Introduction to MV- and Effect-algebras, I

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Précis

- Mundici (1986) surprisingly connected up MV-algebras (arising from Łukasiewicz many-valued logics, first discovered in the 1920's) with G. Elliott's classification program of AF C*-algebras via countable dimension groups. Mundici and his school also developed MV-algebras into a major area of mathematics.
- In 1990's, the algebraic theory of quantum effects and quantum measurement theory in physics led to *Effect Algebras* developed by Bennett & Foulis, Eastern European school: Jencova, Pulmannova, etc.

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- Recently: major categorical advances by Bart Jacobs (Nijmegen) and his school, leading to *Effectus Theory*.

What do we want to do?

- Encompass both frameworks using Inverse Semigroup Theory.
- Connect with noncommutative Stone-Duality, étale groupoids, pseudogroups, tilings, etc. via *Boolean Inverse monoids* (Lawson, Lenz, Kellendonk, Resende)
- Generalize AF C*-algebra techniques to develop a theory of AF inverse monoids connecting up MV and effect algebras.

- Theorem (Coordinatization Theorem, L-S)

  Let $\mathcal{A}$ be a denumerable MV algebra. Then there exists a boolean coordinatizing AF inverse monoid S s.t. $Ideals(S) = S/\mathcal{J} \cong \mathcal{A}$.

  Here $\mathcal{J}$ is the standard relation: $a\mathcal{J}b$ iff $SaS = SbS$

M. Lawson , P. Scott, AF Inverse Monoids and the structure of Countable MV Algebras, *J. Pure and Applied Algebra* 221 (2017), pp. 45 – 74

Łukasiewicz many-valued logics

Łukasiewicz (1878-1956) introduced many-valued logics in the 1920's. Studied by the Polish school, e.g. Tarski. What are they?

- "Fuzzy" logics $\mathcal{L}$ with (infinitely-many) truth values in [0,1] (& related ones: truth values in $\mathbb{Q} \cap [0,1]$ or $\mathbb{Q}_{\mathbf{Dyad}} \cap [0,1]$).
- Finite-valued Łukasiewicz logics $\mathcal{L}_n$, with truth values in $\{0, \frac{1}{n-1}, \frac{2}{n-1}, \cdots, \frac{n-2}{n-1}, 1\}$.

Łukasiewicz Logics and their algebras: every 30 years

- **1920's**: Polish school: Łukasiewicz , Lesniewski/Tarski , Post.
- **1950's**: R. McNaughton, **C.C. Chang (MV Algebras)**
- **1980's**: D. Mundici, et.al.
  - MV-Algebras: rich algebraic, topological, & geometric theory.
  - Closely related to (AF) C*-algebras (Bratteli, Elliott).
  - Deep connections with analysis and operator algebras.
- **2010–:**
  - Sheaf Representation: Dubuc/Poveda (2010), Gehrke (2014).
  - Topos Theory & MV-algebras (Caramello: 2014–),
  - Łukasiewicz μ-calculus, M. Mio & A. Simpson (2013)
  - Coordinatization (Lawson-Scott, Wehrung, Mundici) (2015-)

What are MV Algebras?

MV algebras are structures $\mathcal{M} = \langle M, \oplus, \neg, 0 \rangle$ satisfying:

- $\langle M, \oplus, 0 \rangle$ is a commutative monoid.
- $\neg$ is an involution: $\neg\neg x = x$, for all $x \in M$.
- $1 := \neg 0$ is absorbing: $x \oplus 1 = 1$, for all $x \in M$.
- $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

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Example: a Boolean algebra $\mathcal{B} = (B, \vee, \overline{(\ )}, 0)$, where we define $x \oplus y := x \vee y$ and $\neg x = \overline{x}$. The last equation says: $x \vee y = y \vee x$

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- $$(x \multimap y) \multimap y = (y \multimap x) \multimap x$$

Further MV Structure I

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Define:

$$\begin{aligned}
x \multimap y &:= \neg x \oplus y \\
x \otimes y &:= \neg(\neg x \oplus \neg y) \\
x \ominus y &:= x \otimes \neg y \\
x \leqslant y &\text{ iff } \text{for some } z,\ x \oplus z = y
\end{aligned}$$

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Facts:

(i) $\leqslant$ is a partial order.

