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NEW PROOFS OF SOME INTUITIONISTIC PRINCIPLES

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0. Introduction

In this note we shall give short proofs of various metarules of intuitionistic higher-order logic, the rules of Existence, Disjunction, Uniformity and Independence of Premisses among others. Our method is a modification of FREYD's [2] proof of the Existence and Disjunction rules, which was further developed by us [5, 6], yet it also resembles our original method. It should be somewhat more congenial to logicians; rather than translating each rule into its algebraic equivalent, e.g., some statement about projectives, we use the internal logic of the Freyd cover of the free topos [5, § 6]. Whereas FREYD's proof uses essentially the KLEENE-FRIEDMAN method (see [9]), as did the original proofs by BOILEAU [1] and us [5], the proof here involves a higher-order version of the "Aczel slash" (see § 2 below) and is more perspicuous.

This paper is a continuation of [7]; at the same time, the use of the internal logic of Freyd covers simplifies our presentation. We had begun the computation of this internal logic in [7, Corollary 4.4]. Theorem 2.2 below completes the picture; we are endebted to MARTIN HYLAND for pointing out the relationship with the Aczel slash.

1. Type theory and the free topos

The language $\mathscr{L}$ of pure intuitionistic type theory has been described in detail elsewhere [5, 6]. We shall briefly outline the formation rules for a version of $\mathscr{L}$ based on equality. We are given a hierarchy of pure types consisting of three primitive types: 1 (a one-point set), N (the set of natural numbers) and Ω (the set of truth-values), as well as two rules for generating new types from old ones: from A and B form $A \times B$ (the Cartesian product) and PA ($= \Omega^A$, the power-set of A). In addition to countably many variables of each type, we have the following terms, each listed under its type:

1	N	Ω	$A \times B$	PA
$*$	0 Sn	$a = a'$ $a \in \alpha$	$\langle a, b\rangle$	$\{x \in A \mid \varphi(x)\}$

where the displayed terms satisfy: n has type N, α has type PA, $\varphi(x)$ has type Ω, a and a' have type A and b has type B.

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It is well-known how all logical symbols may be defined in terms of equality (see e.g. [6]):

$$\begin{aligned}
\top \text{ (true)} &\equiv * = *,\\
p \wedge q &\equiv \langle p, q\rangle = \langle \top, \top\rangle,\\
p \Rightarrow q &\equiv p \wedge q = p,\\
\forall_{x\in A}\varphi(x) &\equiv \{x \in A \mid \varphi(x)\} = \{x \in A \mid \top\},\\
\bot \text{ (false)} &\equiv \forall_{t\in\Omega} t,\\
p \vee q &\equiv \forall_{t\in\Omega}(((p \Rightarrow t) \wedge (q \Rightarrow t)) \Rightarrow t),\\
\exists_{x\in A}\varphi(x) &\equiv \forall_{t\in\Omega}(\forall_{x\in A}(\varphi(x) \Rightarrow t) \Rightarrow t),\\
\neg p &\equiv p \Rightarrow \bot,\\
\exists !_{x\in A}\varphi(x) &\equiv \exists_{x'\in A}\{x \in A \mid \varphi(x)\} = \{x \in A \mid x = x'\}.
\end{aligned}$$

We stipulate a deducibility relation $\Gamma \vdash_X \varphi$ between finite sets of formulas Γ (possibly $\Gamma = \emptyset$) and formulas φ, where the free variables of Γ and φ are contained in X. The axioms and rules of inference governing $\vdash_X$ are essentially those of intuitionistic predicate calculus, augmented by extensionality, comprehension, PEANO's axioms and obvious axioms for $*$ and $\langle a, b\rangle$. For details see [6] or Appendix I.

It is also well-known how to interpret intuitionistic type theory $\mathscr{L}$ in any topos $\mathscr{A}$ (see, e.g. [5], [3]): the pure type A is interpreted as the object A of $\mathscr{A}$ with the same name and a closed term a of type A is interpreted as an arrow $a\colon 1 \to A$. Thus $*$ is interpreted as the only arrow $1 \to 1$, 0, Sn and $\langle a, b\rangle$ as the arrows $1 \to \mathsf{N}$, $1 \to \mathsf{N}$ and $1 \to A \times B$ usually so denoted, $a = a'$ as the arrow $\delta_A\langle a, a'\rangle\colon 1 \to A \times A \to \Omega$, where δ_A is the characteristic arrow of the monomorphism $\langle 1_A, 1_A\rangle\colon A \to A \times A$, and $a \in \alpha$ as the arrow $\varepsilon_A\langle \alpha, a\rangle\colon 1 \to \mathrm{P}A \times A \to \Omega$, where ε_A is the usual evaluation. Finally, $\{x \in A \mid \varphi(x)\}$ is the arrow $1 \to \mathrm{P}A$ corresponding to the unique arrow $f\colon A \to \Omega$ in $\mathscr{A}$ for which $\varphi(x) \cdot\!=\!\cdot fx$ in $\mathscr{A}[x]$, the predogma obtained from $\mathscr{A}$ by adjoining an indeterminate arrow $x\colon 1 \to A$. (A *predogma* is essentially a Cartesian closed category, except that no powers are stipulated other than $\Omega^A = \mathrm{P}A$.) For a sentence p of $\mathscr{L}$ we write $\mathscr{A} \models p$ (*$\mathscr{A}$ satisfies p*) to mean $p \cdot\!=\!\cdot \top$ as arrows $1 \to \Omega$ in $\mathscr{A}$.

