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INTUITIONIST TYPE THEORY AND THE FREE TOPOS

J. LAMBEK* and P.J. SCOTT

Mathematics Department, McGill University, Montreal, Canada

Dedicated to Saunders MacLane on his seventieth birthday

In this article we study free toposes with the help of intuitionist type theory. Our treatment is self-contained and aims to be accessible to both categorists and logicians. We attempt to explain the relevant logic to the former and the categorical applications to the latter.

Algebraically, free toposes arise as solutions to a universal problem, which amounts to constructing a left adjoint to the forgetful functor Top→Graph. Here "Top" denotes the category of small toposes, which we shall assume to possess a natural number object, with appropriate morphisms. These are essentially the so-called logical functors, except that we insist on them being strict functors which preserve everything on the nose. "Graph" denotes the category of graphs, which we take to be oriented, and functor-like morphisms. The adjoint functor Graph→Top associates to each graph $\mathscr{G}$ the topos $T(\mathscr{G})$ freely generated by $\mathscr{G}$. In particular, when $\mathscr{G}=\emptyset$ is the empty graph, we obtain the so-called free topos $T(\emptyset)$, which is an initial object in Top.

Lawvere has often pointed out the strong connection between topos theory and higher order intuitionist logic. It is precisely in the construction of the free topos that this connection is seen most easily.

In Section 1 we present a formulation of intuitionist type theory with product types and mention the fundamental theorem which comprises three things:

(1) the consistency of intuitionist type theory,

(2) the $\vee$-property which asserts that if $p\vee q$ is provable then either p or q is provable,

(3) The $\exists$-property which asserts that if $\exists_{x\in A}\,\varphi(x)$ is provable, then $\varphi(a)$ is provable for some term a of type A.

Our type-theoretical language contains enough terms to witness all existential theorems; yet it does not contain too many terms, for example, it lacks a description operator.

The fundamental theorem can be proved by several methods:

(a) the cut elimination method of Gentzen–Girard,

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(b) the realizability method of Kleene–Friedman,

(c) the categorical method of Peter Freyd.

In Section 2 we use the realizability method to prove the fundamental theorem and also to extend Troelstra's Uniformity Property to higher order arithmetic.

In Section 3 we collect all kinds of theorems from logic that deal with the representability of functions $\mathbf{N}^k \to \mathbf{N}$ in intuitionist type theory. These will later enable us to discuss certain arrows in the free topos. Most of these results are essentially contained in the book by Kleene.

In Section 4 we give a construction of the free topos which is based on the language of intuitionist type theory developed in Section 1. Constructions using somewhat different languages have been given by Coste, Fourman and Boileau; but the first construction of the free topos is due to Volger, who used an altogether different approach.

We also study the arrows between certain objects in the free topos, namely those objects which correspond to types. In particular, we show that all arrows $1 \to N$ are standard numerals, that all arrows $N^k \to N$ induce recursive functions $\mathbf{N}^k \to \mathbf{N}$ and that not all recursive functions are obtained in this way. These results have already been found by Boileau and the Costes. We also show that all arrows $\Omega \to N$ and $PB \to N$ factor through 1.

The universal property of the free topos had been shown by Volger for his construction, only the morphisms in his category of toposes were not strict functors. None of the other authors established the universal property for their construction or proved it equivalent to Volger's. We therefore devote Section 5 to proving the universal property of the present construction of the free topos. One of the present authors had already shown that Volger's logical functors could be made strict by stipulating that all toposes have canonical subobjects.

Our original intention had been to use methods of mathematical logic to obtain results in category theory, to wit, properties of the free topos. In the mean time Peter Freyd made a fundamental breakthrough, which suggests that the more interesting applications may be in the opposite direction. As an afterthought, we therefore added Section 6, in which the $\exists$-property and the Uniformity Property are proved again, and perhaps with less effort, by Freyd's method.

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1. Intuitionist type theory

We shall present a language $\mathscr{L}_1$ for intuitionist type theory with product types.

Definition 1.1. *Type* symbols are defined inductively as follows:[1]

(i) 1, N, Ω are types;

(ii) if A and B are types, so are $A \times B$ and PA.

[1] It is understood that nothing is a type of $\mathscr{L}_1$ unless its being so follows from (i) and (ii).

Remark. N is the type of natural numbers; Ω is the type of propositions or truth values; 1 is a one element type and may also be regarded as the empty product; $A \times B$ is the type of pairs $\langle a, b\rangle$, where a is of type A and b of type B; finally PA is the type of all subsets of A, that is, of all sets of entities of type A.

