Dynamic Responses of Systems

The most important function of a model devised for measurement or control systems is to be able to predict what the output will be for a particular input.

We are not only interested with a static situation but wanted to see how the output will change with time when there is a change of input or when the input changes with time.

To do so, we need to form equations which will indicate how the system output will vary with time when the input is varying with time. This can be done by the use of differential equations.

Examples of Dynamic Systems

- An example of a first order system is water flowing out of a tank. q_1 is the forcing input.
- Another example, is a thermometer being placed in a hot liquid at some temperature. The rate at which the reading of the thermometer changes with time. T_L is the forcing input.

$$RA\frac{dh}{dt} + pgh = q_1$$

$$RC\frac{dT}{dt} + T = T_L$$

Electromechanical System Dynamics, energy Conversion, and Electromechanical Analogies

Modeling of Dynamic Systems

Modeling of dynamic systems may be done in several ways:

- Use the standard equation of motion (Newton's Law) for mechanical systems
- Use circuits theorems (Ohm's law and Kirchhoff's laws: KCL and KVL)
- Another approach utilizes the notation of energy to model the dynamic system (Lagrange model).

Mathematical Modeling and System Dynamics

Newtonian Mechanics: Translational Motion

- The equations of motion of mechanical systems can be found using Newton's second law of motion. **F** is the vector sum of all forces applied to the body; a is the vector of acceleration of the body with respect to an inertial reference frame; and *m* is the mass of the body.
- To apply Newton's law, the free-body diagram in the coordinate system used should be studied.

$$\sum \mathrm{F} = m\mathrm{a}$$

Translational Motion in Electromechanical Systems

- Consideration of friction is essential for understanding the operation of electromechanical systems.
- Friction is a very complex nonlinear phenomenon and is very difficult to model friction.
- The classical Coulomb friction is a retarding frictional force (for translational motion) or torque (for rotational motion) that changes its sign with the reversal of the direction of motion, and the amplitude of the frictional force or torque are constant.
- Viscous friction is a retarding force or torque that is a linear function of linear or angular velocity.

Newtonian Mechanics: Translational Motion

- For one-dimensional rotational systems, Newton's second law of motion is expressed as the following equation. *M* is the sum of all moments about the center of mass of a body (N-m); *J* is the moment of inertial about its center of mass (kg/m^2); and α is the angular acceleration of the body (rad/s^2).

$$M = j\alpha$$

The Lagrange Equations of Motion

- Although Newton's laws of motion form the fundamental foundation for the study of mechanical systems, they can be straightforwardly used to derive the dynamics of electromechanical motion devices because electromagnetic and circuitry transients behavior must be considered. This means, the circuit dynamics must be incorporated to find augmented models.
- This can be performed by integrating torsional-mechanical dynamics and sensor/actuator circuitry equations, which can be derived using Kirchhoff's laws.
- Lagrange concept allows one to integrate the dynamics of mechanical and electrical components. It employs the scalar concept rather the vector concept used in Newton's law of motion to analyze much wider range of systems than *F =ma.*
- With Lagrange dynamics, focus is on the entire system rather than individual components.
- Γ, D, Π are the total kinetic, dissipation, and potential energies of the system. q_i and Q_i are the generalized coordinates and the generalized applied forces (input).

$$\frac{d}{dt}\left(\frac{d\Gamma}{d\dot{q}_i}\right) - \frac{d\Gamma}{dq_i} + \frac{dD}{d\dot{q}_i} + \frac{d\Pi}{dq_i} = Q_i$$

