

Laplace Transform:

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1. Why we need Laplace transform:

In general, the control systems we analyze are non-linear in time domain, such as the spring-mass-damper system, or more complex machine, ship. There are lots of integral and differential operations. Their series or parallel connects would be very difficult and complex. It is preferred to using some transform to convert the non-linear operation into linear operation. The Laplace Transform is developed to solve this problem. It makes our life easier.

If system input is $X(s)$, system is $H(s)$, then the output $Y(s) = H(s) * X(s)$.

In s domain (we denote the Laplace transformed domain), the series connection is simply a multiplication of transfer functions of two systems. Then, the connections can be simplified and represented by block diagram. The operations are summarized in **Table 2.6**.

In this method, the final time response solution is obtained by:

- a. obtain the differential equations of the system (we need the knowledge of the system, the differential equations for ideal elements are summarized in **Table 2.2**);
- b. obtain the Laplace transformation of the differential equations, which is quite simple (transformation of commonly used equations are summarized in **Table 2.3**);
- c. analyze the system in s domain;
- d. get the final time domain solution by inverse Laplace Transform.

2. Some basic definitions:

a. The Laplace Transform:
$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st} dt$$

The inverse Laplace transform:
$$f(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} dt$$

Although the definition is quite mathematics, the commonly used transformations have been summarized in Table 2.3. We will see some examples in section 3.

b. Transfer function:

In s domain, we define $H(s) = \frac{Y(s)}{X(s)}$ as **Transfer function**, $Y(s) = H(s) * X(s)$.

In general $Y(s) = \frac{num(s)}{den(s)} = \frac{p(s)}{q(s)}$, the denominator polynomial is called the

characteristic equation, the roots of it (poles of the system) determine the character of the system response in time domain.

c. Damping ratio and natural frequency, critical damping:

If
$$Y(s) = \frac{p(s)}{s^2 + as + b} = \frac{p(s)}{s^2 + 2\xi\omega s + \omega^2},$$

Where ξ is defined as **damping ratio**, and ω is the **natural frequency** of the system.

If $\xi = 1$, the roots of the characteristic equation are repeat and real $s_{1,2} = -\omega$, this condition is called **critical damping**.

d. Final value theorem:

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} s \cdot Y(s)$$

3. examples:

a. $x(t)=u(t)$
$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st} dt = \int_{0^-}^{\infty} e^{-st} dt = \int_{0^-}^{\infty} e^{-st} \frac{dst}{s} = \frac{1}{s} \int_{-\infty}^0 e^z dz = \frac{1}{s}$$

$$\text{b. } x(t) = e^{-at}, \quad X(s) = \int_0^{\infty} x(t)e^{-st} dt = \int_0^{\infty} e^{-at} e^{-st} dt = \int_0^{\infty} e^{-(s+a)t} dt = \frac{1}{s+a}$$

$$\text{c. } x^{(1)}(t) = \frac{dx(t)}{dt}, \quad X(s) = \int_0^{\infty} x(t)e^{-st} dt = sX(s) - x(0^-)$$

