capacitor.html



This system has two degrees of freedom (electrical and mechanical : charge q and displacement x from equilibrium)

$$K_e = \frac{1}{2}L\dot{q}^2 + \frac{1}{2}m\dot{x}^2; \; V = \frac{1}{2C}q^2 + \frac{1}{2}Kx^2$$

$$C = \frac{\varepsilon A}{x_o - x}\left(\begin{array}{l} \varepsilon \text{ is the dielectric constant of the air (F/m),} \\ A \text{ is the area of the plate, } x_o - x \text{ is the plate separation} \end{array} \right)$$

$$V = \frac{1}{2\,\varepsilon\,A}(x_o - x)q^2 + \frac{1}{2}Kx^2; \; P = \frac{1}{2}R\dot{q}^2 + \frac{1}{2}B\dot{x}^2$$

$$L = \frac{1}{2}L\dot{q}^2 + \frac{1}{2}m\dot{x}^2 - \frac{1}{2\varepsilon A}(x_o - x)q^2 - \frac{1}{2}Kx^2$$

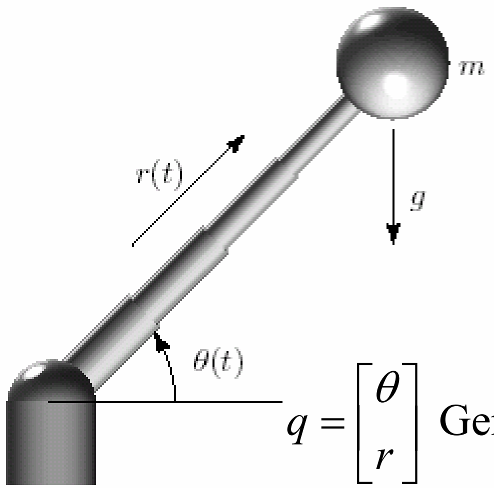
$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}; \ \frac{\partial L}{\partial x} = \frac{q^2}{2\varepsilon A} - Kx; \frac{\partial \mathrm{P}}{\partial \dot{x}} = B\dot{x}$$

$$\frac{\partial L}{\partial \dot{q}} = L\dot{q}; \ \frac{\partial L}{\partial q} = \frac{\left(x_o - x\right)q}{\varepsilon A}; \frac{\partial P}{\partial \dot{q}} = R\dot{q}$$

Then we obtain the two Lagrange equations

$$m\ddot{x} + B\dot{x} + Kx - \frac{q2}{2\,\varepsilon\,A} = f$$

$$L\ddot{q} + R\dot{q} + \frac{1}{\varepsilon\,A}\left(x_o - x\right)q = v$$



Robotic Example

$q = \begin{bmatrix} \theta \\ r \end{bmatrix}$ Generalized coordinates (θ angular position; r radial length; both vary)

$Q = \begin{bmatrix} \tau \\ f \end{bmatrix}$ Applicable forces to each component; τ is the torque; f is the force

$$J = mr^2;\ K_e = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m\dot{r}^2;\ V = mgr\sin(\theta)$$

The power dissipation : $P = \frac{1}{2}B_1\dot{\theta}^2 + \frac{1}{2}B_2\dot{r}^2$

$$L = K_e - V = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m\dot{r}^2 - mgr\sin(\theta)$$

$$\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} \frac{\partial L}{\partial \dot{\theta}} \\ \frac{\partial L}{\partial \dot{r}} \end{bmatrix} = \begin{bmatrix} J\dot{\theta} \\ m\dot{r} \end{bmatrix} = \begin{bmatrix} mr^2\dot{\theta} \\ m\dot{r} \end{bmatrix};\ \frac{\partial L}{\partial q} = \begin{bmatrix} \frac{\partial L}{\partial \theta} \\ \frac{\partial L}{\partial r} \end{bmatrix} = \begin{bmatrix} -mgr\cos(\theta) \\ mr\dot{\theta}^2 - mg\sin(\theta) \end{bmatrix};\ \frac{\partial P}{\partial \dot{q}} = \begin{bmatrix} \frac{\partial P}{\partial \dot{\theta}} \\ \frac{\partial P}{\partial \dot{r}} \end{bmatrix} = \begin{bmatrix} B_1\dot{\theta} \\ B_2\dot{r} \end{bmatrix}$$