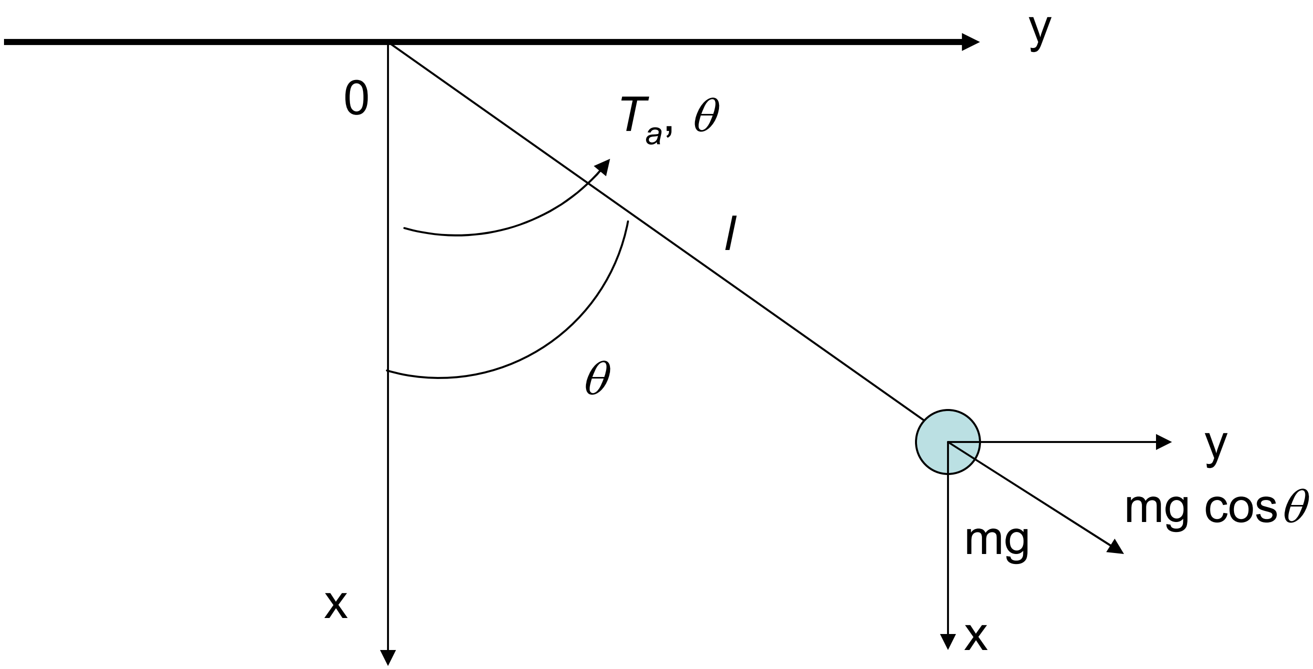
Electrical and Mechanical Counterparts

Energy	**Mechanical**	**Electrical**
Kinetic	**Mass / Inertia** 0.5 mv^2 / 0.5 $j\omega^2$	**Inductor** 0.5 Li^2
Potential	**Gravity:** mgh **Spring:** 0.5 kx^2	**Capacitor** 0.5 Cv^2
Dissipative	**Damper / Friction** 0.5 Bv^2	Resistor Ri^2

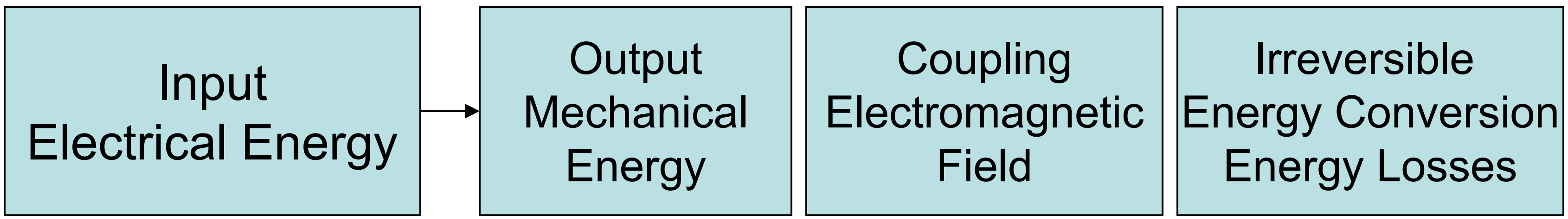
Mathematical Model for a Simple Pendulum

The kinetic energy of the pendulum bob is: $\Gamma = \frac{1}{2} m v^2 = \frac{1}{2} m \left(l \dot{\theta} \right)^2$

The potential energy is: $\Pi = mgh = mgl(1 - \cos\theta)$



Electrical Conversion



Energy Transfer in Electromechanical Systems

For rotational motion, the electromagnetic torque, as a function of current and angular displacement, is : $T_e(i,\theta) = \dfrac{dW_c(i,\theta)}{d\theta}$

Where $W_c = \oint_i \psi di$; where ψ is the flux.

