The Shuffle Hopf Algebra and Noncommutative Full Completeness

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August 30, 1996

Abstract

We present a full completeness theorem for the multiplicative fragment of a variant of noncommutative linear logic known as *cyclic linear logic (CyLL)*, first defined by Yetter. The semantics is obtained by considering dinatural transformations on a category of topological vector spaces which are equivariant under certain actions of a noncocommutative Hopf algebra, called the *shuffle algebra*. Multiplicative sequents are assigned a vector space of such dinaturals, and we show that the space has the denotations of cut-free proofs in $CyLL+MIX$ as a basis. This can be viewed as a fully faithful representation of a free $*$-autonomous category, canonically enriched over vector spaces.

This work is a natural extension of the authors' previous work, "Linear Läuchli Semantics", where a similar theorem is obtained for the commutative logic. In that paper, we consider dinaturals which are invariant under certain actions of the additive group of integers. We also present here a simplification of that work by showing that the invariance criterion is actually a consequence of dinaturality. The passage from groups to Hopf algebras corresponds to the passage from commutative to noncommutative logic.

1 Introduction

This paper is a continuation of a program initiated in [13], where a linear version of Läuchli's semantics for intuitionistic logic is presented. In that paper, we consider actions of the additive group of integers on a category of topological vector spaces. We associate to any sequent in Multiplicative Linear Logic (MLL) a vector space of dinatural transformations which are invariant with respect to certain such actions. Originally, we called these dinaturals $\mathbf{Z}$-*uniform*. In this paper we present a simplification of this notion based on an observation of Plotkin. We will show that in fact uniformity with respect to arbitrary groups is a consequence of dinaturality. We then show that this vector space of dinaturals has as basis the denotations of cut-free proofs of the sequent in the theory $MLL + MIX$. Thus we obtain a *full completeness theorem* in the sense of [2]: our semantics consists entirely of (linear combinations of) denotations of proofs. The fact that cut-free proofs form a basis and not just a spanning set means that our interpretation is faithful, as well as full. In fact, we have a fully faithful representation of a free $*$-autonomous category, canonically enriched over vector spaces. This will be discussed in Remark 2.16 below.

It was observed at the end of [13] that this semantics might be expanded to noncommutative logics by replacing groups with *Hopf algebras*. In [12], the representation theory of Hopf algebras is presented

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as a unifying framework for the analysis of a number of variants of linear logic. By varying the Hopf structure, one obtains models of the commutative, fully noncommutative, cyclic or braided variants. Thus, choosing a Hopf algebra corresponds abstractly to specifying the structural rules of a theory. This is summarized in the following chart:

Theory	Hopf Structure
commutative	Δ cocommutative
braided	quantum group (i.e. quasitriangular) [28]
noncommutative	Δ noncommutative, S invertible
cyclic	Δ noncommutative, $S^2 = id$

The relevance of Hopf algebras is further suggested by the conservativity theorem, Theorem 11.7 of [13], which says that every dinatural which is uniform with respect to the integers is also uniform with respect to arbitrary cocommutative Hopf algebras. Thus by considering general Hopf algebras, it seemed plausible that one could obtain such theorems for noncommutative logics. The full completeness theorem we present here strengthens this analogy, and suggests a general theory which we hope to explore in the future.

The particular variant of linear logic that we will work with is the *cyclic linear logic (CyLL)* of Yetter [38]. This variant is obtained by adding the cyclic exchange rule to the fully noncommutative logic of [3]. The corresponding version of proof net is also described in [38]. This theory has subsequently been used substantially by Retoré in his work on linguistics [31].

The Hopf algebra which provides our semantics is an example of the incidence algebras of [22, 33]. It is also refered to as a *shuffle algebra* in [10], which is the name we have chosen to use. Given a sequent in linear logic, we assign a vector space of dinaturals which are uniform with respect to this Hopf algebra, and show that it is generated by the denotations of (equivalence classes of) proofs in the cyclic fragment.

Nonsymmetric monoidal categories which arise from Hopf algebras have recently become important in quantum physics [27, 28]. Since linear logic is a natural vehicle for describing free monoidal categories [11], then modifying the structural rule of exchange should be the logical analogue of the *quantization* process discussed in these references. This suggests for example a logical interpretation of theorems such as the various *Tannaka-Krein* theorems described in [27, 28, 37].

The particular Hopf algebra chosen is of independent interest in several fields. In the theory of distributed and concurrent computation, an important notion is that of interleaving or merging of input streams of data. Benson [10] observed that this process has a natural algebraic structure, which led him to consider the shuffle algebra. Such structures also arise in a fundamental way in combinatorics [22, 33], as such Hopf algebras provide an algebraic framework for the study of generating functions. Connections to combinatorics are further established via Joyal's notion of *species* [23], a functorial framework for analyzing generating functions. Species were then generalized and given a Hopf-algebraic interpretation by Schmitt in [34]. Thus the representation theory of such structures should have important consequences for both these subjects. An overview of the applications of Hopf algebras to various branches of mathematics is given by Hazewinkel in [21].

These results were first presented at the conference Linear Logic '96 at Keio University in Tokyo. An extended electronic abstract announcing the results has appeared as [14].

Note to the Reader: To avoid repetition of previous work, we assume that the reader has some familiarity with Hopf algebras, linear topology and functorial polymorphism. Appropriate references are [12, 13, 4]. We will begin by reviewing the results of [13].

1.1 Review of Linear Topology

It is well known that $\mathcal{VEC}$, the category of vector spaces, is autonomous, i.e. symmetric monoidal closed. To obtain a $*$-autonomous category of vector spaces, we add a topological structure, due to Lefschetz [26].

Definition 1.1 Let V be a vector space. A topology, τ, on V is *linear* if it satisfies the following three properties:

- Addition and scalar multiplication are continuous, when the field $\mathbf{k}$ is given the discrete topology.
- τ is hausdorff
- $0 \in V$ has a neighborhood basis of open linear subspaces.

Let $\mathcal{TVEC}$ denote the category whose objects are vector spaces equipped with linear topologies, and whose maps are linear continuous morphisms.

$\mathcal{TVEC}$ is a symmetric monoidal closed category, when $V \multimap W$ is defined to be the vector space of linear continuous maps, topologized with the topology of pointwise convergence. (It is shown in [8] that the forgetful functor $\mathcal{TVEC} \rightarrow \mathcal{VEC}$ is tensor-preserving.) Lefschetz proves that the embedding $V \rightarrow V^{\perp\perp}$ is always a bijection, but need not be an isomorphism. We then have:

Theorem 1.2 *(Barr [5]) $\mathcal{RTVEC}$, the full subcategory of reflexive objects in $\mathcal{TVEC}$, is a complete, cocomplete $*$-autonomous category.*

The following definition and theorem can be found in [13].

Definition 1.3 Let G be a group. A *continuous G-module* is a linear action of G on a space V in $\mathcal{TVEC}$, such that for all $g \in G$, the induced map $g\bullet(\ ) : V \rightarrow V$ is continuous. Let $\mathcal{TMOD}(G)$ denote the category of continuous G-modules and continuous equivariant maps. Let $\mathcal{RTMOD}(G)$ denote the full subcategory of reflexive objects.

We have the following result, which in fact holds in the more general context of Hopf algebras [12].

Theorem 1.4 *The category $\mathcal{TMOD}(G)$ is symmetric monoidal closed. The category $\mathcal{RTMOD}(G)$ is $*$-autonomous, and a reflective subcategory of $\mathcal{TMOD}(G)$ via the functor $(\ )^{\perp\perp}$. Furthermore the forgetful functor to $\mathcal{RTVEC}$ preserves the $*$-autonomous structure.*

2 Linear Läuchli Semantics Revisited

In [13], a full completeness theorem is established for $MLL+MIX$ via the notion of a *uniform dinatural*. Here we simplify the presentation somewhat, in that we show that uniformity is actually a consequence of dinaturality, for the original (commutative) setting (see Proposition 2.5 and Corollary 2.7). We also present the proof of faithfulness of our interpretation 2.16, which was left implicit in [13].