The Lagrange equation becomes

$$Q = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial P}{\partial \dot{q}}$$

$$Q = \begin{bmatrix} mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} \\ m\ddot{r} \end{bmatrix} - \begin{bmatrix} -mgr\cos(\theta) \\ mr\dot{\theta}^2 - mg\sin(\theta) \end{bmatrix} + \begin{bmatrix} B_1\dot{\theta} \\ B_2\dot{r} \end{bmatrix}$$

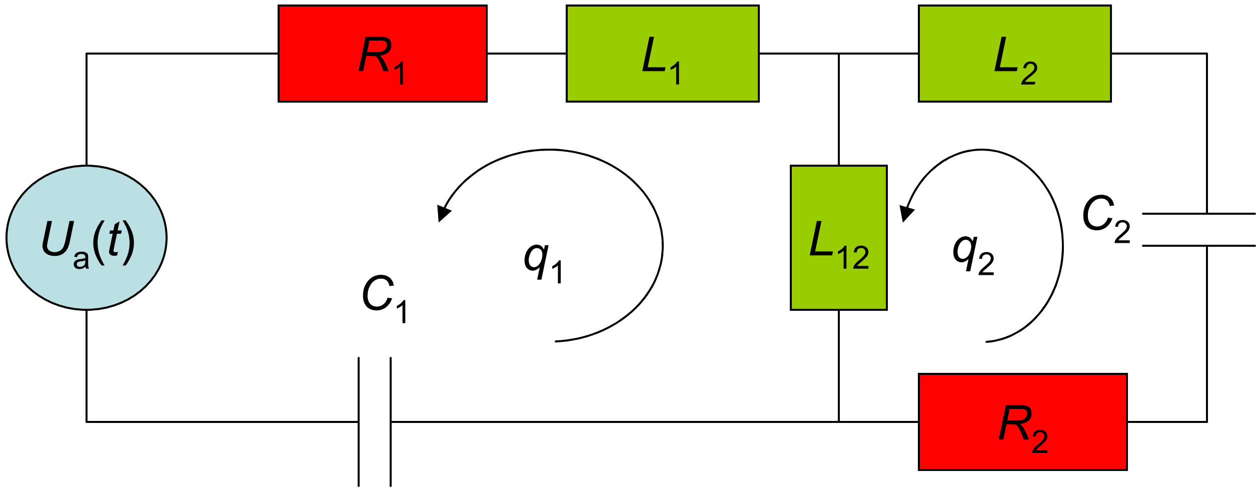
$$\begin{bmatrix} mr^2 & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{r} \end{bmatrix} + \begin{bmatrix} B_1 & 2mr\dot{\theta} \\ -mr\dot{\theta} & B_2 \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ \dot{r} \end{bmatrix} + \begin{bmatrix} mgr\cos(\theta) \\ mg\sin(\theta) \end{bmatrix} = \begin{bmatrix} \tau \\ f \end{bmatrix}$$

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = Q$$

$M(q)$ is the inertia matrix; $V(q,\dot{q})$ is the Coriolis/centripetal vector

$G(q)$ is the gravity vector; Q is the input vector

Example: Two Mesh Electric Circuit



Assume q_1 and q_2 as the independent generalized coordinates, where q_1 is the electric charge in the first loop and q_2 is the electric charge in the second loop.

The generalized force applied to the system is denoted as Q_1

We should know that : $i_1 = \dot{q}_1;\ i_2 = \dot{q}_2;\ q_1 = \frac{i_1}{s};\ q_2 = \frac{i_2}{s};\ Q_1 = U_a(t).$

The total magnetic energy (kinetic energy) is :

$$K_e = \frac{1}{2}L_1\dot{q}_1^2 + \frac{1}{2}L_{12}(\dot{q}_1 - \dot{q}_2)^2 + \frac{1}{2}L_2\dot{q}_2^2$$

$$\frac{\partial K_e}{\partial q_1} = 0;\ \frac{\partial K_e}{\partial \dot{q}_1} = (L_1 + L_{12})\dot{q}_1 - L_{12}\dot{q}_2$$

$$\frac{\partial K_e}{\partial q_2} = 0;\ \frac{\partial K_e}{\partial \dot{q}_2} = (L_2 + L_{12})\dot{q}_2 - L_{12}\dot{q}_1$$