Electromechanical Analogies

- From Newton's law or using Lagrange equations of motions, the second-order differential equations of translational-dynamics and torsional-dynamics are found as

$$m\frac{d^2x}{dt^2} + B_v\frac{dx}{dt} + k_s x = F_a(t)\ \text{(Translational dynamics)}$$

$$j\frac{d^2\theta}{dt^2} + B_m\frac{d\theta}{dt} + k_s\theta = T_a(t)\ \text{(Torsional dynamics)}$$

For a series RLC circuit, find the characteristic equation and define the analytical relationships between the characteristic roots and circuitry parameters.

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = \frac{1}{L}\frac{dv_a}{dt}$$

$$s^2 + \frac{R}{L} s + \frac{1}{LC} = 0$$

The characteristic roots are

$$s_1 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Translational Damper, B_v (N-sec)

Appied force $F_a(t)$ in Newton

Linear velocity $v(t)$ (m/sec)

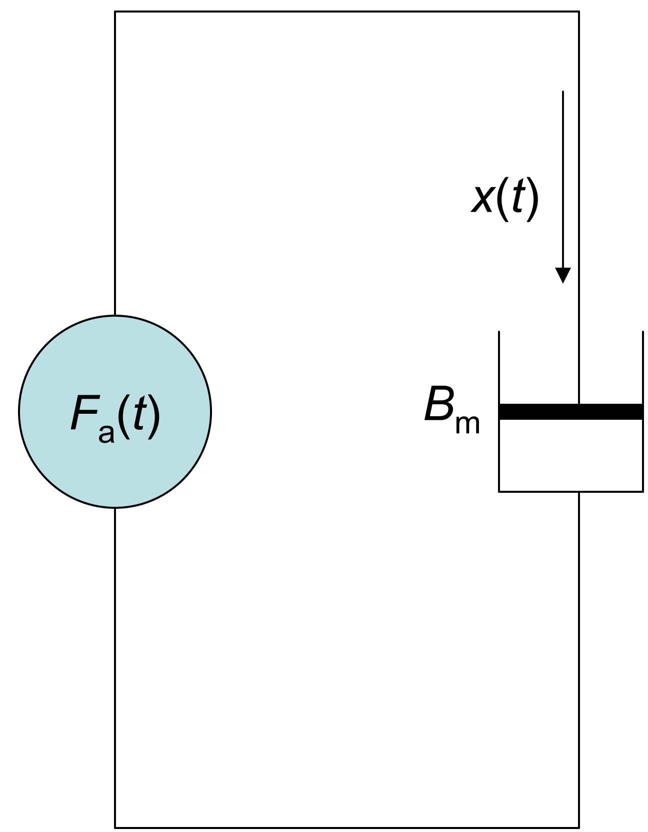
Linear position $x(t)$ (m)

$$F_a(t) = B_m v(t)$$

$$v(t) = \frac{1}{B_m} F_a(t)$$

$$F_a(t) = B_m v(t) = B_m \frac{dx(t)}{dt}$$

$$x(t) = \frac{1}{B_v} \int_{t_0}^{t} F_a(t) dt$$



Translational Spring, k (N)

Appied force $F_a(t)$ in Newton

Linear velocity $v(t)$ (m/sec)

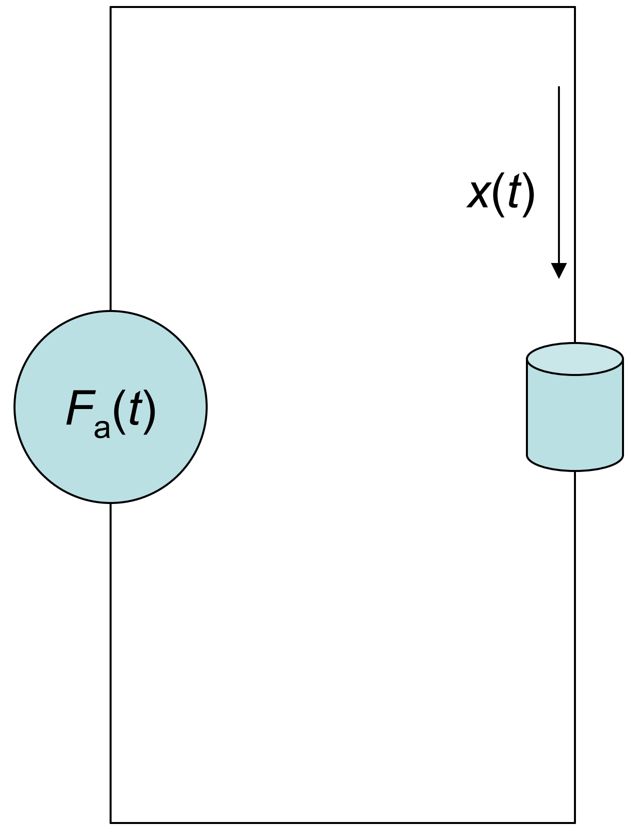
Linear position $x(t)$ (m)