In the language $\mathscr{L}$ of *pure* type theory it was tacitly understood that there are no other types and terms than those obtainable from the indicated formation rules. It was also understood that the deducibility relation $\vdash_X$ discussed in Appendix I is the smallest relation satisfying the stated axioms and rules of inference. By contrast, when we remove these tacit restrictions, we obtain many *applied* type theories. Among them there is the *internal logic* $\mathscr{L}(\mathscr{A})$ of a topos $\mathscr{A}$. Its types are the objects of $\mathscr{A}$ and its terms of type A are the arrows $a\colon 1 \to A$ in $\mathscr{A}$. Furthermore, the deducibility relation $\vdash_X$ for $\mathscr{L}(\mathscr{A})$ is defined as follows. Without loss in generality we may assume that X contains just a single variable x of type A. (For we may replace variables $x_1, \ldots, x_m$ of types $A_1, \ldots, A_m$ by a single variable x of type $A = A_1 \times \ldots \times A_m$.) Moreover, we may write p_i as $f_i x$ and q as gx, where f_i and g are arrows $A \to \Omega$. Then $p_1, \ldots, p_m \vdash_X q$ means that, for all objects C and all arrows $h\colon C \to A$ in $\mathscr{A}$, if $f_i h \cdot\!=\!\cdot \top$ for $i = 1, \ldots, n$ then also $gh \cdot\!=\!\cdot \top$ in $\mathscr{A}$. In particular, when $n = 0$ and $m = 0$ (hence $A = 1$), we interpret $\vdash q$ as $q \cdot\!=\!\cdot \top$ in $\mathscr{A}$, for which we also write $\mathscr{A} \models q$. Clearly $\mathscr{L}(\mathscr{A})$ is an extension of $\mathscr{L}$.

The closed term model of $\mathscr{L}$ can be made into a topos $\mathscr{F}$, the so-called *free* topos, actually the topos freely generated by the empty graph. $\mathscr{F}$ is initial in the category of toposes (with canonical subobjects) and strict logical morphisms. We assume that all toposes contain a natural numbers object. Briefly, the construction of $\mathscr{F}$ is as follows: the objects are the names of "sets" in $\mathscr{L}$, i.e. closed terms of type PA for arbitrary pure types A; the morphisms are names of "provably functional relations", i.e. closed terms of type P$(A \times B)$ which are provably (in $\mathscr{L}$) the graphs of functions between objects. Equality of objects and morphisms means provable equality in $\mathscr{L}$.

Freyd used the universal property of the free topos to give an ingenious proof of the following fundamental principles:

(EP) *If* $\vdash \exists_{x \in A} \varphi(x)$ *then* $\vdash \varphi(a)$ *for some closed term* a *of type* A.

(DP) *If* $\vdash p \vee q$ *then* $\vdash p$ *or* $\vdash q$.

For details and history of these results see [5] and for generalizations see [5, 7]. Briefly, Freyd's proof proceeds as follows:

(i) Translate (EP) and (DP) into algebraic properties of the free topos, namely that the terminal object 1 of $\mathscr{F}$ is projective and indecomposable.

(ii) Given a topos $\mathscr{A}$, construct its "Freyd cover" $\hat{\mathscr{A}}$, see § 2 below. There is always a strict logical functor $\hat{\mathscr{A}} \to \mathscr{A}$; moreover, in $\hat{\mathscr{A}}$, 1 is an indecomposable projective.

(iii) Being initial, $\mathscr{F}$ is a retract of $\hat{\mathscr{F}}$; hence the required algebraic properties of $\mathscr{F}$ are inherited from its Freyd cover.

In spite of its conceptual elegance and clarity, Freyd's proof is not particularly short when all the details are carefully worked out [5]. Indeed, a logician might find two objections: First, one must translate syntactical questions into their algebraic equivalents, and in some situations this can be quite delicate [7]. Secondly, the logicians' realizability proofs are more immediate: once the inductive clauses of realizability are known, syntactic results like (EP) and (DP) follow immediately. As it turns out, a version of realizability due to Aczel [8] describes the internal logic of Freyd covers, leading to shorter proofs of (EP) and (DP). The logic of Freyd covers is described in § 2 below, with applications in § 3, and § 4.

2. Freyd covers and their logic

The *Freyd cover* of a category $\mathscr{A}$ is the comma category $\hat{\mathscr{A}} = (\text{Sets}, \Gamma_{\mathscr{A}})$, where $\Gamma_{\mathscr{A}} = \mathscr{A}(1, -)\colon \mathscr{A} \to \text{Sets}$. Its objects are triplets (X, ξ, A), where X is a set, A an object of $\mathscr{A}$ and $\xi\colon X \to \Gamma_{\mathscr{A}}(A)$ a mapping. Morphisms $(X, \xi, A) \to (Y, \eta, B)$ are pairs of arrows $\Phi\colon X \to Y$, $f\colon A \to B$ such that the square

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "\xi"'] & Y \arrow[d, "\eta"] \\ \Gamma(A) \arrow[r, "\Gamma(f)"'] & \Gamma(B) \end{tikzcd}$$

commutes. $\hat{\mathscr{A}}$ is a topos with natural numbers whenever $\mathscr{A}$ is. In addition there is a strict logical functor $G\colon \hat{\mathscr{A}} \to \mathscr{A}$, where $G(X, \xi, A) = A$ on objects and $G(\Phi, f) = f$ on arrows.

The objects $\hat{A}$ of $\hat{\mathscr{A}}$ lying over objects A of $\mathscr{A}$ have the form $\hat{A} = (S_A, \lambda_A, A)$, where S_A is defined by induction on A:

$$S_1 = \{*\}, \qquad S_{\mathsf{N}} = \mathsf{N}, \qquad S_{A\times B} = S_A \times S_B, \qquad S_{PA} = \hat{\mathscr{A}}(\hat{A}, \hat{\Omega})$$

and

$$S_\Omega = \Gamma_{\mathscr{A}}(\Omega) \uplus \{\top\} = (\Gamma_{\mathscr{A}}(\Omega) \times \{0\}) \cup (\{\top\} \times \{1\}).$$

Moreover, λ_A is the obvious mapping $S_A \to \Gamma_{\mathscr{A}}(A)$.

A closed term $\hat{a}$ of type $\hat{A}$ in $\mathscr{L}(\hat{\mathscr{A}})$ is interpreted in $\hat{\mathscr{A}}$ as an arrow $\hat{a}\colon \hat{1} \to \hat{A}$. Clearly then $\hat{a}$ is given by a commutative square:

$$\begin{tikzcd}
\{*\} \arrow[r, "\sigma_a"] \arrow[d] & S_A \arrow[d, "\lambda_A"] \\
\Gamma(1) \arrow[r, "\Gamma(a)"'] & \Gamma(A)
\end{tikzcd}$$

where $a \cdot{=}\cdot G(\hat{a})$. For typographical reasons we have omitted the hat ^ on the subscript A in S_A, σ_A and λ_A.