Use the equation for the total electric energy (potential energy)

$$V = \frac{1}{2}\frac{q_1^2}{C_1} + \frac{1}{2}\frac{q_2^2}{C_2};\ \frac{\partial V}{\partial q_1} = \frac{q_1}{C_1} \text{ and } \frac{\partial V}{\partial q_2} = \frac{q_2}{C_2}$$

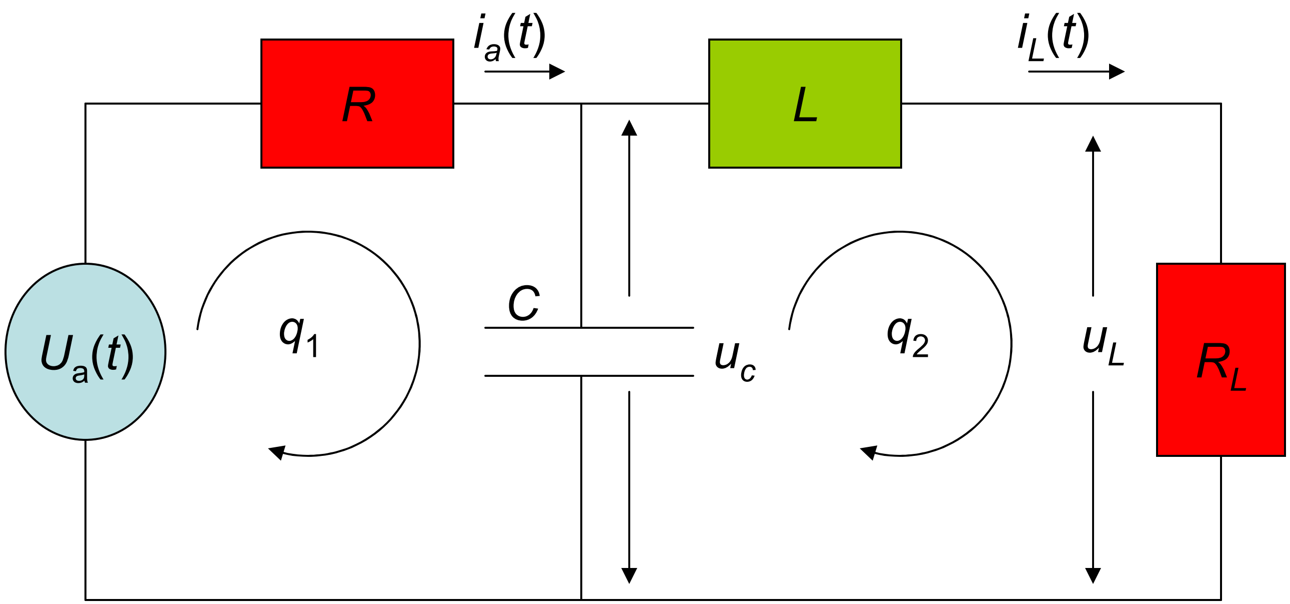
The total heat energy dissipated : $P = \frac{1}{2}R_1\dot{q}_1^2 + \frac{1}{2}R_2\dot{q}_2^2;\ \frac{\partial P}{\partial \dot{q}_1} = R_1\dot{q}_1$ and $\frac{\partial P}{\partial \dot{q}_2} = R_2\dot{q}_2$

$$\frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_1}\right) - \frac{\partial K_e}{\partial q_1} + \frac{\partial P}{\partial \dot{q}_1} + \frac{\partial V}{\partial q_1} = Q_1;\ \frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_2}\right) - \frac{\partial K_e}{\partial q_2} + \frac{\partial P}{\partial \dot{q}_2} + \frac{\partial V}{\partial q_2} = 0$$

$$(L_1 + L_{12})\ddot{q}_1 - L_{12}\ddot{q}_2 + R_1\dot{q}_1 + \frac{q_1}{C_1} = U_a;\ -L_{12}\ddot{q}_1 + (L_2 + L_{12})\ddot{q}_2 + R_2\dot{q}_2 + \frac{q_2}{C_2} = 0$$

$$\ddot{q}_1 = \frac{1}{(L_1 + L_{12})}\left(-\frac{q_1}{C_1} - R_1\dot{q}_1 + L_{12}\ddot{q}_2 + U_a\right);\ \ddot{q}_2 = \frac{1}{(L_2 + L_{12})}\left(L_{12}\ddot{q}_1 - \frac{q_2}{C_2} - R_2\dot{q}_2\right)$$