$$F_a(t) = k_s x(t)$$

$$x(t) = \frac{1}{k_s} F_a(t)$$

$$v(t) = \frac{dx(t)}{dt} = \frac{1}{k_s} \frac{dF_a(t)}{dt}$$

$$F_a(t) = k_s \int_{t_0}^{t} v(t) dt$$



Rotational Damper, B_m (N-m-sec/rad)

Appied torque $T_{\mathrm{a}}(t)$ (N - m)

Angular velocity $\omega(t)$ (rad/sec)

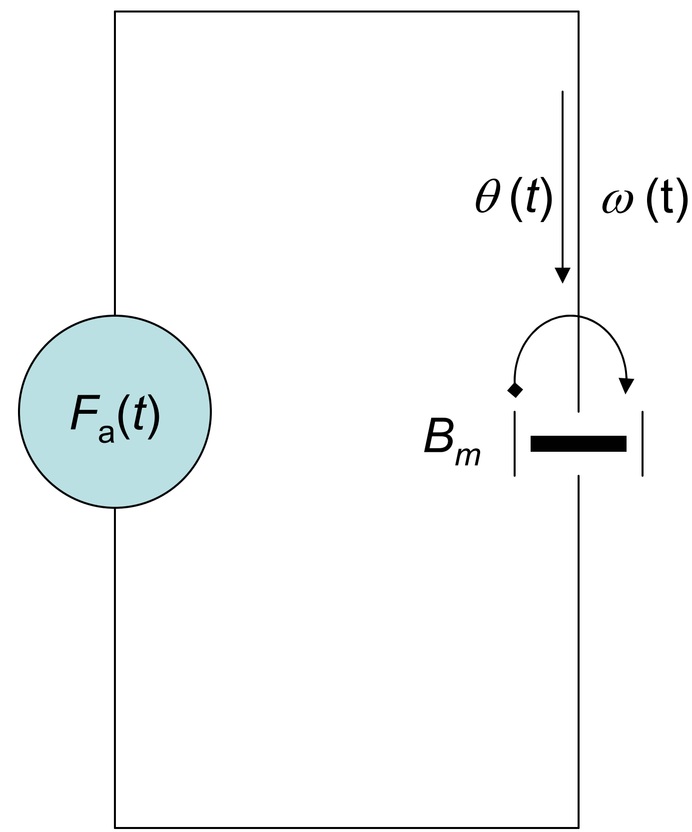
Angular displacement $\theta(t)$ (rad)

$$T_a(t) = B_m \omega(t)$$

$$\omega(t) = \frac{1}{B_m} T_a(t)$$

$$T_a(t) = B_m \omega(t) = B_m \frac{d\theta(t)}{dt}$$

$$\theta(t) = \frac{1}{B_m} \int_{t0}^{t} T_a(t) dt$$



Rotational Spring, k_s (N-m-sec/rad)

Appied torque $T_a(t)$ (N - m)

Angular velocity $\omega(t)$ (rad/sec)