Definition 2.1 Let $\mathcal{C}$ be a category, and $F, G : \mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathcal{C}$ functors. A *dinatural transformation* is a family of $\mathcal{C}$-morphisms $\theta = \{\theta_X : FXX \rightarrow GXX \mid X \in |\mathcal{C}|\ \}$ such that for any $f : X \rightarrow Y$, the following diagram commutes:

$$\begin{tikzcd}
 & FXX \arrow[r, "\theta_X"] & GXX \arrow[dr, "GXf"] & \\
FYX \arrow[ur, "FfX"] \arrow[dr, "FYf"'] & & & GXY \\
 & FYY \arrow[r, "\theta_Y"] & GYY \arrow[ur, "GfY"'] &
\end{tikzcd}$$

i.e. equationally,

$$GXf \circ \theta_X \circ FfX = GfY \circ \theta_Y \circ FYf \tag{1}$$

Note that functoriality of F implies that for arrows $X' \xrightarrow{g} X$ and $X \xrightarrow{f} Y$, $FXX \xrightarrow{Fgf} FX'Y = FXX \xrightarrow{FgX} FX'X \xrightarrow{FX'f} FX'Y$.

Let $Dinat(F,G)$ denote the set of dinatural transformations from F to G. If $\vdash \Gamma$ is a one-sided sequent, then $Dinat(\Gamma)$ denotes the set of dinaturals from $\mathbf{k}$ to $\invamp\Gamma$.

Recall from [13] the following definition of uniform dinatural:

Definition 2.2 Let F and $F^{'}$ be definable functors on $\mathcal{RTVEC}$. A dinatural transformation $\theta\colon F \to F'$ is *uniform* for a group G if for every $V_1, \ldots, V_n \in \mathcal{RTMOD}(G)$, the morphism $\theta_{|V_1|,\ldots,|V_n|}$ is a G-map, i.e. is equivariant with respect to the actions induced from the atoms V_i.

Remark 2.3 Gordon Plotkin [30] has recently observed that dinaturality implies uniformity in our sense; that is, *all* dinaturals between *MLL* definable functors are automatically uniform. We shall prove this result below. Of course this permits dropping the word "uniform" in the results of [13] and also puts some of our previous work in a new light.

This observation is based on the original Läuchli setting of Sets and hereditary permutations, where there are intriguing connections to Reynolds' "parametricity" [4, 29] and the theory of logical relations. Indeed, Plotkin and Abadi [29], answering a problem of [4], prove that Reynolds' relational parametricity formally implies dinaturality, in a parametric logical calculus for Girard's system $\mathcal{F}$. We will show that dinaturality implies a version of "naturality for isomorphisms", which in turn implies our G-uniformity condition.

We now show dinaturality implies the following version of "uniformity" of the family θ.

Definition 2.4 A family $\theta = \{\theta_X : FXX \to GXX \mid X \in |\mathcal{C}|\}$ is *uniform for isomorphisms* if for all isomorphisms $f : X \to Y \in |\mathcal{C}|$, the following diagram commutes:

$$\begin{tikzcd}
FXX \arrow[r, "\theta_X"] \arrow[d, "Ff^{-1}f"'] & GXX \arrow[d, "Gf^{-1}f"] \\
FYY \arrow[r, "\theta_Y"] & GYY
\end{tikzcd}$$

Proposition 2.5 *For a family θ defined as above, dinaturality implies uniformity for isomorphisms.*

Proof. Since $Ff^{-1}f = FYf \circ Ff^{-1}X$ and similarly for G, the above diagram translates to the following equation: for all $f : X \to Y \in \mathcal{C}$,

$$GYf \circ Gf^{-1}X \circ \theta_X \;=\; \theta_Y \circ FYf \circ Ff^{-1}X \quad . \tag{2}$$

We show (1) implies (2). From (1), composing both sides on the right with $Ff^{-1}X$ and observing that $(FfX)^{-1} = Ff^{-1}X$, we have:

$$GXf \circ \theta_X \;=\; GXf \circ \theta_X \circ FfX \circ Ff^{-1}X = GfY \circ \theta_Y \circ FYf \circ Ff^{-1}X \quad . \tag{3}$$

From equation (3), composing both sides on the left with $Gf^{-1}Y$ we simplify to:

$$Gf^{-1}Y \circ GXf \circ \theta_X \;=\; \theta_Y \circ FYf \circ Ff^{-1}X \quad . \tag{4}$$

Finally, observe that (4) equals (2) , since their left hand sides both equal to $Gf^{-1}f \circ \theta_X$. $\square$

In the above discussions, we can equally consider n-ary multivariate functors, using $\mathcal{C}^n$ in place of $\mathcal{C}$, in which case X, Y, f, etc. denote vectors of objects and arrows, respectively.

We now specialize to the case $\mathcal{C} = \mathcal{RTVEC}$. We shall be interested in MLL-definable functors. Such functors are given by MLL-formulae $F(\alpha_1, \cdots, \alpha_n)$ built from propositional atoms $\alpha_1, \cdots, \alpha_n$, using the connectives $\otimes, \parr, (-)^\perp$. We shall be interested in instantiations $F(V_1, \cdots, V_n)$ at G-modules V_i (each equipped with a linear automorphic action $v \mapsto g{\bullet}v,\ \ g \in G$).

Proposition 2.6 *Let $F(\alpha_1, \cdots, \alpha_n)$ be an MLL-formula, considered as a definable n-ary multivariate functor. Instantiate each α_i at the G-module $V_i \in \mathcal{RTMOD}(G)$. Then the instantiation $F(V_1, \cdots, V_n)$ is a G-module, with action given by $Ff^{-1}f : F(V_1, \cdots, V_n) \to F(V_1, \cdots, V_n)$, where $f = \langle f_i \rangle$ is the vector of maps whose components f_i are the automorphisms $v \mapsto g{\bullet}v$, for each fixed $g \in G$.*

Proof. By induction on the formula F. If F is an atom α, then $F(V) = V$. So $Ff^{-1}f : V \to V$ equals f, the given action map $v \mapsto g{\bullet}v$, $g \in G$. If F is $\alpha^\perp$, then $Ff^{-1}f : V^\perp \to V^\perp$ is the map $w \mapsto w{\circ}f^{-1}$. We must show this is the action map of $V^\perp = Hom(V, \mathbf{k})$, where $\mathbf{k}$ is given the discrete action. But $(Ff^{-1}f)(w)(v) = (w{\circ}f^{-1})(v) = w(f^{-1}(v)) = w(g^{-1}{\bullet}v) = (g{\bullet}w)(v)$ from the definition of the contragredient action. So $(Ff^{-1}f)(w) = g{\bullet}w$. Finally, $\otimes$ and $\parr$ are straightforward. $\square$

From the definition of uniformity for isomorphisms and the previous proposition, we see that the action maps commute with the components of the family θ. Thus, the action on an instantiation of F is inherited from the actions on its atoms. We immediately conclude:

Corollary 2.7 *Under the previous assumptions, if θ is uniform for isomorphisms then θ is uniform in the sense of linear Läuchli semantics [13].*

Remark 2.8 While the above observation allows one to remove the uniformity condition in the commutative case, uniformity will still clearly be necessary with respect to an appropriate Hopf algebra for noncommutative logic. This follows from the fact that arbitrary symmetries are dinatural. The reason the above proof fails in the noncommutative setting is the action of a general Hopf algebra on a tensor product is *not* the the tensor of actions. In other words, if H is a Hopf algebra, and $h \in \mathsf{H}$ then $h(v \otimes w)$ will generally not be $hv \otimes hw$. This leads one to define the notion of a *grouplike* element of H, which will satisfy this equation. See [35].

We now reformulate the results of [13] in light of this new observation. It is straightforward to verify that $Dinat(F, F')$ is a vector space, under pointwise operations. We call it the *space of proofs* associated to the sequent $F \vdash F'$. Note that we identify formulas with definable functors.

Before obtaining a full completeness theorem, we first obtained a traditional completeness theorem, which is analogous to the results of [25, 20].

Theorem 2.9 (Completeness) *Let $M \vdash N$ be a balanced binary sequent. If the unique cut-free proof structure associated to $M \vdash N$ is not a proof net for the theory $MLL + MIX$, then $Dinat(M, N)$ is a zero dimensional vector space.*

The key lemma in extending this result to a full completeness theorem is:

Lemma 2.10 *Let M, N be MLL formulas. If $Dinat(M, N)$ has dimension greater than 0, then the sequent $M \vdash N$ is balanced.*

Now that we see that only balanced sequents need be considered, we establish the first form of full completeness:

Theorem 2.11 (Full Completeness for Binary Sequents) *If a sequent $M \vdash N$ is binary, then $Dinat(M, N)$ is zero or 1-dimensional, depending on whether its uniquely determined proof structure is a net. In the latter case, every dinatural is a scalar multiple of the denotation of the unique cut-free proof net.*

Note that to any balanced sequent, say $M \vdash N$, we can assign a set of sets of axiom links. This assignment determines a finite list of binary sequents of which $M \vdash N$ is a substitution instance. Suppose this list is: $M_1 \vdash N_1, M_2 \vdash N_2, \ldots$. (The list must be finite.) We define a new vector space, called the *associated binary space* for the sequent $M \vdash N$.

$$\mathcal{ABS}(M,N) = \coprod_i Dinat(M_i, N_i)$$

There is a canonical linear map:

$$\varphi : \mathcal{ABS}(M,N) \longrightarrow Dinat(M,N)$$

On basis elements, this is defined by "equating variables" in the sequent $M_i \vdash N_i$, or more formally, restricting which instantiations we will allow according to the pattern in $M \vdash N$.