Another Example



Use q_1 and q_2 as the independent generalized coordinates :

$$i_a = \dot{q}_1; i_L = \dot{q}_2; u_a(t) = Q_1$$

$$K_e = \frac{1}{2}L\dot{q}_2^2; \quad \frac{\partial K_e}{\partial q_1} = 0; \quad \frac{\partial K_e}{\partial \dot{q}_1} = 0; \quad \frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_1}\right) = 0$$

$$\frac{\partial K_e}{\partial q_2} = 0; \quad \frac{\partial K_e}{\partial \dot{q}_2} = L\dot{q}_2; \quad \frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_2}\right) = L\ddot{q}_2$$

$$\text{The total potential energy is}: V = \frac{1}{2}\frac{(q_1 - q_2)^2}{C}$$

$$\frac{\partial V}{\partial q_1} = \frac{q_1 - q_2}{C} \quad \text{and} \quad \frac{\partial V}{\partial q_2} = \frac{-q_1 + q_2}{C}$$

$$\text{The total dissipated energy is}: P = \frac{1}{2}R\dot{q}_1^2 + \frac{1}{2}R_L\dot{q}_2^2$$

$$\frac{\partial P}{\partial \dot{q}_1} = R\dot{q}_1 \quad \text{and} \quad \frac{\partial P}{\partial \dot{q}_2} = R_L\dot{q}_2$$

$$\frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_1}\right) - \frac{\partial K_e}{\partial q_1} + \frac{\partial P}{\partial \dot{q}_1} + \frac{\partial V}{\partial q_1} = Q_1; \quad \frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_2}\right) - \frac{\partial K_e}{\partial q_2} + \frac{\partial P}{\partial \dot{q}_2} + \frac{\partial V}{\partial q_2} = 0$$

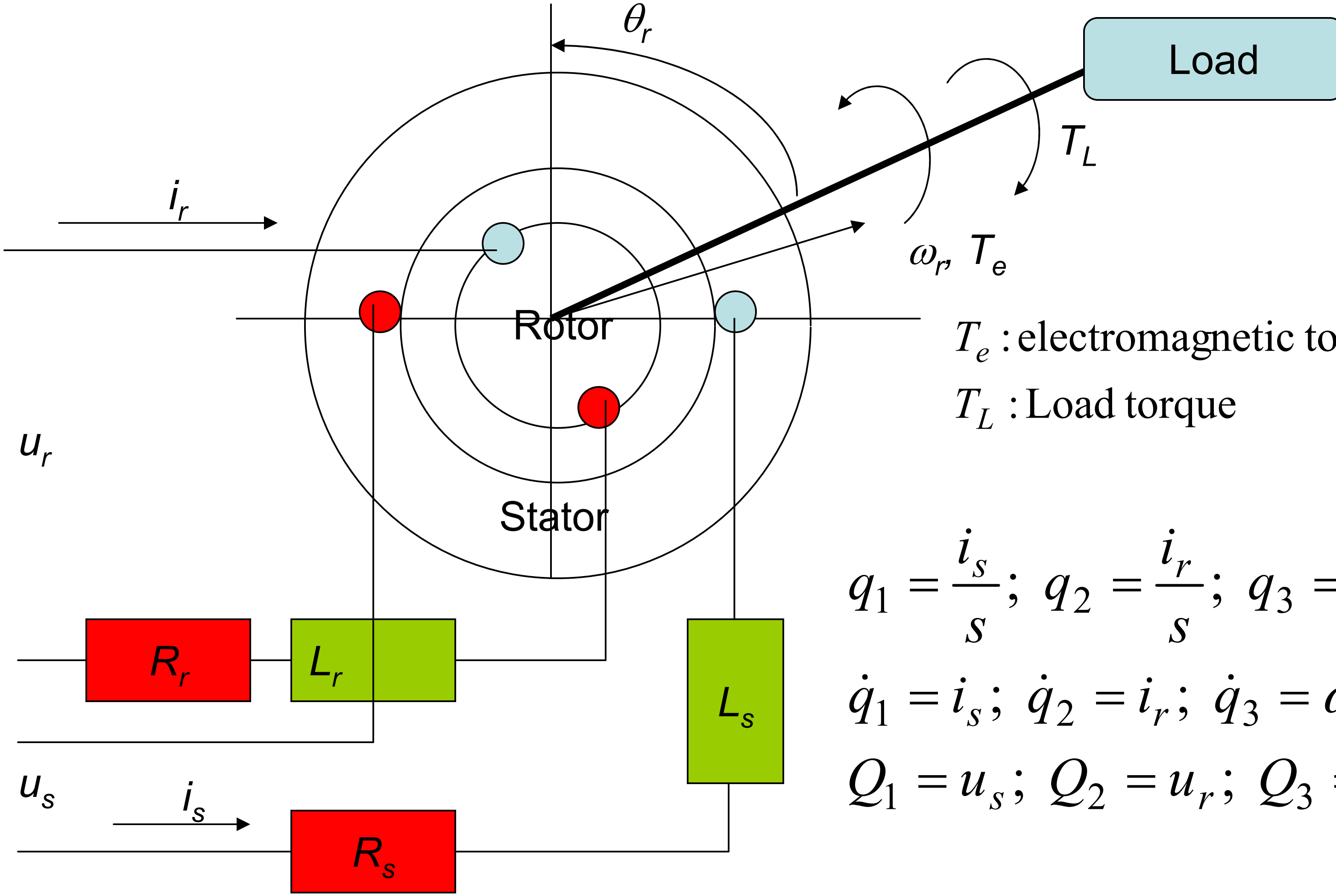
$$R\dot{q}_1 + \frac{q_1 - q_2}{C} = u_a; \quad L\ddot{q}_2 + R_L\dot{q}_2 + \frac{-q_1 + q_2}{C} = 0$$