(ii) $\otimes$ is left adjoint to $\multimap$:
$x \otimes y \leqslant z$ iff $x \leqslant (y \multimap z)$

(iii) $\ominus$ is left adjoint to $\oplus$:
$x \ominus z \leqslant y$ iff $x \leqslant y \oplus z$

Łukasiewicz' Axiom, again

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Lattice Structure ("Additives")

An MV algebra forms a distributive lattice with 0, 1, with:

$$x \vee y := (x \otimes \neg y) \oplus y = (x \ominus y) \oplus y$$
$$x \wedge y := \neg(\neg x \vee \neg y)$$

Fundamental Example of an MV Algebra: $[0, 1]$

For $x, y \in [0, 1]$, define:

1. $\neg x = 1 - x$
2. $x \oplus y = min(1, x + y)$
3. $x \otimes y = max(0, x + y - 1)$

Other models: similarly consider the same operations on:

- $\mathbb{Q} \cap [0, 1]$ and $\mathbb{Q}_{\text{dyad}} \cap [0, 1]$.
- Finite MV algebras $\mathcal{M}_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \cdots, \frac{n-2}{n-1}, 1\}$ (subalgebras of [0,1]). Note: $\mathcal{M}_2 = \{0, 1\}$.

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Fact (Barr)

$([0,1], \otimes, \oplus, 1, 0, \neg)$ *also forms a *-autonomous poset.*

Moreover, it has products ($\wedge$) and thus coproducts ($\vee$).

Example 2: Lattice-Ordered Abelian Groups

- Let $\langle G, +, -, 0, \leqslant \rangle$ be a partially ordered abelian group, i.e. an abelian group with translation invariant partial order.
- If G is lattice-ordered, call G an *ℓ-group*, G^+ its positive cone.
- If G is an ℓ-group and $t \in G$, then $t + (\ )$ preserves $\vee$ and $\wedge$.
- If G is an ℓ-group, an *order unit* $u \in G$ is an *Archimedian element*: $\forall g \in G, \exists n \in \mathbb{N}^+$ s.t. $g \leqslant nu$.
- If G is an ℓ-group with order unit u, define

$$[0, u]_G = \{g \in G \mid 0 \leqslant g \leqslant u\} \quad \text{(just a poset)}$$

Example: $\Gamma(G, u) = ([0, u]_G, \oplus, \otimes, *, 0, 1)$ is an MV algebra, via:

$$\begin{aligned} x \oplus y &:= u \wedge (x + y) \\ x^* &:= u - x \\ x \otimes y &:= (x^* \oplus y^*)^* \end{aligned}$$

$$0 := 0_G \quad \text{and} \quad 1 := u$$

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- If G is an ℓ-group with order unit u, define the G-interval

$$[0, u]_G = \{g \in G \mid 0 \leqslant g \leqslant u\} \quad \text{(just a poset)}$$

G-**Chain**: totally ordered G-interval $[0, u]$.

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Special Cases:

- $\Gamma(\mathbb{R}, 1) = [0,1]$,
- $\Gamma(\mathbb{Q}, 1) = \mathbb{Q} \cap [0,1]$,
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- $\Gamma(\mathbb{Z}, 1) = \mathcal{M}_2 = \{0, 1\}$.

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Let $\mathcal{MV}$ = the category of MV-algebras and MV-morphisms.
$\ell\mathcal{G}_u$ = the category of ℓ-groups and order-unit preserving homs.

Theorem (Mundici, 1986)

Γ *induces an equivalence of categories* $\ell\mathcal{G}_u \cong \mathcal{MV}$: $\quad G \mapsto [0,u]_G$

$\therefore$ For each MV algebra A, there exists ℓ-group G with order unit u, unique up to iso, s.t. $A \cong [0,u]_G$ and $|G| \leqslant max(\aleph_0, |A|)$.
Warning!!