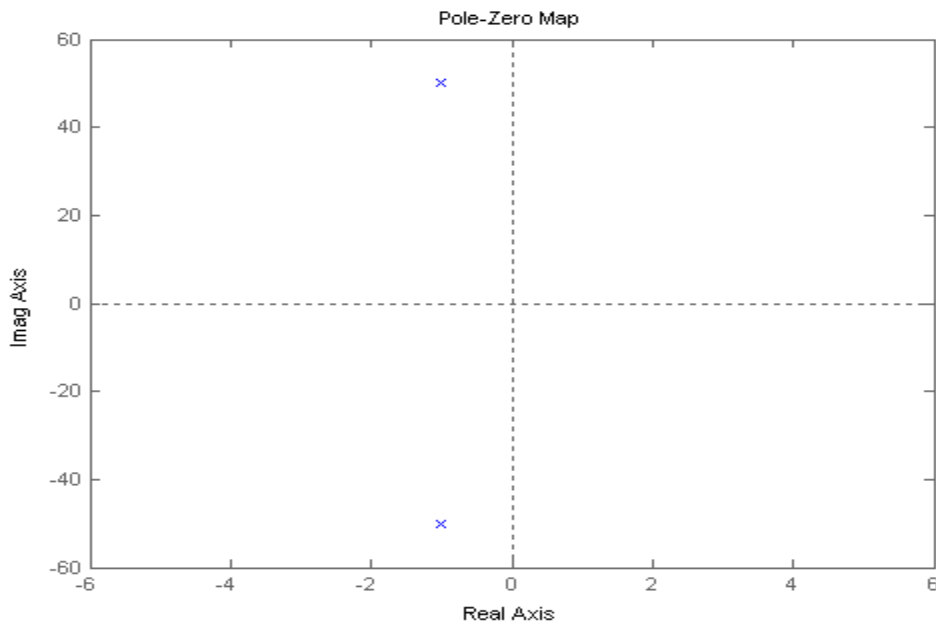
$$\text{d. } M \frac{d^2 y(t)}{dt^2} + b \frac{dy(t)}{dt} + ky(t) = r(t),$$

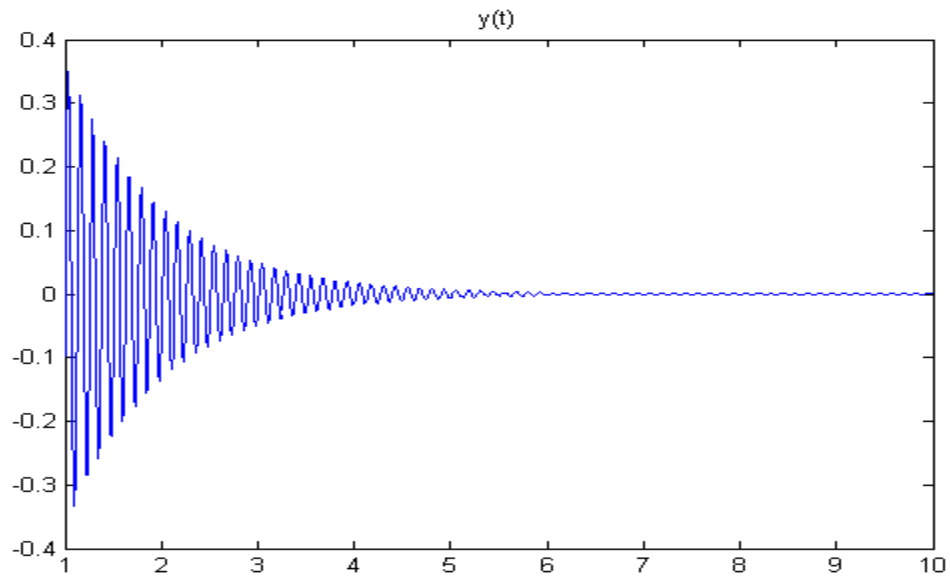
4. zeros/poles

In general, $Y(s) = \frac{\text{num}(s)}{\text{den}(s)} = \frac{p(s)}{q(s)}$, the roots of the numerator are called **zeros**, the roots of the denominators (characteristic function) are called **poles** of the system. At the zeros, the output $Y(s)$ becomes zero, at the poles the output $Y(s)$ becomes infinite. The roots / poles $s = a + jb$, in the complex s domain we can draw the zeros and poles, they graphically portrays the character of the natural transient response of the system.

For example, $H(s) = \frac{50s}{(s+1)^2 + 50^2}$, $X(s) = \frac{1}{s}$, then $Y(s) = \frac{50}{(s+1)^2 + 50^2}$, and

$$y(t) = e^{-t} \sin(50t)$$





if $H(s) = \frac{50s}{(s-1)^2 + 50^2}$, $X(s) = \frac{1}{s}$, $Y(s) = \frac{50}{(s-1)^2 + 50^2}$, then $y(t) = e^t \sin(50t)$