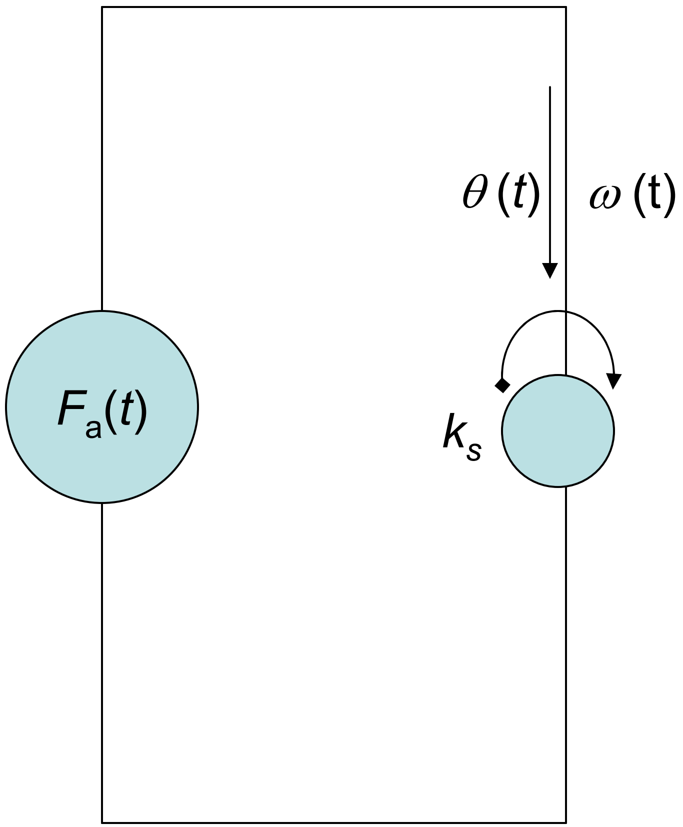
Angular displacement $\theta(t)$ (rad)

$$T_a(t) = B_m \theta(t)$$

$$\theta(t) = \frac{1}{k_s} T_a(t)$$

$$\omega(t) = \frac{d\theta(t)}{dt} = \frac{1}{k_s} \frac{dT_a(t)}{dt}$$

$$T_a(t) = k_s \int_{t_0}^{t} \omega(t) dt$$



Mass Grounded, *m* (kg)

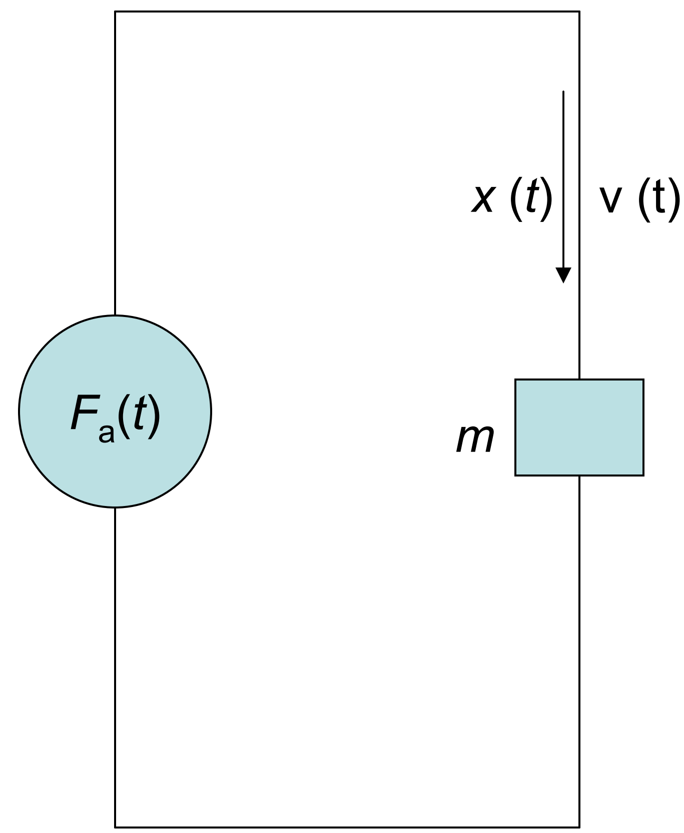
Appied torque $T_{\mathrm{a}}(t)$ (N - m)

Linear velocity $v(t)$ (m/sec)

Linear position $x(t)$ (m)

$$F_a(t) = m\frac{dv}{dt} = m\frac{d^2x(t)}{dt^2}$$

$$v(t) = \frac{1}{m}\int_{t_0}^{t} F_a(t)dt$$



Mass Grounded, *m* (kg)