$$\dot{q}_1 = \frac{1}{R}\left(\frac{-q_1 + q_2}{C} + u_a\right); \quad \ddot{q}_2 = \frac{1}{L}\left(-R_L\dot{q}_2 + \frac{q_1 - q_2}{C}\right)$$

By using Kirchhoff's law, we get

$$\frac{du_c}{dt} = \frac{1}{C}\left(-\frac{u_c}{R} - i_L + \frac{u_a(t)}{R}\right); \quad \frac{di_L}{dt} = \frac{1}{L}\left(u_c - R_L i_L\right)$$

Directly-Driven Servo-System



T_e : electromagnetic torque

T_L : Load torque

$$q_1 = \frac{i_s}{s};\; q_2 = \frac{i_r}{s};\; q_3 = \theta_r;$$

$$\dot{q}_1 = i_s;\; \dot{q}_2 = i_r;\; \dot{q}_3 = \omega_r;$$

$$Q_1 = u_s;\; Q_2 = u_r;\; Q_3 = -T_L$$

The Lagrange equations are expressed in terms of each independent coordinate

$$\frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_1}\right) - \frac{\partial K_e}{\partial q_1} + \frac{\partial P}{\partial \dot{q}_1} + \frac{\partial V}{\partial q_1} = Q_1$$

$$\frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_2}\right) - \frac{\partial K_e}{\partial q_2} + \frac{\partial P}{\partial \dot{q}_2} + \frac{\partial V}{\partial q_2} = Q_2$$

$$\frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_3}\right) - \frac{\partial K_e}{\partial q_3} + \frac{\partial P}{\partial \dot{q}_3} + \frac{\partial V}{\partial q_3} = Q_3$$

The total kinetic energy is the sum of the total electrical (magnetic) and mechanical (moment of inertia) energies

$$K_{ee} = \frac{1}{2} L_s \dot{q}_1^2 + L_{sr} \dot{q}_1 \dot{q}_2 + \frac{1}{2} L_r \dot{q}_2^2 \text{ (Electrical)}; K_{em} = \frac{1}{2} J \dot{q}_3^2 \text{ (Mechanical)}$$

$$K_e = K_{ee} + K_{em} = \frac{1}{2} L_s \dot{q}_1^2 + L_{sr} \dot{q}_1 \dot{q}_2 + \frac{1}{2} L_r \dot{q}_2^2 + \frac{1}{2} J \dot{q}_3^2$$

$$\text{Mutual inductance}: L_{sr}(\theta_r) = \frac{N_s N_r}{\Re_m(\theta_r)}; \; L_M = L_{sr\max} = \frac{N_s N_r}{\Re_m(90^0)}$$

$$L_{sr}(\theta_r) = L_M \cos\theta_r = L_M \cos q_3 \; (L_M \text{ is magnetizing reluctance})$$