In particular, $\hat{\top}\colon \hat{1} \to \hat{\Omega}$ is obtained by setting $\sigma_\top(*) = (\top, 1)$. For a sentence $\hat{p}$ of $\mathscr{L}(\hat{\mathscr{A}})$, if its interpretation $\hat{p}$ in $\hat{\mathscr{A}}$ is

$$\begin{tikzcd}
\{*\} \arrow[r, "\sigma_p"] \arrow[d] & S_\Omega \arrow[d, "\lambda_\Omega"] \\
\Gamma(1) \arrow[r, "\Gamma(p)"'] & \Gamma(\Omega)
\end{tikzcd}$$

then we have:

(1) if $\sigma_p(*) = (\top, 1)$ then $\mathscr{A} \models p$, (2) $\sigma_p(*) = (\top, 1)$ if and only if $\hat{\mathscr{A}} \models \hat{p}$.

The evaluation $\hat{\varepsilon}_A\colon \widehat{PA} \times \hat{A} \to \hat{\Omega}$ has the form:

$$\begin{tikzcd}
S_{PA} \times S_A \arrow[r, "\sigma_\varepsilon"] \arrow[d] & S_\Omega \arrow[d, "\lambda_\Omega"] \\
\Gamma(PA \times A) \arrow[r, "\Gamma(\varepsilon_A)"'] & \Gamma(\Omega)
\end{tikzcd}$$

where $\sigma_\varepsilon((\Phi, h), \alpha) = \Phi(\alpha)$ is given by evaluation in Sets.

We recall the interpretation of equality in Freyd covers:

Proposition 2.1. *If $\hat{a}$ and $\hat{b}$ are closed terms of type $\hat{A}$ in $\mathscr{L}(\hat{\mathscr{A}})$, then $\hat{\mathscr{A}} \models \hat{a} = \hat{b}$ if and only if $\sigma_a(*) = \sigma_b(*)$.*

Proof. See [7, Proposition 4.3].

From the interpretation of equality follows the entire internal logic of $\hat{\mathscr{A}}$.

Theorem 2.2. (internal logic of $\hat{\mathscr{A}}$).

(1) $\hat{\mathscr{A}} \models \hat{p} \wedge \hat{q}$ *if and only if* $\hat{\mathscr{A}} \models \hat{p}$ *and* $\hat{\mathscr{A}} \models \hat{q}$.

(2) $\hat{\mathscr{A}} \models \hat{p} \Rightarrow \hat{q}$ *if and only if* (i) $\mathscr{A} \models p \Rightarrow q$, (ii) *if* $\hat{\mathscr{A}} \models \hat{p}$ *then* $\hat{\mathscr{A}} \models \hat{q}$.

(2′) $\hat{\mathscr{A}} \models \neg\hat{p}$ *if and only if* (i) $\mathscr{A} \models \neg p$, (ii) *not* ($\hat{\mathscr{A}} \models \hat{p}$).

(3) $\hat{\mathscr{A}} \models \hat{p} \vee \hat{q}$ *if and only if* $\hat{\mathscr{A}} \models \hat{p}$ *or* $\hat{\mathscr{A}} \models \hat{q}$.

(4) $\hat{\mathscr{A}} \models \forall_{x \in A}\hat{\varphi}(x)$ *if and only if* (i) $\mathscr{A} \models \forall_{x \in A}\varphi(x)$, (ii) *for all* $\hat{a}\colon \hat{1} \to \hat{A}$ *in* $\hat{\mathscr{A}}, \hat{\mathscr{A}} \models \hat{\varphi}(\hat{a})$.

(5) $\hat{\mathscr{A}} \models \exists_{x \in A}\hat{\varphi}(x)$ *if and only if, for some* $\hat{a}\colon \hat{1} \to \hat{A}$ *in* $\hat{\mathscr{A}}, \hat{\mathscr{A}} \models \hat{\varphi}(\hat{a})$.

In clauses (1) *to* (5), *reference is made to the internal language of* $\hat{\mathscr{A}}$, (see § 1 above).

Proof. For (1) and (2) see [7, Corollary 4.4]. We now prove (4), from which (3), (5) will follow below. Let $\hat{\varphi}(x)$ be a formula of $\mathscr{L}(\hat{\mathscr{A}})$ with x a free variable of type A. As in [4, p. 125; 5] we can interprete $\hat{\varphi}(x)$ in the dogma $\hat{\mathscr{A}}[x]$with an indeterminate x of type $\hat{A}$. In particular, in $\hat{\mathscr{A}}[x]$ we can write $\hat{\varphi}(x) \mathrel{\cdot\underset{x}{=}\cdot} \hat{f}x$, where $\hat{f}\colon \hat{A} \to \hat{\Omega}$ in $\hat{\mathscr{A}}$, and in $\mathscr{A}[x]$ we can write $\varphi(x) \mathrel{\cdot\underset{x}{=}\cdot} fx$, where $f\colon A \to \Omega$ in $\mathscr{A}$. Then $\ulcorner\hat{f}\urcorner\colon \hat{1} \to (\mathrm{P}A)^\wedge$ interprets $\{x \in \hat{A} \mid \hat{\varphi}(x)\}$ in $\hat{\mathscr{A}}$ and ditto $\ulcorner f\urcorner\colon 1 \to \mathrm{P}A$ in $\mathscr{A}$. Moreover, $\ulcorner\hat{f}\urcorner$ must be of the form:

$$\begin{tikzcd}
\{*\} \arrow[r, "\sigma_{\ulcorner f \urcorner}"] \arrow[d] & S_{\mathrm{P}A} \arrow[d, "\lambda_{\mathrm{P}A}"] \\
\Gamma(1) \arrow[r, "\Gamma(\ulcorner f\urcorner)"'] & \Gamma(\mathrm{P}A).
\end{tikzcd}$$

