EXAMPLE 10.2 **A lead compensator for a second-order system**

A unity feedback control system has a loop transfer function

$$L(s) = \frac{K}{s(s+2)}, \tag{10.21}$$

where $L(s) = G_c(s)G(s)$ and $H(s) = 1$. We want to have a steady-state error for a ramp input equal to 5% of the velocity of the ramp. Therefore, we require that

$$K_v = \frac{A}{e_{ss}} = \frac{A}{0.05A} = 20. \tag{10.22}$$

Furthermore, we desire that the phase margin of the system be at least 45°. The first step is to plot the Bode diagram of the uncompensated transfer function

$$G(j\omega) = \frac{K_v}{j\omega(0.5j\omega + 1)} = \frac{20}{j\omega(0.5j\omega + 1)}, \tag{10.23}$$

as shown in Figure 10.10(a). The frequency at which the magnitude curve crosses the 0-dB line is 6.2 rad/s, and the phase margin at this frequency is determined readily from the equation of the phase of $G(j\omega)$, which is

$$\angle G(j\omega) = \phi(\omega) = -90° - \tan^{-1}(0.5\omega). \tag{10.24}$$

At the crossover frequency $\omega = \omega_c = 6.2$ rad/s, we have

$$\phi(\omega) = -162°, \tag{10.25}$$

and therefore the phase margin is 18°. Using Equation (10.24) to evaluate the phase margin is often easier than drawing the complete phase-angle curve, which is shown in Figure 10.10(a). Thus, we need to add a phase-lead network so that the phase margin is raised to 45° at the new crossover (0-dB) frequency. Because the compensation crossover frequency is greater than the uncompensated crossover frequency, the phase lag of the uncompensated system is also greater. We shall account for this additional phase lag by attempting to obtain a maximum phase lead of $45° - 18° = 27°$, plus a small increment (10%) of phase lead to account for the added lag. Thus, we will design a compensation network with a maximum phase lead equal to $27° + 3° = 30°$. Then, calculating α, we obtain

$$\frac{\alpha - 1}{\alpha + 1} = \sin 30° = 0.5, \tag{10.26}$$

and therefore $\alpha = 3$.

The maximum phase lead occurs at ω_m, and this frequency will be selected so that the new crossover frequency and ω_m coincide. The magnitude of the lead network at ω_m is $10 \log \alpha = 10 \log 3 = 4.8$ dB. The compensated crossover frequency is then evaluated where the magnitude of $G(j\omega)$ is -4.8 dB, and thus $\omega_m = \omega_c = 8.4$. Drawing the compensated magnitude line so that it intersects the

0-dB axis at $\omega = \omega_c = 8.4$, we find that $z = \omega_m/\sqrt{\alpha} = 4.8$ and $p = \alpha z = 14.4$. Therefore, the compensation network is

$$G_c(s) = \frac{1}{3}\frac{1 + s/4.8}{1 + s/14.4}. \tag{10.27}$$

The total DC loop gain must be raised by a factor of three in order to account for the factor $1/\alpha = \frac{1}{3}$. Then the compensated loop transfer function is

$$L(s) = G_c(s)G(s) = \frac{20(s/4.8 + 1)}{s(0.5s + 1)(s/14.4 + 1)}. \tag{10.28}$$

To verify the final phase margin, we can evaluate the phase of $G_c(j\omega)G(j\omega)$ at $\omega = \omega_c = 8.4$ and thus obtain the phase margin. The phase angle is then

$$\begin{aligned}\phi(\omega_c) &= -90° - \tan^{-1} 0.5\omega_c - \tan^{-1}\frac{\omega_c}{14.4} + \tan^{-1}\frac{\omega_c}{4.8} \\ &= -90° - 76.5° - 30.0° + 60.2° \\ &= -136.3°. \end{aligned} \tag{10.29}$$

Therefore, the phase margin for the compensated system is 43.7°. If we desire to have exactly a 45° phase margin, we would repeat the steps with an increased value of α—for example, with $\alpha = 3.5$. In this case, the phase lag increased by 7° between $\omega = 6.2$ and $\omega = 8.4$, and therefore the allowance of 3° in the calculation of α was not sufficient. The step response of this system yields a 28% overshoot with a settling time of 0.75 second.