Lattice-Ordered Abelian Groups

Examples of MV algebras $\Gamma(G,u) = ([0,u]_G, \oplus, \otimes, *, 0, 1)$:

- $\Gamma(\mathbb{R}, 1) = [0,1]$,
- $\Gamma(\mathbb{Q}, 1) = \mathbb{Q} \cap [0,1]$,
- $\Gamma(\frac{1}{n-1}\mathbb{Z}, 1) = \mathcal{M}_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \cdots, \frac{n-2}{n-1}, 1\}$ (called a *Łukasiewicz chain*).
- $\Gamma(\mathbb{Z}, 1) = \mathcal{M}_2 = \{0,1\}$.

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Theorem (Mundici, 1986)

Γ *induces an equivalence of categories* $\ell\mathcal{G}_u \cong \mathcal{MV}$

In particular, for every MV algebra A, there exists an ℓ-group G with order unit u, unique up to isomorphism, such that $A \cong \Gamma(G,u)$, and $|G| \leqslant max(\aleph_0, |A|)$.

Some Theorems for Infinite Łukasiewicz logic

Theorem (Chang Completeness, 1955-58)

1. *Every MV algebra is a subdirect product of MV Chains.*
2. *An MV equation holds in* $[0, 1]$ *iff it holds in all MV algebras.*

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Corollary (Existence of Free MV-Algebras)

The free MV algebra $\mathcal{F}_\kappa$ *on* κ *free generators is the smallest MV-algebra of functions* $[0,1]^\kappa \to [0,1]$ *containing all projections (as generators) and closed under the pointwise operations.*

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Theorem (McNaughton, 1950: earlier than Chang!)

The free MV algebra $\mathcal{F}_n$ *is exactly the algebra of* McNaughton Functions*: continuous, piecewise (affine-)linear polynomial functions (in n vbls, with integer coefficients):* $[0,1]^n \to [0,1]$.

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Corollary: an MV equation holds in $[0,1]$ *iff it holds in* $[0,1] \cap \mathbb{Q}$

Some Theorems for Infinite Łukasiewicz logic

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Theorem (Mundici, 1987)

Satisfiability in infinite Łukasiewicz logic is NP complete.

Proof uses interesting features of "differential semantics", based on calculus of several variables (gradients, partial derivatives, tangent planes, etc) over McNaughton Functions, together with careful numerical approximations.

Some Algebra of MV algebras

1. An ideal of MV-algebra A is a subset $I \subseteq A$ containing 0, down-closed (wrt $\leqslant$), and closed under $\oplus$.
2. Usual theory of ideals/kernels/congruence/HSP theorems, etc.
3. A is a subdirect product $A \rightarrowtail \prod_{i \in I} A_i$ iff there is a family of ideals $\{J_i \mid i \in I\}$ s.t. (i) $A_i \cong A/J_i$ and (ii) $\bigcap_{i \in I} J_i = \{0\}$.
4. $\mathcal{F}_n$, the free MV-algebra on n generators is given by MV-terms in n-free variables. An algebra A is *finitely presented* iff $A \cong \mathcal{F}_n/I$, for some finitely generated ($\equiv$ principal) ideal I.
5. $Rad(A) =$ the intersection of all maximal ideals of A. A is *semisimple* iff $Rad(A) = \{0\}$. All MV algebras are semisimple, and hence have no *infinitesimals* (in a suitable sense).
6. Tensor Products, colimits, spectral spaces, etc.

Some Geometry of MV-Algebras

Mundici & colleagues (Marra, Cabrer, Spada, et.al.) have shown deep connections to algebraic geometry and topology.

1. If $P \subseteq \mathbb{R}^n$, the convex hull $conv(P) = \{\sum_i r_i v_i \mid v_i \in P, r_i \in \mathbb{R}^+, \sum_i r_i = 1\}$.
2. P is called:
   - 2.1 *convex* iff $P = conv(P)$.
   - 2.2 *a polytope* iff $P = conv(F)$, $F \subseteq \mathbb{R}^n$ finite.
   - 2.3 *a rational polytope* iff it's a polytope and $F \subseteq \mathbb{Q}^n$.
   - 2.4 *a (compact) polyhedron* iff it's a union of finitely many polytopes in $\mathbb{R}^n$.
   - 2.5 *a rational polyhedron* iff it's a union of finitely many rational polytopes. (These are subsets of $\mathbb{R}^n$ definable by MV-terms.)