From the definition of $\forall$ in $\mathscr{L}$ or $\mathscr{L}(\hat{\mathscr{A}})$ (see § 1), we have:

$\hat{\mathscr{A}} \models \forall_{x \in A}\hat{\varphi}(x)$ if and only if $\hat{\mathscr{A}} \models \{x \in \hat{A} \mid \hat{\varphi}(x)\} = \{x \in \hat{A} \mid \top\}$ i.e. $\hat{\mathscr{A}} \models \ulcorner\hat{f}\urcorner = \ulcorner\hat{\top}\hat{O}_A\urcorner$, where $\hat{O}_A\colon \hat{A} \to \hat{1}$ is the terminal arrow in $\hat{\mathscr{A}}$. By Proposition 2.1, this is the case if and only if

$$\sigma_{\ulcorner f\urcorner}(*) = \sigma_{\ulcorner \top O_A\urcorner}(*).$$

Now $\sigma_{\ulcorner f\urcorner}(*) = (\Phi, f) \in S_{\mathrm{P}A}$, where $\Phi\colon S_A \to S_\Omega$ and $f\colon A \to \Omega$. Hence the displayed equation holds if and only if $\Phi(\alpha) = (\top, 1)$ for all $\alpha \in S_A$ and $f \mathrel{\cdot=\cdot} \top O_A\colon A \to 1 \to \Omega$ in $\mathscr{A}$. The second condition asserts that $\{x \in A \mid \varphi(x)\} \mathrel{\cdot=\cdot} \{x \in A \mid \top\}$ in $\mathscr{A}$, that is, $\mathscr{A} \models \forall_{x \in A}\varphi(x)$. We shall now prove that the first condition holds if and only if $\hat{\mathscr{A}} \models \hat{a} \in \ulcorner\hat{f}\urcorner$, that is, $\mathscr{A} \models \hat{\varphi}(\hat{a})$, for all $\hat{a}\colon \hat{1} \to \hat{A}$.

Indeed, the arrow $\hat{a}$ is given by the commutative square

$$\begin{tikzcd}
\{*\} \arrow[r, "* \mapsto \alpha"] \arrow[d] & S_A \arrow[d, "\lambda_A"] \\
\Gamma(1) \arrow[r, "\Gamma(a)"'] & \Gamma(A)
\end{tikzcd}$$

and is completely determined by $\alpha \in S_A$. Hence quantification over arrows $\hat{a}$ is equivalent to quantification over $\alpha \in S_A$. Moreover, $\hat{\mathscr{A}} \vDash \hat{a} \in \ulcorner\hat{f}\urcorner$ if and only if $\varepsilon_A\langle\ulcorner\hat{f}\urcorner, \hat{a}\rangle \cdot\!=\!\cdot\ \hat{\top}$, that is, the composite

$$\begin{tikzcd}
\{*\} \arrow[r, "\langle\sigma_{\ulcorner f\urcorner},\, * \mapsto \alpha\rangle"] \arrow[d] & S_{\mathrm{P}A} \times S_A \arrow[r, "\sigma_\varepsilon"] \arrow[d] & S_\Omega \arrow[d] \\
\Gamma(1) \arrow[r, "\Gamma(\langle\ulcorner f\urcorner, a\rangle)"'] & \Gamma(\mathrm{P}A \times A) \arrow[r, "\Gamma(\varepsilon_A)"'] & \Gamma(\Omega)
\end{tikzcd}$$

is equal to $\hat{\top}$. Since the top row determines the rest, this is the case if and only if $\sigma_\varepsilon(\sigma_{\ulcorner f\urcorner}(*), \alpha) = (\top, 1)$, that is, $\Phi(\alpha) = (\top, 1)$, since $\sigma_{\ulcorner f\urcorner}(*) = (\Phi, f)$. This completes the proof of (4).

Both (3) and (5) of Theorem 2.2 follow from (4), which has just been proved. For example, we shall show (5).

Indeed, we recall from § 1 that $\hat{\mathscr{A}} \vDash \exists_{x\in A}\hat{\varphi}(x)$ if and only if $\hat{\mathscr{A}} \vDash \forall_{t\in\Omega}(\forall_{x\in A}(\hat{\varphi}(x) \Rightarrow t) \Rightarrow t)$. In view of (4), this holds if and only if

(i) $\mathscr{A} \vDash \exists_{x\in A}\varphi(x)$,

and

(ii) for every $\hat{p}\colon \hat{1} \to \hat{\Omega}$, $\hat{\mathscr{A}} \vDash \forall_{x\in A}(\hat{\varphi}(x) \Rightarrow \hat{p}) \Rightarrow \hat{p}$.

We have to show that the conjunction of (i) and (ii) is equivalent to:

(iii) For some $\hat{a}\colon \hat{1} \to \hat{A}$, $\hat{\mathscr{A}} \vDash \hat{\varphi}(\hat{a})$.

By applying the logical functor $\hat{\mathscr{A}} \to \mathscr{A}$ to (iii), we see that (iii) implies (i). Also clearly (iii) implies (ii). Conversely, assume (i) and (ii) and let $\hat{p}$ be the commutative square

$$\begin{tikzcd}
\{*\} \arrow[r, "* \mapsto (\top, 0)"] \arrow[d] & S_\Omega \arrow[d] \\
\Gamma(1) \arrow[r, "\Gamma(\top)"'] & \Gamma(\Omega).
\end{tikzcd}$$

Using (2) of Theorem 2.2, we infer from (ii) above that

(iv) if $\hat{\mathscr{A}} \vDash \forall_{x\in A}(\hat{\varphi}(x) \Rightarrow \hat{p})$ then $\hat{\mathscr{A}} \vDash \hat{p}$.