Definition 2.12 We call those elements of $Dinat(M,N)$ of the form $\varphi(\mathcal{S})$ for a (necessarily unique) $\mathcal{S} \in \mathcal{ABS}(M,N)$ *diadditive.*

Equivalently, a diadditive dinatural transformation is a transformation which is a linear combination of substitution instances of binary dinaturals. It is still an open question as to whether every dinatural between definable functors is diadditive.

We wish to note the following lemma which was not mentioned in [13]. It establishes that the interpretations of distinct proofs are linearly independent, and thus our interpretation is faithful.

Lemma 2.13 *Let $\vdash \Gamma$ be a balanced, nonbinary sequent. Let $\vdash \Gamma_1, \vdash \Gamma_2, \ldots, \vdash \Gamma_n$ be the binary correct sequents which have Γ as a substitution instance. Then the set of dinatural interpretations in $Dinat(\Gamma)$ of the unique cut-free proofs of $\vdash \Gamma_1, \vdash \Gamma_2, \ldots, \vdash \Gamma_n$ are linearly independent.*

Proof. Since the vector space structure of $Dinat(\Gamma)$ is computed pointwise, it suffices to find a single instantiation for which the interpretations are linearly independent.

We suppose that the sequent has a single literal which appears $2n$ times. The proof we will give is easily extended to the more general case. We instantiate the literal at an n-dimensional vector space with chosen basis $\{e_j\}_{j=1}^n$.

Let Γ_i be one of the associated binary sequents. Its interpretation at this instantiation will be of the following form:

$$\sum_{1 \leq j_1, j_2, \ldots, j_n \leq n} v_{j_1, j_2, \ldots, j_n}$$

where $v_{j_1, j_2, \ldots, j_n}$ is a tensor product of basis elements or duals of basis elements such that:

1. Each e_{j_i} and each $e^*_{j_i}$ appears exactly once.

2. Each e_{j_i}, $e^*_{j_i}$ pair appear in positions corresponding to the pairings of the axiom links.

Among the terms in this summation will occur several such that each j_i is instantiated at a distinct integer. Clearly such tensors will not occur in the interpretation of any other of the binary sequents, thus establishing linear independence. □

The notion of diadditive dinatural transformation then gives us our full completeness theorem.

Theorem 2.14 (Full Completeness) *Let F and F' be formulas in multiplicative linear logic, interpreted as definable multivariant functors on $\mathcal{RTVEC}$. Then the vector space of diadditive dinatural transformations has as basis the denotations of cut-free proofs in the theory MLL+MIX.*

We obtain the following corollary by the methods outlined in [19, 11].

Corollary 2.15 *Diadditive dinatural transformations compose. Thus we obtain an (indexed) $*$-autonomous category by taking as objects formulas, interpreted as multivariant functors. Morphisms will be diadditive dinatural transformations.*

Remark 2.16 (Fully Faithful Representation Theorem) The above results may be interpreted as a fully faithful representation theorem as follows. Let $\mathcal{F}$ denote the free $*$-autonomous category with MIX on a discrete graph of atomic types. Let $\mathcal{LIN}(\mathcal{F})$ denote the category whose objects are the same as in $\mathcal{F}$, but whose homsets are the vector spaces generated by the corresponding homsets in $\mathcal{F}$. Composition is then as in $\mathcal{F}$, and extended linearly. Then the above theorem may be reinterpreted as saying that the free functor from $\mathcal{LIN}(\mathcal{F})$ to the category described in 2.15 is an equivalence.

3 Biautonomous categories

Models of (intuitionistic) commutative linear logic are symmetric monoidal closed categories. If we drop the requirement that the tensor be symmetric, then one should consider categories with two internal HOM's. Thus we should have adjunctions of the form:

$$HOM(A \otimes B, C) \cong HOM(B, A \multimap C)$$
$$HOM(A \otimes B, C) \cong HOM(A, C \multimapinv B)$$

This is the definition of *biautonomous* category, and is the basis of, for example, the Lambek calculus [24]. Analogously, to define a nonsymmetric analogue of categories with dualizing objects one needs two duals, $A^\perp$ and ${}^\perp A$. (The dualizing object for each will be the same.)

These will be subject to the isomorphisms:

$${}^\perp(A^\perp) \cong ({}^\perp A)^\perp \cong A$$

More specifically, a biautonomous category has a canonical morphism:

$$A \longrightarrow {}^\perp(A^\perp) \cong ({}^\perp A)^\perp$$

and if this map is an isomorphism, then we have a *bi-$*$-autonomous* category. This definition is presented along with a noncommutative analogue of the Chu construction in [9]. Rosenthal presents examples of such categories in [32]. Hopf algebraic models are presented in [12]. We now discuss a variant of this notion, which is appropriate for the logic we are modeling.

Definition 3.1 If in a bi-$*$-autonomous category, the dualizing object, $\perp$, has the property that:

$${}^\perp A \cong A^\perp$$

then $\perp$ is said to be *cyclic*. A bi-$*$-autonomous category with such a dualizing object is also said to be *cyclic*.

Such a $*$-autonomous category will generally not be symmetric, but will validate the following weaker form of the exchange rule:

$$\frac{\vdash A_1, A_2, \ldots, A_n}{\vdash A_{\sigma(1)}, A_{\sigma(2)}, \ldots, A_{\sigma(n)}}$$

where σ is a *cyclic* permutation. This leads us to define the following variant of noncommutative linear logic.

4 Sequent Calculus

Yetter proposes *cyclic linear logic (CyLL)* in [38]. He presents a posetal semantics, which he calls *Girard quantales* and presents a completeness theorem, similar to the phase space completeness theorem of [17]. He also presents the proof net syntax we describe below. We present the sequent calculus for the multiplicative fragment with the MIX rule adjoined, hereafter refered to as $CyLL + MIX$.

Structural Rules

(1) *Cyclic Exchange* $\dfrac{\vdash \Gamma}{\vdash \sigma(\Gamma)}$ for any cyclic permutation σ of Γ.

(2) *Mix* $\dfrac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}$

Logical Rules

(3) $\vdash A, A^{\perp}$ *id*

(4) $\dfrac{\vdash \Gamma, A \quad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ Cut$

(5) $\dfrac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\ \otimes$

(6) $\dfrac{\vdash A, B, \Gamma}{\vdash A \mathbin{⅋} B, \Gamma}\ ⅋$

Given the nature of the exchange rule for this fragment, it is natural to represent the formulas of a sequent as lying on the perimeter of a circle, or as labelling radial lines on a disk. Since we will only consider such structures up to a rotation, then we will not need any explicit representation of the cyclic exchange rule [38, 31]. It is possible to represent nets by an inductive procedure analogous to that of the commutative case [17, 15, 16]. Rather than describe the construction in detail, we present an example of a cyclic net in *figure 1*, and refer the reader to [38, 31] for the details of the definition. The net presented corresponds to a deduction of the following sequent*:

$$\vdash \delta^{\perp}, (\alpha \otimes \delta^{\perp}) \otimes \alpha, \alpha^{\perp} ⅋ (\beta \otimes \gamma), \gamma^{\perp}, \beta^{\perp} \otimes \delta, \alpha^{\perp} \otimes \gamma, \varepsilon^{\perp}, \varepsilon \otimes (\gamma^{\perp} \otimes \delta)$$

There are several important features of this proof net to notice. First, the arcs inside the smaller circle are the cyclic analogue of axiom links. It is crucial that these links can be drawn in such a way that they do not cross. This planarity condition (along with the usual correctness criterion) ensures that the structure can be sequentialized.

4.1 Some Circular Reasoning

We begin by recalling some basic definitions from linear sequent calculus [2].

Definition 4.1 A sequent $\vdash \Gamma$ is *balanced* if each atom occurs an even number of times, with proper variance. A balanced sequent is *binary* if each atom occurs exactly twice.

We now give a helpful criterion for provability in $CyLL + MIX$.

Definition 4.2 Given a sequent Γ, its *underlying list of literals* is the list of literals obtained by erasing all commas and connectives.