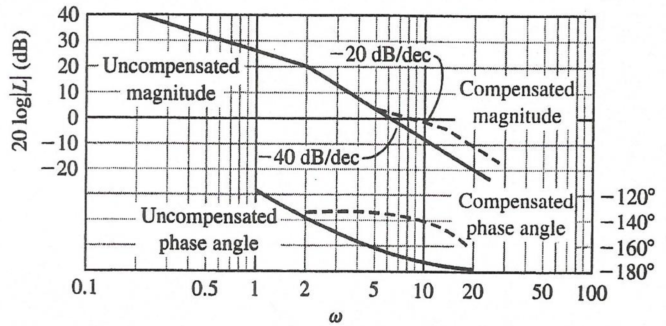
The Nichols diagram for the compensated and uncompensated system is shown in Figure 10.10(b). The reshaping of the frequency response locus is clear on this diagram. Note the increased phase margin for the compensated system as well as the reduced magnitude of $M_{p\omega}$, the maximum magnitude of the closed-loop frequency response. In this case, $M_{p\omega}$ has been reduced from an uncompensated value of +12 dB to a compensated value of approximately +3.2 dB. Also, we note that the closed-loop 3-dB bandwidth of the compensated system is equal to 12 rad/s compared with 9.5 rad/s for the uncompensated system. ■

Looking again at Examples 10.1 and 10.2, we note that the system design is satisfactory when the asymptotic curve for the magnitude $20 \log|GG_c|$ crosses the 0-dB line with a slope of −20 dB/decade.

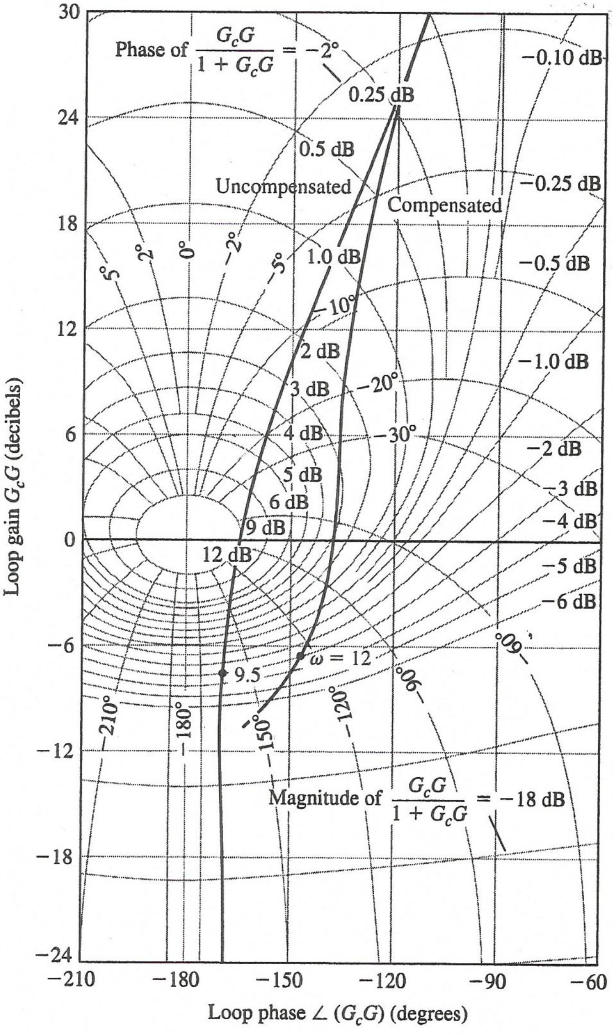
E-LEAD DESIGN USING THE ROOT LOCUS

The design of the phase-lead compensation network can also be readily accomplished using the root locus. The phase-lead network has a transfer function

$$G_c(s) = \frac{s + 1/\alpha\tau}{s + 1/\tau} = \frac{s + z}{s + p}, \tag{10.30}$$


40
30
20
10
0
−10
−20
20 log|L| (dB)
Uncompensated
magnitude
−20 dB/dec
Compensated
magnitude
−40 dB/dec
Uncompensated
phase angle
Compensated
phase angle
−120°
−140°
−160°
−180°
φ(ω)
0.1
0.5
1
2
5
10
20
50
100
ω

(a)


Phase of $\frac{G_cG}{1+G_cG}$ = −2°
0.25 dB
−0.10 dB
0.5 dB
Uncompensated
Compensated
−0.25 dB
1.0 dB
−0.5 dB
−10°
2 dB
−1.0 dB
3 dB
−20°
4 dB
−30°
−2 dB
5 dB
6 dB
−3 dB
9 dB
−4 dB
12 dB
−5 dB
−6 dB
ω = 12
9.5
−210°
−180°
−150°
−120°
−90°
−60°
5°
2°
0°
−2°
−5°
Magnitude of $\frac{G_cG}{1+G_cG}$ = −18 dB
30
24
18
12
6
0
−6
−12
−18
−24
Loop gain G_cG (decibels)
−210
−180
−150
−120
−90
−60
Loop phase ∠ (G_cG) (degrees)

(b)