Now, by choice of $\hat{p}$, it is not true that $\hat{\mathscr{A}} \models \hat{p}$. Therefore, it is not true that $\hat{\mathscr{A}} \models \forall_{x \in A}(\hat{\varphi}(x) \Rightarrow \hat{p})$. Again, using (4), by Theorem 2.2, we conclude that either not $\mathscr{A} \models \forall_{x \in A}(\varphi(x) \Rightarrow \top)$ or, for some $\hat{a}\colon \hat{1} \to \hat{A}$, not $\hat{\mathscr{A}} \models \hat{\varphi}(\hat{a}) \Rightarrow \hat{p}$. Since obviously $\mathscr{A} \models \forall_{x \in A}(\varphi(x) \Rightarrow \top)$, it follows from (2) that there is an $\hat{a}\colon \hat{1} \to \hat{A}$ such that $\hat{\mathscr{A}} \models \hat{\varphi}(\hat{a})$ but not $\hat{\mathscr{A}} \models \hat{p}$. Thus (iii) holds, as was to be shown, and our proof is complete.

The inductive clauses of the internal logic of $\hat{\mathscr{A}}$ had previously been discussed in connection with realizability and Kripke models and are known as the Aczel slash [8, p. 333]. Logicians are referred to Appendix II for a comparison between our version of the Aczel slash and related concepts in the literature.

Since $\mathscr{L}(\hat{\mathscr{A}})$ extends $\mathscr{L}$, we omit the hat ^ in formulas coming from the pure language $\mathscr{L}$ in what follows.

3. Applications

We can now efficiently reprove many of our previous results in [5, 7] and obtain several new ones. We first give new proofs of (DP) and (EP) based on Theorem 2.2.

(DP) *If* $\vdash p \vee q$ *then* $\vdash p$ *or* $\vdash q$.

Proof. Suppose $\vdash p \vee q$. Then, in particular, $\hat{\mathscr{F}} \models p \vee q$. By the theorem, $\hat{\mathscr{F}} \models p$ or $\hat{\mathscr{F}} \models q$. Applying the logical functor $G\colon \hat{\mathscr{F}} \to \mathscr{F}$, we find that $\mathscr{F} \models p$ or $\mathscr{F} \models q$, that is, $\vdash p$ or $\vdash q$.

(EP) *If* $\vdash \exists_{x \in A}\varphi(x)$ *then* $\vdash \varphi(a)$ *for some closed term* a *of type* A.

Proof. Suppose $\vdash \exists_{x \in A}\varphi(x)$, then $\hat{\mathscr{F}} \models \exists_{x \in A}\varphi(x)$. So, by the theorem, for some $\hat{a}\colon 1 \to \hat{A}$, $\hat{\mathscr{F}} \models \varphi(\hat{a})$. Applying the logical functor $G\colon \hat{\mathscr{F}} \to \mathscr{F}$ and writing $G(\hat{a}) = \bar{a}$, we find that there is an arrow $\bar{a}\colon 1 \to A$ such that $\mathscr{F} \models \varphi(\bar{a})$. To deduce from this that $\vdash \varphi(a)$ for a term a of type A, some work has to be done. While the arrow $\bar{a}$ is given by a term ϱ of the language, this is of type $\mathrm{P}(1 \times A)$ and denotes a provable functional relation. We still have to eliminate the description term "the unique $x \in A$ such that $\langle *, x\rangle \in \varrho$", that is, we must show that $\vdash \varrho = \{\langle *, x\rangle \in 1 \times A \mid x = a\}$ for a term a of type A. That this can be done for the language $\mathscr{L}$ has been shown elsewhere [5, Lemma 6.3; 7, § 2].

Next we shall look at (EP) *with parameters*. Suppose $\vdash \forall_{x \in A}\, \exists_{y \in B}\varphi(x, y)$, that is, $\vdash_x \exists_{y \in B}\varphi(x, y)$, where x is a variable of type A. We shall now assume that $A = \Omega$, but the case $A = \mathrm{P}C$ may be treated quite similarly. Consider the topos $\mathscr{F}(x)$ obtained from the topos $\mathscr{F}$ by adjoining an indeterminate arrow $x\colon 1 \to \Omega$. In its Freyd cover $\mathscr{F}(x)^\wedge$ there is an arrow $\hat{x}\colon 1 \to \hat{\Omega}$ given by the commutative square:

$$\begin{array}{ccc} \{*\} & \xrightarrow{* \mapsto (x, 0)} & S_\Omega \\ \downarrow & & \downarrow \\ \Gamma(1) & \xrightarrow[\Gamma(x)]{} & \Gamma(\Omega). \end{array}$$

By the universal property of $\mathscr{F}(x)$, there is a unique logical functor $\mathscr{F}(x) \to \mathscr{F}(x)^\wedge$ mapping x onto $\hat{x}$. Since $\vdash_x \exists_{y\in B}\varphi(x, y)$, we have $\mathscr{F}(x)^\wedge \models \exists_{y\in B}\varphi(\hat{x}, y)$, the variable x being interpreted by the arrow $\hat{x}$. It follows from (5) of Theorem 2.2 that $\mathscr{F}(x)^\wedge \models \varphi(\hat{x}, \hat{\beta}(x))$, where $\hat{\beta}(x)\colon \hat{1} \to \hat{B}$ in $\mathscr{F}(x)^\wedge$. Hence $\mathscr{F}(x) \models \varphi(x, \bar{\beta}(x))$, where $\bar{\beta}(x)\colon 1 \to B$ in $\mathscr{F}(x)$.

In the special case $B = \mathsf{N}$, the arrow $\bar{\beta}(x)\colon 1 \to \mathsf{N}$ must correspond to a standard numeral, hence $\bar{\beta}(x)$ is of the form $S^n 0 \equiv \bar{n}$ [5, Lemma 6.5]. Therefore $\vdash \forall_{x\in\Omega}\varphi(x, \bar{n})$ for some $n \in \mathsf{N}$. We have thus proved TROELSTRA's *Uniformity Rule*:

(UR) *If* $\vdash \forall_{x\in\Omega} \exists_{y\in\mathsf{N}}\varphi(x, y)$ *then* $\vdash \exists_{y\in\mathsf{N}} \forall_{x\in\Omega}\varphi(x, y)$.

(UR) also holds if Ω is replaced by a pure type of the form PC, the argument being similar.

A curious result, which is proved in a similar fashion, is the following:

Indecomposability of Ω. *If* $\vdash \forall_{x\in\Omega}(\varphi(x) \vee \psi(x))$ *then* $\vdash \forall_{x\in\Omega}\varphi(x)$ *or* $\vdash \forall_{x\in\Omega}\psi(x)$.