Definition 4.3 A binary sequent is called *cyclic* if its underlying list of literals satisfies the following cyclicity condition:

*This example of a cyclic net appeared in Retoré's paper [31]

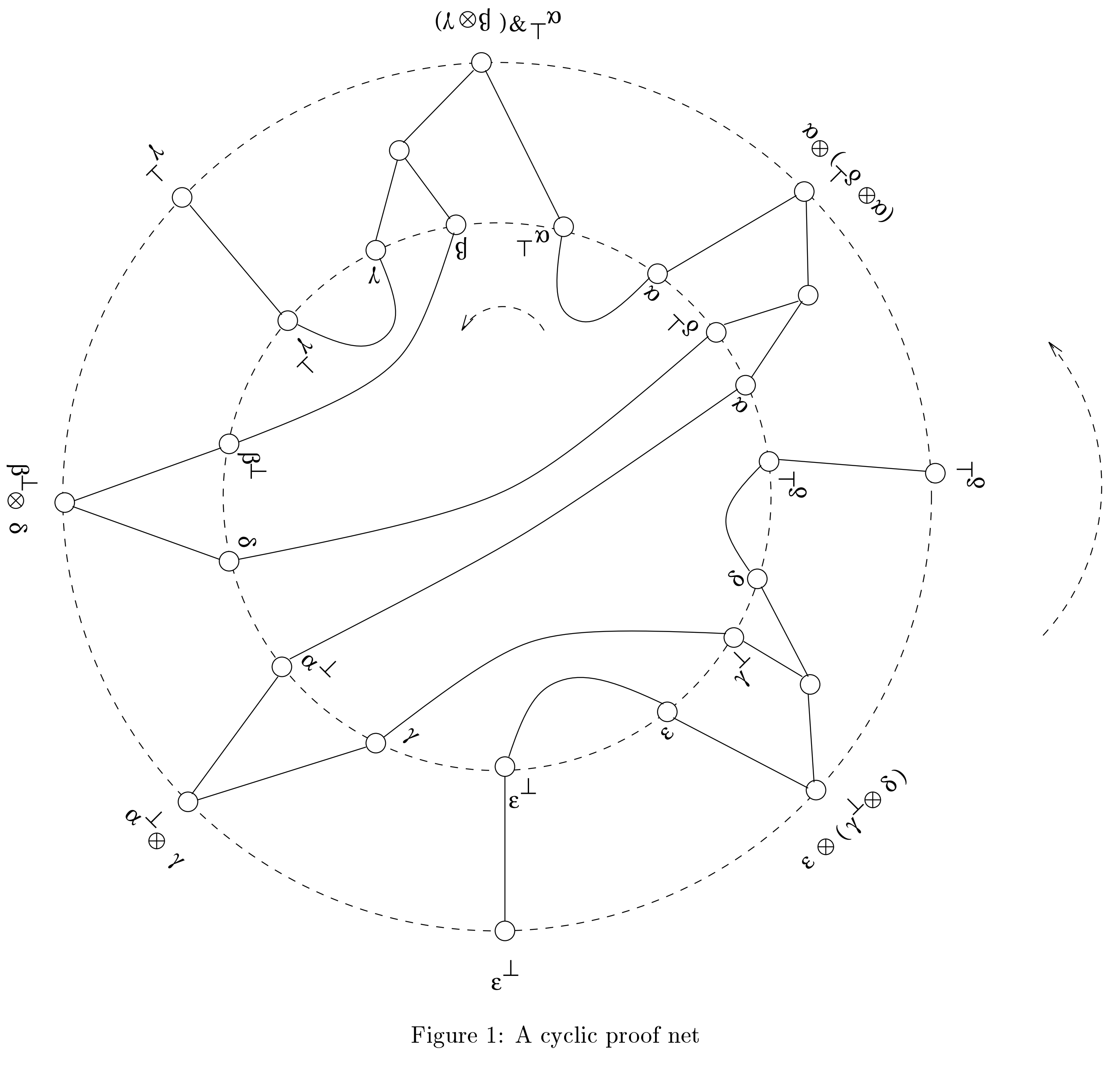

Figure 1: A cyclic proof net

(Cyc) We imagine this list of literals written on the perimeter of a circle. Start at some literal l_j, and travel around the circle either clockwise or counterclockwise. Before reaching $l_j^{\perp}$, if you encounter any other literal l_i, $i \neq j$, you will also encounter $l_i^{\perp}$.

We may also have occasion to refer to the underlying literal list as being cyclic. Equivalently, a literal list is cyclic if, whenever a literal lies between α and $\alpha^{\perp}$ in the literal list, then so does its dual.

Lemma 4.4 *A binary sequent is derivable in $CyLL + MIX$ if and only if it is cyclic and derivable in $MLL + MIX$.*

Proof. It is straightforward to verify that all of the sequent rules for this fragment preserve the property in question. The converse follows from sequentialization for Yetter's nets, cf. [31], and the planarity condition described above. □

5 Hopf algebras and Representations

5.1 Algebras and Coalgebras

In this section we give a quick summary of the necessary background in bialgebras and Hopf algebras. For suitable introductions, see [1, 35, 21].

Definition 5.1 A *Hopf algebra* is a $\mathbf{k}$-vector space†, H, equipped with an algebra structure and a compatible coalgebra structure (= bialgebra) and an antipode satisfying the appropriate equations [21, 35]. The following chart summarizes the necessary structure. All maps shown are linear.

	Structure	*Equations*
Algebra	$m\colon \mathsf{H} \otimes \mathsf{H} \to \mathsf{H}$ (multiplication) $\eta\colon \mathbf{k} \to \mathsf{H}$ (unit)	Associativity and Unit: $m\circ(m \otimes id) = m\circ(id \otimes m)$ and $\eta(1)$ is 2-sided unit for m.
Coalgebra	$\Delta\colon \mathsf{H} \to \mathsf{H} \otimes \mathsf{H}$ (comultiplication) $\varepsilon : \mathsf{H} \to \mathbf{k}$ (counit)	Coassociativity with counit for comultiplication (dual to algebra structure).
Bialgebra	Algebra + Coalgebra	Δ and ε are algebra homs. (Equivalently m, η are coalgebra homs.)
Antipode	$S\colon \mathsf{H} \to \mathsf{H}$	Inverse to $id_{\mathsf{H}} : \mathsf{H} \to \mathsf{H}$ under convolution

Here convolution refers to the operation on $Hom_{\mathbf{k}}(H, H)$ defined by $(f * g)(c) = m((f \otimes g)(\Delta c))$. The identity for the convolution operation is given by $\eta\epsilon : \mathsf{H} \to \mathsf{H}$. All equations above are naturally presented as commutative diagrams, cf. [35, 21]. We say a Hopf algebra is *(co)commutative* if the (co)multiplication is (co)commutative (i.e. the appropriate diagram or its dual commutes [35, 21].) □

The canonical example of a Hopf algebra is the group algebra $\mathbf{k}[\mathrm{G}]$, the vector space generated by the elements of a group G. The algebra structure is induced by the group multiplication, the coalgebra is the diagonal $\Delta(g) = g \otimes g$, the counit $\varepsilon(g) = 1$ and the antipode is induced by $S(g) = g^{-1}$. Thus, Hopf algebras can be thought of as a generalization of the notion of group. We shall study another algebra, the *shuffle Hopf algebra*, in section 6 below.

† We will assume throughout this paper that $\mathbf{k}$ is a discrete field of characteristic 0

5.2 H-Modules

The action of a group G on a vector space V is a group homomorphism $\rho : G \to Aut(V)$, where $Aut(V)$ is the group of linear automorphisms of V. This forms a category, with morphisms being the equivariant maps, i.e. linear maps $f : V \to W$ commuting with the action. Still more generally, we may speak of the action of a Hopf algebra H on a vector space V. This is a linear map $\rho : \mathsf{H} \otimes V \to V$ satisfying the analog of the action equations above:

Definition 5.2 Given a Hopf algebra H, a *module* over H is a vector space V, equipped with a $\mathbf{k}$-linear map called an H-*action* $\rho\colon \mathsf{H} \otimes V \to V$ such that the following diagrams commute:

$$\begin{tikzcd}
\mathsf{H} \otimes \mathsf{H} \otimes V \arrow[r, "id \otimes \rho"] \arrow[d, "m \otimes id"] & \mathsf{H} \otimes V \arrow[d, "\rho"] \\
\mathsf{H} \otimes V \arrow[r, "\rho"] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
V & \mathsf{H} \otimes V \arrow[l, "\rho"'] \\
 & \mathbf{k} \otimes V \arrow[u, "\eta \otimes id"'] \arrow[ul, "\cong"]
\end{tikzcd}$$

We will generally denote an H-action by concatenation, e.g. $\rho(h \otimes v) = hv$. Then the above diagrams translate, respectively, to: $(h{\bullet}h')v = h(h'v)$ and $\eta(1)v = v$, for all $h, h' \in \mathsf{H}, v \in V$. We shall frequently denote $\eta(1)$ by 1_{H}.