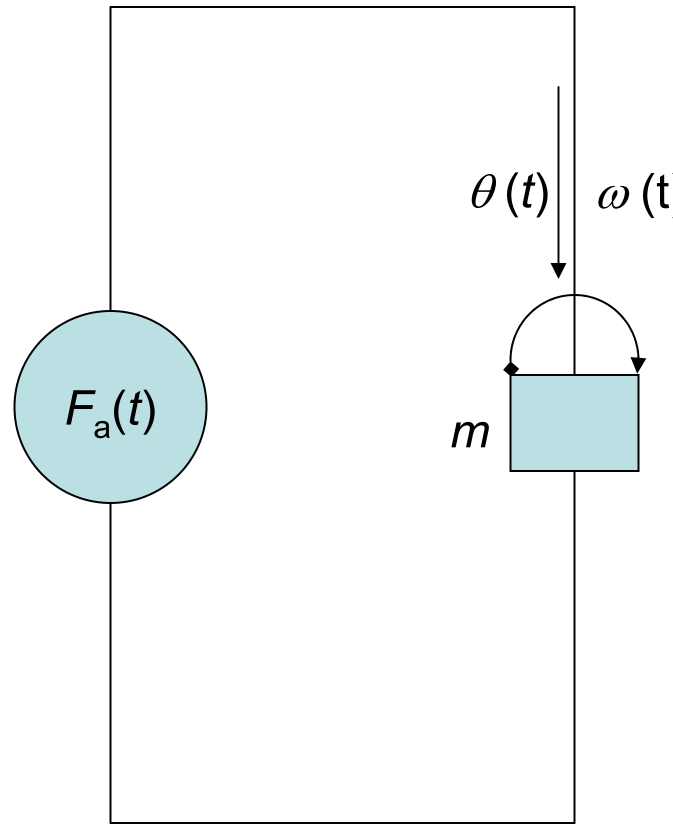
Appied torque $T_{\mathrm{a}}(t)$ (N - m)

Angular velocity $\omega(t)$ (rad/sec)

Angular displacement $\theta(t)$ (rad)

$$T_a(t) = J\frac{d\omega}{dt} = J\frac{d^2\theta(t)}{dt^2}$$

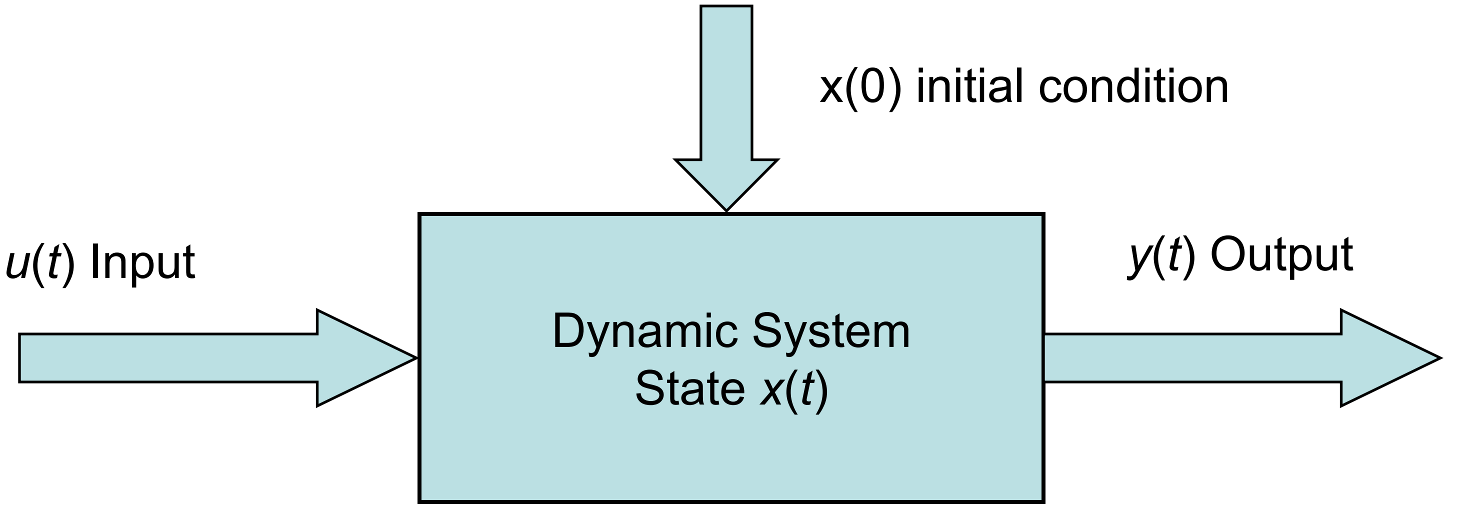
$$\omega(t) = \frac{1}{J}\int_{t0}^{t} T_a(t)dt$$