Proof. Suppose $\vdash \forall_{x\in\Omega}(\varphi(x) \vee \psi(x))$, that is, $\vdash_x \varphi(x) \vee \psi(x)$. Then $\mathscr{F}(x)^\wedge \models \varphi(\hat{x}) \vee \psi(\hat{x})$. By Theorem 2.2 (3), $\mathscr{F}(x)^\wedge \models \varphi(\hat{x})$ or $\mathscr{F}(x)^\wedge \models \psi(\hat{x})$. Hence $\mathscr{F}(x) \models \varphi(x)$ or $\mathscr{F}(x) \models \psi(x)$, that is, $\vdash_x \varphi(x)$ or $\vdash_x \psi(x)$, from which the result follows.

Again, the result also holds if Ω is replaced by PC. These two rules state that Ω and PC are indecomposable objects in the free topos.

The Existence Property Modulo p is the following rule, valid for certain closed formulas p:

If $p \vdash \exists_{x\in A}\varphi(x)$ *then* $p \vdash \varphi(a)$ *for some closed term* a *of type* A.

In [7] this rule was carefully examined and shown to be equivalent to the following [7, Corollary 3.4]:

(IP) *If* $\vdash p \Rightarrow \exists_{x\in A}\varphi(x)$ *then* $\vdash \exists_{x\in A}(p \Rightarrow \varphi(x))$.

An interesting (and still open!) question is to characterize those formulas p for which (IP) holds. Algebraically, this says that p determines a projective subobject of 1 in the free topos $\mathscr{F}$. We had made some progress on this problem in [7] and shall now reconsider our results in the light of Theorem 2.2.

We shall require the notion of the *free topos* $\mathscr{F}/p$ *on the assumption* p. It may be constructed syntactically just like the free topos $\mathscr{F}$, except that we use the language $\mathscr{L}_p$ whose deduction relation is $p \vdash_X$, that is, deduction on the assumption p.

We recall [7, § 6].

Definition 3.1. (i) p is *Freydian* if $(\mathscr{F}/p)^\wedge \models p$ or, equivalently, $\mathscr{F}/p$ is a retract of $(\mathscr{F}/p)^\wedge$. (ii) p is *hereditary* if, for all nondegenerate toposes $\mathscr{A}$, if $\mathscr{A} \models p$ then $\hat{\mathscr{A}} \models p$.

Proposition 3.2. *If* p *is Freydian then* p *satisfies* (IP).

Proof. Suppose $p \vdash \exists_{x\in A}\varphi(x)$. If p is Freydian, $(\mathscr{F}/p)^\wedge \models p$, hence $(\mathscr{F}/p)^\wedge \models \exists_{x\in A}\varphi(x)$. By Theorem 2.2 (5), $(\mathscr{F}/p)^\wedge \models \varphi(\hat{a})$ for some arrow $\hat{a}\colon 1 \to \hat{A}$ in $(\mathscr{F}/p)^\wedge$. Hence $\mathscr{F}/p \models \varphi(\bar{a})$, where $\bar{a}\colon 1 \to A$ in $\mathscr{F}/p$. We would like to replace the arrow $\bar{a}$ by a term a of type A, so that $\mathscr{F}/p \models \varphi(a)$, that is, $p \vdash \varphi(a)$. To do this we must prove a syntactical lemma on eliminability of description. By [7, Lemma 2.1] it suffices to show that arrows $1 \to \mathsf{N}$ in $\mathscr{F}/p$ correspond to standard numerals. Now, since p is Freydian, $\mathscr{F}/p$ is a retract of $(\mathscr{F}/p)^{\wedge}$ and so $\mathscr{F}/p$ inherits this property from $(\mathscr{F}/p)^{\wedge}$.

We recall the following:

Proposition 3.3. (i) *If p is hereditary then either $\vdash \neg p$ or p is Freydian.* (ii) *If p is hereditary then p satisfies* (IP).

Proof. See [7].

The hereditary formulas are easier to handle than the Freydian ones; e.g., they are closed under the inductive clauses of *Harrop Formulas* [7, Theorem 6.3].

Proposition 3.4. (i) *$\perp$ is hereditary.* (ii) *If p and q are hereditary, then so is $p \wedge q$.* (iii) *If q is hereditary, then so is $p \Rightarrow q$ for any p.* (iv) *If $\varphi(x)$ is hereditary, then so is $\forall_{x\in A}\varphi(x)$. Here $\varphi(x)$ is called hereditary provided, for any nondegenerate topos $\mathscr{A}$ and every $\hat{a}\colon \hat{1} \to \hat{A}$ in $\hat{\mathscr{A}}$, if $\mathscr{A} \models \varphi(a)$ then $\hat{\mathscr{A}} \models \varphi(\hat{a})$, where $a = G(\hat{a})$, G being the logical functor $\hat{\mathscr{A}} \to \mathscr{A}$.*

Proof. The only new assertion here is (iv). Suppose $\varphi(x)$ is hereditary and $\mathscr{A} \models \forall_{x\in A}\varphi(x)$, $\mathscr{A}$ being nondegenerate. We claim that $\hat{\mathscr{A}} \models \forall_{x\in A}\varphi(x)$. In view of Theorem 2.2 (4), we need only check that, for any $\hat{a}\colon \hat{1} \to \hat{A}$, $\hat{\mathscr{A}} \vdash \varphi(\hat{a})$. Since $\varphi(x)$ is hereditary, this follows from $\mathscr{A} \vdash \varphi(a)$, a consequence of the assumption.

The set of Freydian formulas does not have such nice properties; for example, it is not closed under conjunction. However, we do have the following:

Proposition 3.5 (Freyd). (i) *$\neg p$ is Freydian if and only if not $\vdash \neg\neg p$.* (ii) *$p \Rightarrow q$ is Freydian if not $(p \Rightarrow q) \vdash p$.*

Proof. See [7, Theorem 6.2].