If (V, ρ) and (W, τ) are modules, then *a map of modules,* sometimes called an H-*map,* is a $\mathbf{k}$-linear map $f\colon V \to W$ such that the following commutes:

$$\begin{tikzcd}
\mathsf{H} \otimes V \arrow[r, "id \otimes f"] \arrow[d, "\rho"'] & \mathsf{H} \otimes W \arrow[d, "\tau"'] \\
V \arrow[r, "f"] & W
\end{tikzcd}$$

i.e. in the above notation, $f(hv) = hf(v)$ for all $h \in \mathsf{H}, v \in V$. We thus obtain a category $\mathcal{MOD}(\mathsf{H})$.

The above definition is a straightforward generalization from group representations; indeed, the latter arises as the special case $\mathsf{H} = \mathbf{k}[G]$. A similar remark applies to the Hopf algebra associated to a Lie algebra [1].

If U and V are modules, then $U \otimes V$ has a natural module structure given by:

$$\mathsf{H} \otimes U \otimes V \xrightarrow{\Delta \otimes id} \mathsf{H} \otimes \mathsf{H} \otimes U \otimes V \xrightarrow{c_{23}} \mathsf{H} \otimes U \otimes \mathsf{H} \otimes V \xrightarrow{\rho \otimes \rho} U \otimes V$$

Denote this module as $U \otimes_{\mathsf{H}} V$. We will frequently drop the subscript if there is no chance of confusion.

Theorem 5.3 *$\mathcal{MOD}(\mathsf{H})$ is a monoidal category. If the Hopf algebra is cocommutative, then the tensor product is symmetric. The unit for the tensor is given by the ground field with the module structure induced by the counit of H.*

Definition 5.4 Given an arbitrary Hopf algebra H with bijective antipode, and two H-modules, A and B, we will define two new H-modules, $A \multimap B$ and $B \multimapinv A$, as follows. In both cases, the underlying space will be $A \multimap_{\mathbf{k}} B$, the space of $\mathbf{k}$-linear maps.

The action on $B \multimapinv A$ is defined by:

$$(hf)(a) = \sum h_1 f(S(h_2)a) \qquad (5)$$

and the action on $A \multimap B$ is defined by:

$$(hf)(a) = \sum h_2 f(S^{-1}(h_1)a) \tag{6}$$

where $\Delta(h) = \sum h_1 \otimes h_2$.

The following is proved by Majid in [28].

Theorem 5.5 *Let* H *be a Hopf algebra with bijective antipode. Then with the actions defined above,* $\mathcal{MOD}(\mathsf{H})$ *is a biautonomous category. The adjoint relation:*

$$HOM(A \otimes B, C) \cong HOM(B, A \multimap C)$$

holds whether or not the antipode is bijective. In the case of a cocommutative Hopf algebra, the two internal HOM*'s are equal.*

Now we will consider a topological variant of this result.

Definition 5.6 The category $\mathcal{TMOD}(\mathsf{H})$ is defined as follows. Objects are modules (V, ρ) such that V is equipped with a linear topology, and such that the action of H on V is continuous, for each element $h \in \mathsf{H}$. Maps are H-maps which are also continuous. Define $\mathcal{RTMOD}(\mathsf{H})$ to be the full subcategory of reflexive objects.

The following results are presented in [12]. They are a straightforward generalization of the results of [28].

Theorem 5.7 *Let* H *be a Hopf algebra with bijective antipode. Then* $\mathcal{RTMOD}(\mathsf{H})$ *is a bi-$*$-autonomous category. Furthermore, if* H *has an involutive antipode, i.e.* $S^2 = id$*, then* $\mathcal{RTMOD}(\mathsf{H})$ *is a cyclic $*$-autonomous category.*

6 The Shuffle Algebra

The particular Hopf algebra which will provide our semantics is known as the *shuffle algebra*. It is an example of an *incidence algebra* [33] and is of fundamental importance in several areas, see [10, 22]. The terminology below is motivated by thinking of shuffling a deck of cards.

Let X be a set and X^* the free monoid generated by X. We denote words (= strings) in X^* by $w, w', \cdots$ and occasionally $z, z' \ldots$ Elements $x, y, \cdots \in X$ are identified with words of length 1, the empty word (= unit of the monoid) is denoted by ϵ, and the monoid multiplication is given by concatenation of strings. We denote the length of word w by $|w|$. Let $\mathbf{k}[X^*]$ be the free $\mathbf{k}$-vector space generated by X. We consider $\mathbf{k}[X^*]$ endowed with the following Hopf algebra structure [21, 10]:

(i) $\mathcal{A} = \mathbf{k}[X^*]$ is an *algebra*, i.e. comes equipped with an associative $\mathbf{k}$-linear multiplication (with unit) $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$:

$$w \otimes w' \quad \mapsto \quad w \bullet w' = \sum_{u \in Sh(w,w')} u \tag{7}$$

where $Sh(w, w')$ denotes the set of "shuffled" words of length $|w| + |w'|$ obtained from w and w'. Here, a *shuffle* of $w = a_1 \cdots a_m$ and $w' = a'_1 \cdots a'_n$ is a word of length $m + n$, say $w'' = c_1 \cdots c_{m+n}$ such that each of the a_i and a'_j occurs once in w'' ; moreover, within w'', a_i and a'_j occur in their original sequential order. For example, if $w = aba$ and $w' = bc$, we obtain the following set of shuffled words (where the letters from w' are underlined)

$$aba\underline{bc}, ab\underline{b}a\underline{c}, a\underline{b}ba\underline{c}, \underline{b}aba\underline{c}, ab\underline{bc}a, a\underline{b}b\underline{c}a, \underline{b}ab\underline{c}a, a\underline{bc}ba, \underline{b}a\underline{c}ba, \underline{bc}aba$$

Thus the summation $w \bullet w'$ is equal to

$$ababc + 2abbac + babac + 2abbca + babca + abcba + bacba + bcaba$$

Note that we will always denote the shuffle multiplication with $\bullet$, as opposed to the monoid multiplication, for which we use concatenation.

The unit $\eta : \mathbf{k} \to \mathcal{A}$ arises by mapping $1 \mapsto \epsilon$.

(ii) $\mathcal{A} = \mathbf{k}[X^*]$ is a *coalgebra*, i.e. comes equipped with a coassociative comultiplication (with counit) $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, defined as:

$$\Delta(w) = \sum_{w_1 w_2 = w} w_1 \otimes w_2 \tag{8}$$

Note that in the equation $w_1 w_2 = w$ we are using the original monoid multiplication of X^*. The above pair $w_1 w_2$ is called a *cut* of w.

The counit $\varepsilon : \mathcal{A} \to \mathbf{k}$ is defined by:

$$\varepsilon(w) = \begin{cases} 1 & \text{if } w = \epsilon \\ 0 & \text{else} \end{cases} \tag{9}$$

Finally, there is an *antipode* defined as

$$S(w) = (-1)^{|w|}\overline{w} \tag{10}$$

where $\overline{w}$ denotes the word w written backwards.

Proposition 6.1 *$\mathcal{A} = \mathbf{k}[X^*]$ with the above structure forms a Hopf algebra with involutive antipode. Thus $\mathcal{RTMOD}(\mathcal{A})$ is a cyclic $*$-autonomous category.*

6.1 $\mathcal{A}$-Modules

We now give several examples of $\mathcal{A}$-modules, where $\mathcal{A}$ is the shuffle algebra.

- The simplest example of an $\mathcal{A}$-action, which we will frequently have occasion to use, is the zero action on an arbitrary $V \in \mathcal{RTVEC}$ defined by:

$$wv = \begin{cases} v & \text{if } w = \epsilon \\ 0 & \text{else} \end{cases}$$

- Let V be an object of $\mathcal{RTVEC}$ and $v \in V$. Given any element $w \in X^*$, define

$$wv = \frac{v}{|w|!}$$

  It follows from the fact that the number of shuffles of w with w' is given by $\frac{(|w|+|w'|)!}{|w|!|w'|!}$, that this is indeed an action.