Steady-State Analysis

- **State:** The state of a dynamic system is the smallest set of variables (called state variables) so that the knowledge of these variables at $t = t_0$, together with the knowledge of the input for $t \geq t_0$, determines the behavior of the system for any time $t \geq t_0$.
- **State Variables:** The state variables of a dynamic system are the variables making up the smallest set of variables that determine the state of the dynamic system.
- **State Vector:** If n state variables are needed to describe the behavior of a given system, then the n state variables can be considered the n components of a vector x. Such vector is called a state vector.
- **State Space:** The n-dimensional space whose coordinates axes consist of the x_1 axis, x_2 axis, .., x_n axis, where x_1, x_2, .., x_n are state variables, is called a state space.
- **State-Space Equations:** In state-space analysis we are concerned with three types of variables that are involved in the modeling of dynamic system: input variables, output variables, and state variables.

State Variables of a Dynamic System



The state variables describe the future response of a system, given the present state, the excitation inputs, and the equations describing the dynamics

Electrical Example: An *RLC* Circuit

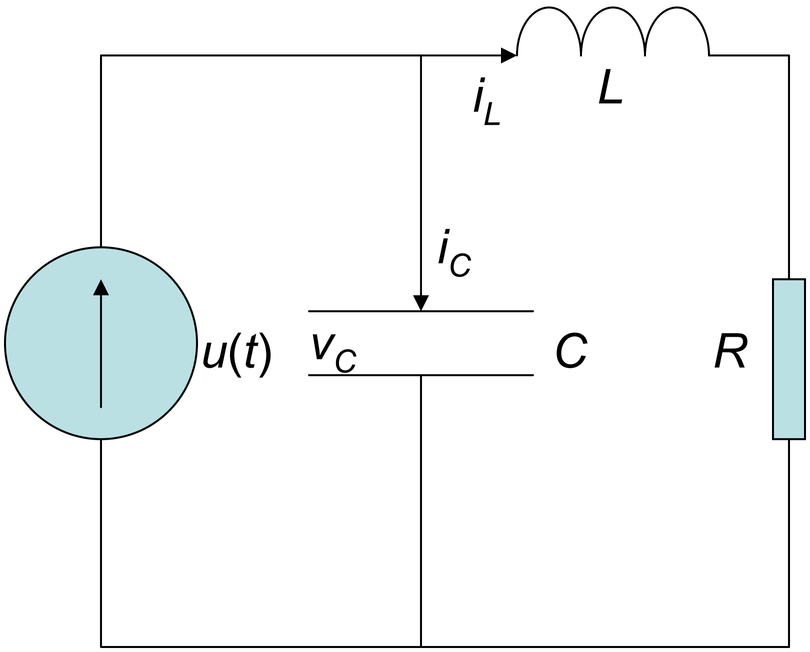
$x_1 = v_C(t); x_2 = i_L(t)$

$\xi = (1/2)Li_L^2 + (1/2)Cv_c^2$

$x_1(t_0)$ and $x_2(t_0)$ is the total initial energy of the network

USE KCL at the junction

$$i_c = C\frac{dv_c}{dt} = +u(t) - i_L$$



The State Differential Equation

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n + b_{11}u_1 + ... + b_{1m}u_m$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n + b_{21}u_1 + ... + b_{2m}u_m$$

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n + b_{n1}u_1 + ... + b_{nm}u_m$$

State Vector

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ . & . & . \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11}....b_{1m} \\ .... \\ b_{n1}....b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ . \\ u_m \end{bmatrix}$$

$\dot{\mathrm{x}} = \mathrm{Ax} + \mathrm{Bu}$ (State Differential Equation)

$\mathrm{y} = \mathrm{Cx} + \mathrm{Du}$ (Output Equation)

A : State matrix; B : input matrix

C : Output matrix; D : direct transmission matrix

The Output Equation

$$y_1 = h_{11}x_1 + h_{12}x_2 + \ldots + h_{1n}x_n$$

$$y_2 = h_{21}x_1 + h_{22}x_2 + \ldots + h_{2n}x_n$$

$$y_b = a_{b1}x_1 + a_{b2}x_2 + \ldots + a_{bn}x_n$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ . \\ y_b \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{1n} \\ h_{21} & h_{22} & h_{2n} \\ . & . & . \\ h_{b1} & h_{b2} & h_{bn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} = \mathrm{H}x$$