We remark that both Propositions 3.4 and 3.5 establish (IP) for "stable" p, that is, for those propositions p for which $\vdash \neg\neg p \Rightarrow p$. Clearly, the set of stable formulas has the closure properties of Proposition 3.4. One may ask whether there are any hereditary formulas which are not stable.

Unfortunately, the sets of Freydian and hereditary formulas are not directly comparable.

Proposition 3.6. (i) *$\perp$ is hereditary but not Freydian.* (ii) *$\neg\neg\beta \Rightarrow \beta$ is Freydian but not hereditary, where β is the Boolean axiom*

$$\beta \equiv \forall_{t\in\Omega}(\neg\neg t \Rightarrow t).$$

Proof. (i) $\perp$ is hereditary by Proposition 3.4. It cannot be Freydian, else the Freyd cover $(\mathscr{F}/\perp)^{\wedge}$ would be degenerate; but no Freyd cover is degenerate.

(ii) $\neg\neg\beta \Rightarrow \beta$ is Freydian by Proposition 3.5, since not $\neg\neg\beta \Rightarrow \beta \vdash \neg\neg\beta$. To prove this, suppose $\neg\neg\beta \Rightarrow \beta \vdash \neg\neg\beta$. Since $\neg\beta \vdash \neg\neg\beta \Rightarrow \beta$, it would follow that $\neg\beta \vdash \neg\neg\beta$, that is, $\vdash \neg\neg\beta$. This is known to be false.

As for the fact that $\neg\neg\beta \Rightarrow \beta$ is not hereditary, note that Sets $\models \neg\neg\beta \Rightarrow \beta$. However, we claim that not Sets$^{\wedge} \models \neg\neg\beta \Rightarrow \beta$. To this end recall [7, Proposition 5.4] that

Sets$^\wedge \cong$ Sets$^{\underline{2}}$. Using Theorem 2.2 (2′) above, one easily verifies that Sets$^{\underline{2}} \models \neg\neg\beta$, whereas not Sets$^{\underline{2}} \models \beta$, as Sets$^{\underline{2}}$ is known to be not Boolean. Therefore, it is not true that Sets$^{\underline{2}} \models \neg\neg\beta \Rightarrow \beta$.[1])

4. Markov's Rule

By *Markov's Rule at type A* we mean the following rule:

MR(A) *If* $\vdash \forall_{x\in A}(\varphi(x) \vee \neg\varphi(x))$ *and* $\vdash \neg\forall_{x\in A}\neg\varphi(x)$ *then* $\vdash \exists_{x\in A}\varphi(x)$.

The usual form of Markov's Rule is MR(N) and may be proved as follows.

Suppose $\vdash \neg\forall_{x\in \mathsf{N}}\neg\varphi(x)$, then Sets $\models \neg\forall_{x\in \mathsf{N}}\neg\varphi(x)$. Since Sets is Boolean, Sets $\models$ $\models \exists_{x\in \mathsf{N}}\varphi(x)$. Therefore Sets $\models \varphi(\bar{n})$, where $\bar{n} = S^n 0$, for some $n \in \mathsf{N}$. Hence not $\vdash \neg\varphi(\bar{n})$. Now suppose $\vdash \forall_{x\in \mathsf{N}}(\varphi(x) \vee \neg\varphi(x))$, then $\vdash \varphi(\bar{n}) \vee \neg\varphi(\bar{n})$, hence $\vdash \varphi(\bar{n})$ or $\vdash \neg\varphi(\bar{n})$, by (DP). Therefore $\vdash \varphi(\bar{n})$, and so $\vdash \exists_{x\in \mathsf{N}}\varphi(x)$.

We shall prove that Markov's Rule holds at any pure type A. First note that any pure type (regarded as an object in the free topos) is isomorphic to a type of the form

$$\mathsf{N}^k \times \mathrm{P}(A_1) \times \ldots \times \mathrm{P}(A_n), \quad (k \geqq 0,\ n \geqq 0)$$

since $\Omega \cong \mathrm{P}(1)$. Now $\mathsf{N}^k \cong 1$ or $\mathsf{N}^k \cong \mathsf{N}$. Moreover, $\mathrm{P}(A)$ is injective in any topos and any product of injectives is injective. Therefore every pure type is isomorphic to Q or $\mathsf{N} \times Q$ where Q is injective. If $n = 0$, then $Q \cong 1$.

Lemma 4.1. *In the free topos $\mathscr{F}$, injective pure types are indecomposable, that is, if* $\vdash \forall_{x\in Q}(\varphi(x) \vee \psi(x))$ *then* $\vdash \forall_{x\in Q}\varphi(x)$ *or* $\vdash \forall_{x\in Q}\psi(x)$.

Proof. Suppose Q is an injective pure type, regarded as an object of $\mathscr{F}$. Then the singleton morphism $\iota_Q\colon Q \to \mathrm{P}(Q)$ splits, that is, there is an arrow $e\colon \mathrm{P}(Q) \to Q$ in $\mathscr{F}$ such that $e\iota_Q \cdot\!=\!\cdot\ 1_Q$. Now suppose $\vdash \forall_{x\in Q}(\varphi(x) \vee \psi(x))$. Let $y\colon 1 \to \mathrm{P}(Q)$ be an indeterminate arrow, then $\mathscr{F}(y) \models \varphi(e(y)) \vee \psi(e(y))$. Since $\mathrm{P}(Q)$ is indecomposable (see § 3), we have $\mathscr{F}(y) \models \varphi(e(y))$ or $\mathscr{F}(y) \models \psi(e(y))$, hence $\mathscr{F}(x) \models \varphi(e(\iota_Q x))$ or $\mathscr{F}(x) \models$ $\models \psi(e(\iota_Q x))$. Since $e\iota_Q \cdot\!=\!\cdot\ 1_Q$, it follows that $\vdash \forall_{x\in Q}\varphi(x)$ or $\vdash \forall_{x\in Q}\psi(x)$.

Proposition 4.2. MR(A) *holds for all pure types* A.

Proof. We consider two cases, MR(Q) and MR($\mathsf{N} \times Q$), where Q is injective, hence indecomposable.