- For a more general example, given a vector space V, equipped with a $\mathbf{Z}$-action, we can define an $\mathcal{A}$-action by:

$$wv = \frac{|w|v}{|w|!}$$

- Finally, choose any element $x \in X$. We can define an $\mathcal{A}$ action by:

$$wv = \begin{cases} \frac{v}{n!} & \text{if } w = x^n \\ 0 & \text{else} \end{cases}$$

In the sequel, we will use various combinations of these actions.

7 The $*$-Autonomous Structure of $\mathcal{RTMOD}(\mathcal{A})$

Let us now look in more detail at the structure of $\mathcal{RTMOD}(\mathcal{A})$. We will give explicit formulas for the negation, tensor and par in this category, and then describe the action of $\mathcal{A}$ on dinaturals.

7.1 Negation and Tensor

It is straightforward to describe the action of a word w on an element of $V^\perp$.

Proposition 7.1 *Given a* $\mathbf{k}$*-linear function* $f\colon V \to \mathbf{k}$ *and an element* $v \in V$*,*

$$(wf)(v) = f((-1)^{|w|}\overline{w}v) \quad . \tag{11}$$

Proof. Using equations (5) and (10), we have

$$\begin{aligned}(wf)(v) &= \sum_{w_1 w_2 = w} w_1 f(S(w_2)v) \\ &= \sum_{w_1 w_2 = w} w_1 f((-1)^{|w_2|}\overline{w_2}v)\end{aligned}$$

But $f((-1)^{|w_2|}\overline{w_2}v)$ is an element of the field $\mathbf{k}$, whose structure is induced by the counit, cf. (9). It follows that the only nontrivial term in the last sum is when $w_1 = \epsilon$ and $w_2 = w$. $\square$

If $U, V \in \mathcal{RTMOD}(\mathcal{A})$ then the action on $U \otimes V$ is given by:

$$w(v \otimes v') = \sum (w_1 v) \otimes (w_2 v') \tag{12}$$

where $\Delta(w) = \sum w_1 \otimes w_2$, the sum over all possible cuts of w. Similarly, to calculate the action of a word on an iterated tensor product, say $V_1 \otimes V_2 \otimes \ldots \otimes V_n$, calculate $\Delta^{n-1}(w)$, and calculate componentwise.

7.2 The Par Structure

Recall that in a noncommutative logic, the usual linear DeMorgan rule takes on the following form: $A \parr B \cong (B^\perp \otimes A^\perp)^\perp$. We now describe the action of a word on an element of $A \parr B$.

Lemma 7.2 *Suppose that* $f : B^\perp \otimes A^\perp \to \mathbf{k}$*, that* $w \in X^*$*,* $u \in B^\perp$ *and* $v \in A^\perp$*. Then the action of* w *on* f *may be described as follows:*

$$(wf)(u \otimes v) = \sum_{w_1 w_2 = w} f(u(w_2-) \otimes v(w_1-))$$

where for any word w'*,* $u(w'-)$ *denotes the function* $a \mapsto u(w'a)$ *and similarly for* $v(w'-)$*.*

Proof. The proof proceeds by the following calculation.

$$\begin{aligned}(wf)(u \otimes v) &= f((-1)^{|w|}\overline{w}(u \otimes v)) \\ &= (-1)^{|w|} f(\overline{w}(u \otimes v)) \\ &= (-1)^{|w|} f(\sum_{z_1 z_2 = \overline{w}} z_1 u \otimes z_2 v) \\ &= (-1)^{|w|} f(\sum_{z_1 z_2 = \overline{w}} u((-1)^{|z_1|}\overline{z_1}-) \otimes v((-1)^{|z_2|}\overline{z_2}-)) \\ &= (-1)^{|w|} f(\sum_{z_1 z_2 = \overline{w}} (-1)^{|z_1|+|z_2|} u(\overline{z_1}-) \otimes v(\overline{z_2}-))\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{|w|} f(\sum_{z_1 z_2 = \overline{w}} (-1)^{|w|}\, u(\overline{z_1}-) \otimes v(\overline{z_2}-)) \\
&= (-1)^{2|w|} f(\sum_{z_1 z_2 = \overline{w}} u(\overline{z_1}-) \otimes v(\overline{z_2}-)) \\
&= f(\sum_{z_1 z_2 = \overline{w}} u(\overline{z_1}-) \otimes v(\overline{z_2}-)) \\
&= \sum_{z_1 z_2 = \overline{w}} f(u(\overline{z_1}-) \otimes v(\overline{z_2}-)) \\
&= \sum_{w_1 w_2 = w} f(u(w_2-) \otimes v(w_1-)) \quad ,
\end{aligned}
$$

observing that $z_1 z_2 = \overline{w}$ if and only if $\overline{z_2}\ \overline{z_1} = w$.

□

The key thing to note is the similarity to the calculation for tensor product. To calculate the action of a word w on $A ⅋ B$, we calculate $\Delta(w) = \sum w_1 \otimes w_2$, and allow the w_1 terms to act in the A position, and the w_2 terms to act in the B position. A similar remark applies to iterated ⅋'s.

7.3 Sample Calculation

To illustrate calculations in this category, we present a typical example.

Let $V \in \mathcal{RTVEC}$. There is a canonical element:

$$\iota \in (V \otimes V^{\perp})^{\perp}$$

defined by $\iota(v \otimes u) = u(v)$.

We will show that for $x, y \in X$, $(xy\iota)(v \otimes u) = 0$, for all $v \in V, u \in V^{\perp}$. In fact, the result follows for general reasons, but it is illustrative of how the coalgebra structure interacts with the shuffle multiplication. Recall that the shuffle multiplication is denoted by $\bullet$.

$$
\begin{aligned}
(xy\iota)(v \otimes u) &= \iota(yx(v \otimes u)) \\
&= \iota(yxv \otimes u + yv \otimes xu + v \otimes yxu) \\
&= \iota(yxv \otimes u(-) - [yv \otimes u(x-)] + v \otimes u(xy-)) \\
&= u(yxv) - u((x \bullet y)v) + u(xyv) \\
&= u(yxv) - u(xyv + yxv) + u(xyv) \\
&= u(yxv) - (u(xyv) + u(yxv)) + u(xyv) \\
&= 0
\end{aligned}
$$

7.4 The Action of a Word on a Dinatural

Suppose that we have a balanced, binary sequent with n atoms and thus $2n$ literals, say $\vdash \Gamma$. Following the ideas of *functorial polymorphism* [4, 13], we will interpret Γ as a multivariant functor denoted $[\![\Gamma]\!]$. When an object of $\mathcal{RTVEC}$ is chosen for each atom in Γ, then we have an *instantiation* of $[\![\Gamma]\!]$. Suppose also that we have a dinatural transformation $\theta : \mathbf{k} \to [\![\Gamma]\!]$ interpreting the sequent $\vdash \Gamma$. This can be viewed as a parametrized family of elements of the instantiations of $[\![\Gamma]\!]$. If each atom is interpreted as a space equipped with an $\mathcal{A}$-action, then this induces an action of $\mathcal{A}$ on $[\![\Gamma]\!]$ at this instantiation. To calculate the action of a word $w \in X^*$ on θ, we begin by calculating $\Delta^{2n-1}(w)$. For example, suppose θ interprets the sequent $\vdash \alpha^{\perp} \otimes \beta, \beta^{\perp} ⅋ \alpha$. To determine the action of the word $w = xy$ on θ, we first calculate $\Delta^3 : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$, obtained by iterating Δ.

In the following display, we have written out the element $\Delta^3(xy) \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ in ten rows. The element in question is the sum of the ten rows. We have also divided the element into four columns and labelled each column with the literal on which it will act.

$$\Delta^3(xy) \quad = \quad \begin{array}{ccccccccc}
\alpha^\perp & \otimes & \beta & , & \beta^\perp & ⅋ & \alpha & \\
\hline
xy & \otimes & \epsilon & \otimes & \epsilon & \otimes & \epsilon & + \\
x & \otimes & y & \otimes & \epsilon & \otimes & \epsilon & + \\
x & \otimes & \epsilon & \otimes & y & \otimes & \epsilon & + \\
x & \otimes & \epsilon & \otimes & \epsilon & \otimes & y & + \\
\epsilon & \otimes & xy & \otimes & \epsilon & \otimes & \epsilon & + \\
\epsilon & \otimes & x & \otimes & y & \otimes & \epsilon & + \\
\epsilon & \otimes & x & \otimes & \epsilon & \otimes & y & + \\
\epsilon & \otimes & \epsilon & \otimes & xy & \otimes & \epsilon & + \\
\epsilon & \otimes & \epsilon & \otimes & x & \otimes & y & + \\
\epsilon & \otimes & \epsilon & \otimes & \epsilon & \otimes & xy &
\end{array}$$

In general, we refer to the summands in $\Delta^{2n-1}(w)$ as *tensor expressions*, and the words which comprise a tensor expression as *tensor terms.* Now to calculate the action of a word on θ, we calculate the action of each associated tensor expression, and add. To calculate the action of a tensor expression, we determine whether a given term is in a slot corresponding to a covariant or contravariant literal. In the latter case, we apply the antipode to that term. Then the summand acts "pointwise", each term acting in the slot to which it is assigned.