$$K_e = \frac{1}{2} L_s \dot{q}_1^2 + L_M \dot{q}_1 \dot{q}_2 \cos q_3 + \frac{1}{2} L_r \dot{q}_2^2 + \frac{1}{2} J \dot{q}_3^2$$

$$\text{The following partial derivatives result}: \frac{\partial K_e}{\partial q_1} = 0; \frac{\partial K_e}{\partial \dot{q}_1} = L_s \dot{q}_1 + L_M \dot{q}_2 \cos q_3$$

$$\frac{\partial K_e}{\partial q_2} = 0; \frac{\partial K_e}{\partial \dot{q}_2} = L_M \dot{q}_1 \cos q_3 + L_r \dot{q}_2; \frac{\partial K_e}{\partial q_3} = -L_M \dot{q}_1 \dot{q}_2 \sin q_3; \frac{\partial K_e}{\partial \dot{q}_3} = J \dot{q}_3$$

We have only a mechanical potential energy: Spring with a constant k_s

$$\text{The potential energy of the spring with constant } k_s : V = \frac{1}{2} k_s q_3^2$$

$$\frac{\partial V}{\partial q_1} = 0; \frac{\partial V}{\partial q_2} = 0; \frac{\partial V}{\partial q_3} = k_s q_3$$

$$\text{The total heat energy dissipated is expressed as} : P = P_E + P_M$$

$$P_E = \frac{1}{2} R_s \dot{q}_1^2 + \frac{1}{2} R_r \dot{q}_2^2; \; P_M = \frac{1}{2} B_m \dot{q}_3^2$$

$$P = \frac{1}{2} R_s \dot{q}_1^2 + \frac{1}{2} R_r \dot{q}_2^2 + \frac{1}{2} B_m \dot{q}_3^2$$

$$\frac{\partial P}{\partial \dot{q}_1} = R_s \dot{q}_1; \; \frac{\partial P}{\partial \dot{q}_2} = R_r \dot{q}_2; \text{ and } \frac{\partial P}{\partial \dot{q}_3} = B_m \dot{q}_3$$

$$\text{Substituting the original values, we have three differential equations for servo-system}$$

$$L_s \frac{di_s}{dt} + L_M \cos\theta_r \frac{di_r}{dt} - L_M i_r \sin\theta_r \frac{d\theta_r}{dt} + R_s i_s = u_s$$

$$L_r \frac{di_r}{dt} + L_M \cos\theta_r \frac{di_s}{dt} - L_M i_s \sin\theta_r \frac{d\theta_r}{dt} + R_r i_r = u_r$$

$$J \frac{d^2\theta_r}{dt^2} + L_M i_s i_r \sin\theta_r + B_m \frac{d\theta_r}{dt} + k_s \theta_r = -T_L$$

The last equation should be written in terms of rotor angular velocity ($\frac{d\theta_r}{dt} = \omega$).

Also, using stator current and rotor current, angular velocity, and position as state variables

$$\frac{di_s}{dt} = \frac{1}{L_s L_r - L_M^2 \cos^2 \theta_r}\left(-R_s L_r i_s - \frac{1}{2} L_M^2 i_s \omega_r \sin 2\theta_r + R_r L_M i_r \cos\theta_r + L_r L_M \omega_r \sin\theta_r + L_r u_s - L_M \cos\theta_r u_r\right)$$

$$\frac{di_r}{dt} = \frac{1}{L_s L_r - L_M^2 \cos^2 \theta_r}\left(-R_s L_M i_s - \frac{1}{2} L_s L_M i_s \omega_r \sin\theta_r - R_r L_s i_r - \frac{1}{2} L_M^2 i_r \omega_r \sin 2\theta - L_M \cos\theta_r u_s + L_s u_r\right)$$

$$\frac{d\omega_r}{dt} = \frac{1}{J}(-L_M i_s i_r \sin\theta_r - B_m \omega_r - k_s \theta_r - T_L)$$

$$\frac{d\theta_r}{dt} = \omega_r$$

Considering the third equation : $\frac{d\omega_r}{dt} = \frac{1}{J}(-L_M i_s i_r \sin\theta_r - B_m \omega_r - k_s \theta_r - T_L)$

We can obtain the expression for the electromagnetic torque T_e developed :

$$T_e = -L_M i_s i_r \sin\theta_r$$

More Application

Application of Lagrange equations of motion in the modeling of two-phase induction motor and generator.

Application of Lagrange equations of motion in the modeling of permanent-magnet synchronous machines.

Transducers