MR(Q). Suppose $\vdash \forall_{x\in Q}(\varphi(x) \vee \neg\varphi(x))$, then $\vdash \forall_{x\in Q}\varphi(x)$ or $\vdash \forall_{x\in Q}\neg\varphi(x)$. Now suppose $\vdash \neg\forall_{x\in Q}\neg\varphi(x)$, then the second alternative is out, and so $\vdash \forall_{x\in Q}\varphi(x)$. Let t be a term of type Q, then $\vdash \varphi(t)$, hence $\vdash \exists_{x\in Q}\varphi(x)$.

MR($\mathsf{N} \times Q$). Suppose $\vdash \forall_{x\in \mathsf{N}}\forall_{y\in Q}(\varphi(x, y) \vee \neg\varphi(x, y))$ and $\vdash \neg\forall_{x\in \mathsf{N}}\forall_{y\in Q}\neg\varphi(x, y)$. It follows from the second assumption that Sets $\models \exists_{x\in \mathsf{N}}\exists_{y\in Q}\varphi(x, y)$, and therefore there is a numeral $\bar{n} = S^n 0$ such that Sets $\models \exists_{y\in Q}\varphi(\bar{n}, y)$, hence not $\vdash \forall_{y\in Q}\neg\varphi(\bar{n}, y)$. It follows from the first assumption that $\vdash \forall_{y\in Q}(\varphi(\bar{n}, y) \vee \neg\varphi(\bar{n}, y))$, and so, since Q

[1]) A false assertion crept into [7, Example 7.6]. Contrary to the claim there, it is not true that Sets$^2 \models \neg\beta$. In fact, Sets$^2 \models \neg\neg\beta$; for Sets$^2 \cong$ Sets$^\wedge$, and, by Theorem 2.2 Sets$^\wedge \models \neg\neg\beta$ if and only if Sets $\models \neg\neg\beta$, which is true. Nonetheless, the nonconstructive proof in the example, using $\neg\neg\beta \Rightarrow \beta$, is correct.

is indecomposable, that $\vdash \forall_{y \in Q} \varphi(\bar{n}, y)$ or $\vdash \forall_{y \in Q} \neg \varphi(\bar{n}, y)$. Since the second alternative has already been ruled out, the former must hold. Let t be a term of type Q, then $\vdash \varphi(\bar{n}, t)$, hence $\vdash \exists_{x \in \mathbf{N}} \exists_{y \in Q} \varphi(x, y)$, as was to be proved.

Appendix I. Rules for intuitionistic type theory based on equality

1. Structural rules

$$p \vdash_X p;$$

$$\frac{\Gamma \vdash_X p \quad \Gamma, p \vdash_X q}{\Gamma \vdash_X q}; \qquad \frac{\Gamma \vdash_X q}{\Gamma, p \vdash_X q}; \qquad \frac{\Gamma \vdash_X q}{\Gamma \vdash_{X \cup \{y\}} q};$$

$$\frac{\Gamma(y) \vdash_{X \cup \{y\}} \varphi(y)}{\Gamma(b) \vdash_X \varphi(b)} \quad \text{[assume } b \text{ free for } y \text{ in } \varphi(y) \text{ and } \Gamma(y)].$$

2. Pure equality rules

$$\vdash_X a = a;$$

$$a = b, \quad \varphi(a) \vdash_X \varphi(b) \quad \text{[assume } a \text{ and } b \text{ free for } x \text{ in } \varphi(x)];$$

$$\frac{\Gamma, p \vdash_X q; \ \Gamma, q \vdash_X p}{\Gamma \vdash_X p = q}.$$

3. Other logical rules

$$\frac{\Gamma \vdash_{X \cup \{x\}} \varphi(x) \Rightarrow x \in \alpha}{\Gamma \vdash_X \{x \in A \mid \varphi(x)\} = \alpha};$$

$$\langle a, b \rangle = \langle c, d \rangle \vdash_X a = c; \qquad \langle a, b \rangle = \langle c, d \rangle \vdash_X b = d.$$

4. Other product rules

$$\vdash_x x = * \quad \text{[assume } x \text{ of type } 1];$$

$$\frac{\Gamma, z = \langle x, y \rangle \vdash_{\{x,y,z\}} \varphi(z)}{\Gamma \vdash_z \varphi(z)}.$$

5. Peano's rules

$$\mathrm{S}x = 0 \vdash_x p; \qquad \mathrm{S}x = \mathrm{S}y \vdash_{\{x,y\}} x = y;$$

$$\frac{\Gamma \vdash \varphi(0); \ \Gamma, \varphi(x) \vdash_x \varphi(\mathrm{S}x)}{\Gamma \vdash_x \varphi(x)}.$$

Appendix II. Comparison of our version of the Aczel slash with the literature

The original version of the Aczel slash [8, p. 332] was meant to describe the logic of certain Kripke models for first order logic and arithmetic. To obtain a higher order analogue, we write

$$\Gamma \mid p \text{ for } (\mathscr{F}/\Gamma)^{\wedge} \models p,$$

where $\mathscr{F}/\Gamma$ is the free topos "with assumptions Γ", that is, the term model of the language of pure types with deducibility relation $\Gamma \vdash$. Using Theorem 2.2 above, we see that $\Gamma \mid p$ satisfies exactly the usual clauses of the Aczel slash [8, p. 333], except, of course, that the atomic formulas are handled somewhat differently in higher order logic.

In both the Kleene slash and the Aczel slash for first order logic, an important rôle is played by those formulas p for which $p \mid p$. In our theory these formulas are also important; for, by the above, $p \mid p$ if and only if $(\mathscr{F}/p)^\wedge \models p$, that is, p is Freydian (see Definition 3.1).

It follows from Propositions 3.2 and 3.3 that (IP) holds not only for the Freydian formulas, but also for the hereditary ones. It is only the latter class which contains all Harrop formulas.

Finally, Ščedrov and Scott [9] showed that Freyd's proof of (EP) and (DP) is virtually the same as the proof based on Friedman's realizability [5]. Indeed, our $|p$, that is, $\hat{\mathscr{F}} \models p$, is equivalent to "$p$ is Kleene-Friedman realizable and $\vdash p$".

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