Example 7.3 Let $\theta : \alpha^\perp \otimes \beta, \beta^\perp ⅋ \alpha$ be the dinatural transformation given by $\theta(f \otimes u, g) = rg(f \otimes u)$, where the variables are $v : \alpha, f : \alpha^\perp, u : \beta, g : (\alpha^\perp \otimes \beta)^\perp$, and r is any scalar. (Note that θ may equivalently be thought of as $\theta : \alpha^\perp \otimes \beta, (\alpha^\perp \otimes \beta)^\perp$.)

Consider the following actions, with $u : \beta$ and $v : \alpha$.

$$\alpha - action \qquad wv = \begin{cases} \frac{v}{n!} & \text{if } w = x^n \\ 0 & \text{otherwise} \end{cases}$$

$$\beta - action \qquad wu = \begin{cases} \frac{u}{n!} & \text{if } w = y^n \\ 0 & \text{otherwise} \end{cases}$$

We shall calculate the action of the word xy on θ, by calculating the action of each row of $\Delta^3(xy)$ above and then summing the total. In row 1, there is an xy (which is a non-power of x) in the $\alpha^\perp$ slot, so θ is killed, i.e. $\theta \mapsto 0$. In row 2, x is in the $\alpha^\perp$ slot, which multiplies by -1, y is in the β slot, which multiplies by 1; hence the action of row 2 maps $\theta \mapsto -\theta$. In row 3, x, y appear in negative slots, contributing (-1) each, so the action is $\theta \mapsto \theta$. In each of the rows 4-10, there is either a y in an α slot, or an x in a β slot. In all those cases, θ is killed, i.e. $\theta \mapsto 0$. The total action is $\theta \mapsto -\theta + \theta$, i.e. $\theta \mapsto 0$.

8 Full Completeness for $CyLL$

The notion of G-uniformity can be extended in an evident way to H-*uniformity*, where H is an arbitrary Hopf algebra. As in the commutative case, we call $\mathcal{A}{-}Dinat(F, F')$ the *space of cyclic proofs* associated to the sequent $F \vdash F'$, where $\mathcal{A}$ is the shuffle Hopf algebra.

Given a θ interpreting a one-sided sequent $\vdash \Gamma$, we must determine when it is $\mathcal{A}$-uniform. Since θ is of the form $\theta : \mathbf{k} \rightarrow |\Gamma|$ and the module structure of $\mathbf{k}$ is determined by the counit of $\mathcal{A}$, we conclude:

Lemma 8.1 *θ is $\mathcal{A}$-uniform if and only if*

$$w\theta = \begin{cases} \theta & \text{if } w = \epsilon \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

We immediately demonstrate that it is sufficient to consider balanced sequents.

Lemma 8.2 *If a sequent $\vdash \Gamma$ has a nonzero $\mathcal{A}$-uniform dinatural, then it is balanced.*

Strictly speaking, this proposition is a consequence of the observation (from the last section) that all dinaturals are **Z**-uniform, and the fact that the proposition holds for such dinaturals, see Lemma 2.10. However, we add a direct proof since it illustrates calculations with this Hopf algebra.

Proof. Suppose that Γ is such a sequent, and θ is a nonzero dinatural. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the atoms appearing in Γ. For each α_i, pick a distinct x_i in X.

Define an $\mathcal{A}$-action on α_i by:

$$wv = \begin{cases} \frac{v}{n!} & \text{if } w = x_i^n \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

To calculate the action of x_i on θ, we must repeatedly apply Δ to x_i. The result will be a sum of tensor expressions of the form:

$$\epsilon \otimes \epsilon \otimes \ldots \otimes x_i \otimes \ldots \otimes \epsilon$$

Such a term acting on θ will be 0, except in the cases where the position of x_i corresponds to the position of α_i. In this case, the result will either be θ or $-\theta$ depending on whether the α_i is in a covariant or contravariant position. Thus if $x_i\theta = 0$, it must be the case that the covariant and contravariant occurrences are in bijection. Thus the sequent is balanced. $\square$

Now that we have established that a sequent with a nontrivial $\mathcal{A}$-uniform semantics must be balanced, our full completeness result will follow from two facts. They are as follows: (all sequents considered are binary.)

1. If $\vdash \Gamma$ is a sequent with a noncyclic list of literals, then there are no $\mathcal{A}$-uniform dinaturals.
2. Every $\mathcal{A}$-uniform dinatural is **Z**-uniform

The first of these will be proved in the next subsection. The second is now an immediate consequence of 2.7. We leave it to the reader as a difficult combinatorial exercise to prove the second fact directly.

Our full completeness result would then be an immediate consequence. It follows from our previous work [13], summarized in section 2, that every $\mathcal{A}$-uniform dinatural is the denotation of an $MLL{+}MIX$ proof (up to scalar multiplication.) The first fact then says that a **Z**-uniform dinatural is $\mathcal{A}$-uniform if and only if its associated proof is a cyclic proof. Clearly every proof in *CyLL+MIX* is a proof in *MLL+MIX*. Furthermore, the faithfulness argument of Lemma 2.13 evidently extends to this setting. Therefore we may conclude our full completeness theorem. We now present these ideas in more detail.

8.1 Noncyclic Sequents

Lemma 8.3 *Suppose that* $\vdash \Gamma$ *is a binary sequent which is not cyclic. Then* $dim\,(\mathcal{A} - Dinat(\Gamma)\,) = 0$.

Proof. If Γ is not cyclic, then there exists atoms α and β such that either β or $\beta^\perp$ occurs between α and $\alpha^\perp$ but its dual does not. Suppose that α is instantiated at A and β at B. Choose $x, y \in X$ and define actions on A and B as follows:

$A - action$: $$wv = \begin{cases} \frac{v}{n!} & \text{if } w = x^n \\ 0 & \text{otherwise} \end{cases}$$

$B - action$: $$wv' = \begin{cases} \frac{v'}{n!} & \text{if } w = y^n \\ 0 & \text{otherwise} \end{cases}$$

For all other atoms in the atom list, we define the action by $wv = 0$ for all nonempty words.

Now consider $\theta \in \mathcal{A} - Dinat(\Gamma)$, instantiated as above. We consider the action of the word xyx on θ. As usual, we calculate all tensor expressions for the word xyx. It is clear that the only one which does not annihilate θ is:

$$\epsilon \otimes \ldots \otimes x \otimes \ldots \otimes y \otimes \ldots \otimes x \otimes \ldots \otimes \epsilon$$

where the letters appear in the slots corresponding to A, B and $A^\perp$ respectively. (We are assuming for convenience that the β occurring between α and $\alpha^\perp$ is covariant, though this makes no difference.) Evidently, the action of this tensor expression is given by $\theta \mapsto \theta$. Thus the total action of the word xyx on θ is given by $\theta \mapsto \theta$. Since θ is $\mathcal{A}$-uniform, we conclude that $\theta = 0$. □

9 Main Results

By the previous discussion, we may now state:

Theorem 9.1 (Completeness for $\mathcal{A}$-Dinaturals) *Let $M \vdash N$ be a balanced binary sequent. If the unique cut-free proof structure associated to $M \vdash N$ is not a proof net for the theory $CyLL + MIX$, then $\mathcal{A} - Dinat(M, N)$ is a zero dimensional vector space.*

Theorem 9.2 (Full Completeness for Cyclic Binary Sequents) *If a sequent $M \vdash N$ is binary, then $\mathcal{A} - Dinat(M, N)$ is zero or 1-dimensional, depending on whether its uniquely determined proof structure is a cyclic net. In the latter case, every $\mathcal{A}$-dinatural is a scalar multiple of the denotation of the unique cut-free proof net.*

Theorem 9.3 (Cyclic Full Completeness) *Let F and F' be formulas in multiplicative linear logic, interpreted as definable multivariant functors on $\mathcal{RTVEC}$. Then the vector space of diadditive $\mathcal{A}$-uniform dinatural transformations has as basis the denotations of cut-free proofs in the theory CyLL+MIX.*

As usual, we are able to obtain the following corollary.