$\dot{\mathrm{x}} = \mathrm{Ax} + \mathrm{Bu}$ (State Differential Equation)

$\mathrm{y} = \mathrm{Hx} + \mathrm{Du}$ (Output Equation)

A : State matrix; B : input matrix

H : Output matrix; D : direct transmission matrix

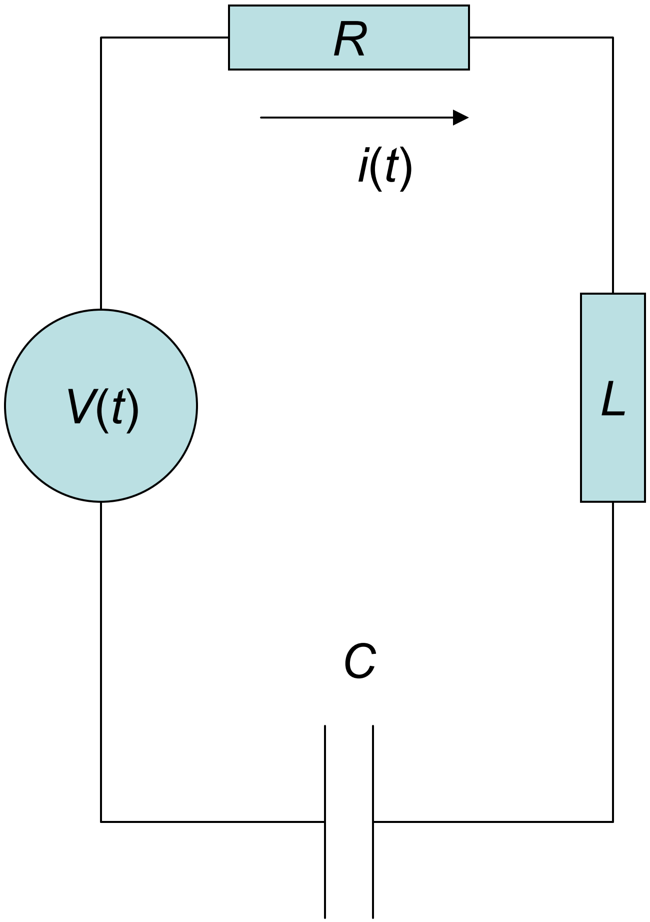
Example 1: Consider the given series RLC circuit. Derive the differential equations that map the circuitry dynamics.

$$C\frac{dv_c}{dt} = i$$

$$L\frac{di}{dt} = -v_c - Ri + v(t)$$

$$\frac{dv_c}{dt} = \frac{1}{C}i$$

$$\frac{di}{dt} = \frac{1}{L}\left(-v_c - Ri + v(t)\right)$$



Example 2: Using the state-space concept, find the state-space model and analyze the transient dynamics of the series *RLC* circuit.

$$\frac{dv(t)}{dt} = \frac{1}{C} i$$

$$\frac{di}{dt} = \frac{1}{L}(-v_c - Ri + v(t))$$

$$x_1(t) = v_c(t); x_2(t) = i(t) \text{ (These are the states)}$$

$$v(t) \text{ is the control}$$

$$\frac{dx}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} \frac{dv_c}{dt} \\ \frac{di}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} v_c \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v_a = Ax + Bu$$

Continue with Values..

- Assume R = 2 ohm, L = 0.1 H, and C = 0.5 F, find the following coefficients.

$$A = \begin{bmatrix} 0 & 2 \\ -10 & -20 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

- The initial conditions are assumed to be $v_c(t_0)=v_{c0}$=15 V; and $I\,(t_0) = i_0$ = 5 A.

$$x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \end{bmatrix}$$

- Let the voltage across the capacitor be the output; $y(t)$= $v_c(t)$. The output equation will be

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v_c \\ i \end{bmatrix} = \mathrm{H}x; \mathrm{H} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- The expanded output equation in y

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v_c \\ i \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} v_a = Hx + Du$$

- The circuit response depends on the value of $v\,(t)$