What about maps between rational polyhedra?

Some Geometry of MV-Algebras

- For $P \subseteq \mathbb{R}^n$, $f : P \to \mathbb{R}$ is a $\mathbb{Z}$-*map* if it's a McNaughton Function over $\mathbb{R}$ (instead of $[0,1]$)). Ditto, if $P, Q \subseteq \mathbb{R}^n$, $P \xrightarrow{f} Q$ is a $\mathbb{Z}$-map if its components are. (These are the continuous transformations of polyhedra definable by tuples of MV terms!)

Theorem (Marra& Spada, APAL, 2012)

The category of finitely presented MV-algebras and homs is dually equivalent to the opposite of the category of rational polyhedra and $\mathbb{Z}$-maps: $MV_{fp} \cong Poly_{\mathbb{Q}}^{op}$

There's a huge and sophisticated literature of these types of results. Interestingly, there is a strong analogy with a remarkable independent series of papers by the algebraic topologist W. M. Beynon (1974-77) on related topological dualities for ℓ-groups.

Typical Beynon Theorem

Theorem (Beynon, 1977)

The full subcategory of the category of finitely generated lattice-ordered Abelian groups consisting of projective lattice-ordered Abelian groups is equivalent to the dual of the category whose objects are rational Euclidean closed polyhedral cones, and whose morphisms are piecewise homogeneous linear maps with integer coefficients.

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1. W. M. Beynon, Combinatorial aspects of piecewise linear maps, J. London Math. Soc. (2) (1974), 719-727.
2. W. M. Beynon: Duality theorems for finitely generated vector lattices, Proc. London Math. Soc. (3) 31 (1975), 114-128.
3. W. M. Beynon, Applications of Duality in the theory of finitely generated lattice-ordered abelian groups, Can.J. Math, 1977

From Marra & Mundici, 2003: MV- vs $\ell-$

MV	ℓ
Chang's Theorem (1959) [22]	**Weinberg's Theorem** (1963) [102]
The variety of MV algebras is generated by $[0,1] \cap \mathbb{Q}$. (Corollary 3.3.)	The variety of ℓ-groups is generated by $\mathbb{Z}$. (Corollary 5.5.)
McNaughton's Theorem (1951) [67]	**Beynon's Theorem, I** (1974) [13]
Every McNaughton function of n variables belongs to $\mathcal{M}_n$. (Theorem 8.1.)	Every ℓ-function of n variables belongs to $\mathcal{A}_n$. (Subsection 4.4, *passim.*)
Free representation (1951-59) [22, 67]	**Free representation** (1963-74) [102, 13]
$\mathcal{M}_n$ is the free MV algebra over n free generators, i.e. projection functions. (Subsection 3.1, *passim.*)	$\mathcal{A}_n$ is the free ℓ-group over n free generators, i.e. projection functions. (Subsection 4.4, *passim.*)
MV Nullstellensatz (1959) [104, 22]	**ℓ-Nullstellensatz** (1975) [14]
TFAE: 1. A is fin. gen. semisimple. 2. $\mathbb{I}(\mathbb{V}(J)) = J$ if $A \cong \mathcal{M}_n/J$. (Theorem 3.2.)	TFAE: 1. G is fin. gen. Archimedean. 2. $\mathbb{I}(\mathbb{V}(\mathfrak{o})) = \mathfrak{o}$ if $G \cong \mathcal{A}_n/\mathfrak{o}$. (Subsection 4.4, *passim.*)
Wójcicki's Theorem (1973) [103]	**Baker's Theorem** (1968) [9]
Every finitely presented MV algebra is semisimple. (Theorem 3.4.)	Every finitely presented ℓ-group is Archimedean. (Subsection 4.4, *passim.*)

Next Lecture

In the next lecture, we will introduce Effect Algebras, Mundici's Theorem II, and our Coordinatization Program (joint work with Mark Lawson).