Corollary 9.4 *$\mathcal{A}$-uniform diadditive dinatural transformations compose. Thus we obtain an (indexed) cyclic $*$-autonomous category by taking as objects formulas, interpreted as multivariant functors. Morphisms will be uniform diadditive dinatural transformations.*

In [13] Theorem 11.7, we show that a diadditive dinatural transformation between multivariant functors is uniform with respect to the actions of arbitrary cocommutative Hopf algebras. We now present a similar result for the cyclic setting.

Theorem 9.5 *A diadditive dinatural transformation which is uniform with respect to the shuffle Hopf algebra is uniform with respect to arbitrary involutive Hopf algebras, i.e. those for which $S^2 = id$.*

Proof. Our full completeness theorem implies that the only dinaturals which are uniform with respect to the shuffle Hopf algebra are the structure maps of a cyclic $*$-autonomous category. Any such maps will be uniform with respect to all involutive Hopf algebras. □

10 Future Directions

The next avenue we hope to explore is extending our approach to include the additive connectives. The categories we have considered thus far are inadequate for the consideration of **MALL** in that product and coproduct are isomorphic, i.e. $\mathcal{RTVEC}$ has all finite biproducts. This problem is avoided by considering *normed vector spaces* [36], p. 96. We define a category $\mathcal{BAN}_1$ whose objects are Banach spaces, i.e. complete normed vector spaces, and whose morphisms are linear maps of norm less than or equal to 1. This is a symmetric monoidal closed category, when the tensor product is taken to be the completed projective tensor [36, 6]. One can then apply the Chu construction to $\mathcal{BAN}_1$ [7]. In so doing, we obtain a $*$-autonomous category of topological vector spaces in which products and coproducts no longer coincide. Explicitly, if $V, W \in \mathcal{BAN}_1$, then we have the following formulas:

$$\begin{array}{rrcl} \textit{Products-} & ||(v,w)|| & = & max\{||v||, ||w||\} \\ \textit{Coproducts-} & ||(v,w)|| & = & ||v|| + ||w|| \end{array}$$

These correspond to the ℓ_∞ and ℓ_1 norms respectively. Given our previous work, this seems a promising candidate for a full completeness theorem for **MALL**.

Girard, in a recent series of talks and preprint [18], has proposed the notion of a *coherent Banach space* in which the additive structure is modeled as above, and the exponentials are modeled *via* the notion of analytic functions on a Banach space. He also proposes a new version of linear sequent calculus in which the proof rules are labeled by scalars. He then shows that his semantics is sound for this theory.

11 Acknowledgement

The authors would like to thank Arnaud Fleury and Christian Retoré for their helpful comments on Yetter's Cyclic Linear Logic. We thank Gordon Plotkin for his astute comments, which led to the new results of section 2. We also thank Esfan Haghverdi for help in LaTeX-ing figure 1.

Special thanks are due to Mitsu Okada and the other organizers of Linear Logic '96 for giving us the opportunity to present these results.

References

[1] K. Abé, *Hopf Algebras*, Cambridge University Press, (1977).

[2] S. Abramsky, R. Jagadeesan, Games and Full Completeness for Multiplicative Linear Logic, *J. Symbolic Logic*, Vol. 59, No.2 (1994), pp. 543-574.

[3] V.M. Abrusci, Phase Semantics and Sequent Calculus for Pure Noncommutative Classical Linear Propositional Logic, *J. Symbolic Logic* Vol. 56 (1991), pp. 1403-1456.

[4] E. Bainbridge, P. Freyd, A. Scedrov, P. Scott, Functorial Polymorphism, *Theoretical Computer Science* 70, (1990), pp. 1403-1456.

[5] M. Barr, Duality of Vector Spaces, *Cahiers de Top. et Géom. Diff.* 17, (1976), pp. 3-14.

[6] M. Barr, Duality of Banach Spaces, *Cahiers de Top. et Géom. Diff.* 17, (1976), pp. 15-32.

[7] M. Barr, $*$-*Autonomous Categories*, Springer Lecture Notes in Mathematics 752, (1980).

[8] M. Barr, Appendix to [12] (1994).

[9] M. Barr, Noncommutative $*$-autonomous Categories, preprint, (1993).

[10] D.B. Benson, Bialgebras: Some Foundations for Distributed and Concurrent Computation, *Fundamenta Informaticae* 12, p. 427-486, (1989).

[11] R. Blute, Linear Logic, Coherence and Dinaturality, *Theoretical Computer Science* 115, (1993), pp. 3-41.

[12] R. Blute, Hopf Algebras and Linear Logic, *Mathematical Structures in Computer Science 6*, (1996), pp. 189-217.

[13] R.F. Blute, P.J. Scott, Linear Läuchli Semantics, *Annals of Pure and Applied Logic* 77, (1996), pp.101-142.

[14] R.F. Blute, P.J. Scott, A Noncommutative Full Completeness Theorem (Extended Abstract), *Electronic Notes in Theoretical Computer Science 3* (1996), Elsevier Science B.V.

[15] V. Danos, L. Regnier, The Structure of Multiplicatives, *Arch. Math. Logic* 28 (1989) pp. 181-203

[16] A. Fleury, C. Rétoré, The *MIX* Rule, *Mathematical Structures in Computer Science* 4, p. 273-285 (1994).

[17] J.Y. Girard, Linear Logic, *Theoretical Computer Science* 50, p. 1-102 (1987).

[18] J.Y. Girard, Coherent Banach Spaces, preprint, (1996) and lectures delived at Keio University, Linear Logic '96, April 1996

[19] J. Y. Girard, A. Scedrov, P. Scott, Normal Forms and Cut-free Proofs as Natural Transformations, in : *Logic From Computer Science*, Mathematical Science Research Institute Publications 21, (1991), pp. 217-241. (Also available by anonymous ftp from: theory.doc.ic.ac.uk, in: papers/Scott).

[20] V. Harnik, M. Makkai, Lambek's Categorical Proof Theory and Läuchli's Abstract Realizability, *Journal of Symbolic Logic* 57 (1992), pp. 200-230.

[21] M. Hazewinkel, Introductory Recommendations for the Study of Hopf Algebras in Mathematics and Physics, CWI Quarterly, *Centre for Mathematics and Computer Science, Amsterdam* Vol. 4, No. 1, March 1991.

[22] S. Joni, G.C. Rota, Coalgebras and Bialgebras in Combinatorics, *Studies in Applied Mathematics* 61, p. 93-139, (1979).

[23] A. Joyal, Une Théorie Combinatoire des Séries formelles, *Advances in Mathematics* 42, pp. 1-82 (1981)

[24] J. Lambek, Bilinear Logic in Algebra and Linguistics, (1993).

[25] H. Läuchli, An Abstract Notion of Realizability for which Intuitionistic Predicate Calculus is Complete, *Intuitionism and Proof Theory*, North-Holland (1970), pp. 227-234.

[26] S. Lefschetz, Algebraic Topology, American Mathematical Society Colloquium Publications 27, (1963).

[27] S. Majid, Physics for Algebraists: Noncommutative and Noncocommutative Hopf Algebras by a Bicrossproduct Construction, *Journal of Algebra* 130, p. 17-64 (1990).

[28] S. Majid, Quasitriangular Hopf Algebras and Yang-Baxter Equations, *International Journal of Modern Physics* 5, p. 1-91, (1990).

[29] G. Plotkin, M. Abadi, A Logic for Parametric Polymorphism, *Typed Lambda Calculus and Applications*, Springer Lecture Notes in Computer Science 664, pp. 361-375, (1993).

[30] G. Plotkin, *private communication*, August 1996.

[31] C. Retoré, Des Réseaux De Démonstration Pour La Linguistique, manuscript (1996).

[32] K. I. Rosenthal, $*$-Autonomous Categories of Bimodules, *Journal of Pure and Applied Algebra* 97, p. 189-202, (1994).

[33] W. Schmitt, Antipodes and Incidence Coalgebras, *Journal of Combinatorial Theory* 46, p. 264-290, (1987).

[34] W. Schmitt, Hopf Algebras of Combinatorial Structures, *Canadian Journal of Mathematics* 45, pp. 412-428, (1993)

[35] M. Sweedler, *Hopf Algebras*, Benjamin Press, (1969).

[36] F. Treves, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, (1967)

[37] K. Ulbrich, On Hopf Algebras and Rigid Monoidal Categories, *Israel Journal of Mathematics* 71, p. 252-256, (1989).

[38] D. Yetter, Quantales and (Noncommutative) Linear Logic, *Journal of Symbolic Logic* 55, (1990), pp. 41-64.