The language $\mathscr{L}_1$ contains a countably infinite set of variables of each type, ordinary parentheses and also the following symbols:

$$*;\ \langle\ ,\ \rangle;\ \in;\ \{\ ,\ \};\ 0;\ S;\ \top;\ \bot;\ \wedge;\ \vee;\ \Rightarrow;\ \forall;\ \exists.$$

Definition 1.2. *Terms* in $\mathscr{L}_1$ are defined as follows:[2]

(1) variables of type A are terms of type A;

(2) $*$ is a term of type 1;

(3) if a is a term of type A and b of type B then $\langle a, b\rangle$ is a term of type $A \times B$;

(4) if a is a term of type A and α a term of type PA then $a \in \alpha$ is a term of type Ω;

(5) if $\varphi(x)$ is a term of type Ω, possibly containing the free variable x of type A, then $\{x \in A \mid \varphi(x)\}$ is a term of type PA;

(6) 0 is a term of type N;

(7) if n is a term of type N, then so is Sn;

(8) $\top$ and $\bot$ are terms of type Ω;

(9) if p and q are terms of type Ω, then so are $p \wedge q$, $p \vee q$ and $p \Rightarrow q$;

(10) if $\varphi(x)$ is a term of type Ω, possibly containing the free variable x of type A, then $\forall_{x \in A}\ \varphi(x)$ and $\exists_{x \in A}\ \varphi(x)$ are terms of type Ω.

Remarks. Parentheses are employed as usual. The notions *free* variable, *bound* variable and *closed* term are defined as usual. Terms of type Ω are also called *formulas*. As usual we write

$$\neg p \quad \text{for } p \Rightarrow \bot,$$

$$p \Leftrightarrow q \quad \text{for } (p \Rightarrow q) \wedge (q \Rightarrow p),$$

$$a = b \quad \text{for } \forall_{u \in PA}\ (a \in u \Leftrightarrow b \in u).$$

$\mathscr{L}_1$ is not just a language, but also a deductive system. For each set X of variables, we introduce a relation $p \vdash_X q$ between formulas p and q whose free variables are contained in X. The relation $\vdash_X$ is subject to four groups of axioms (and is assumed to be the smallest such relation).

Structural Axioms

(1) $\vdash_X$ is reflexive and transitive;

(2) if $p \vdash_X q$ and $X \subseteq Y$, then $p \vdash_Y q$;

(3) if $\varphi(y) \vdash_{X \cup \{y\}} \psi(y)$, where y is a variable of type B, then $\varphi(b) \vdash_X \psi(b)$, where b, a term of type B, contains at most the free variables in X and b is substitutable for y in $\varphi(y)$ and $\psi(y)$.

[2] It is understood that nothing is a term of $\mathscr{L}_1$ unless its being so follows from (1) to (10).

Logical Axioms

(1) $p \vdash_X \top$;

(2) $\bot \vdash_X p$;

(3) $p \vdash_X q \wedge r$ if and only if $p \vdash_X q$ and $p \vdash_X r$;

(4) $p \vee q \vdash_X r$ if and only if $p \vdash_X r$ and $q \vdash_X r$;

(5) $p \wedge q \vdash_X r$ if and only if $p \vdash_X \Rightarrow r$;

(6) $p \vdash_X \forall_{y \in B}\, \psi(y)$ if and only if $p \vdash_{X \cup \{y\}} \psi(y)$;

(7) $\exists_{y \in B}\, \psi(y) \vdash_X p$ if and only if $\psi(y) \vdash_{X \cup \{y\}} p$.

In stating the following axioms, we write

$$\vdash_X p \quad \text{for } \top \vdash_X p; \qquad \vdash p \quad \text{for } \top \vdash_{\emptyset} p.$$

Nonlogical Axioms

(a) *Comprehension*

$$\vdash_X \forall_{x \in A}\, (x \in \{x \in A \mid \varphi(x)\} \Leftrightarrow \varphi(x)).$$

(b) *Extensionality*

$$\vdash \forall_{u \in PA}\, \forall_{v \in PA}\, (\forall_{x \in A}(x \in u \Leftrightarrow x \in v) \Rightarrow u = v);$$

$$\vdash \forall_{s \in \Omega}\, \forall_{t \in \Omega}\, ((s \Leftrightarrow t) \Rightarrow s = t).$$

(c) *Products*

$$\vdash \forall_{x \in 1}\, x = *;$$

$$\vdash \forall_{z \in A \times B}\, \exists_{x \in A}\, \exists_{y \in B} z = \langle x, y \rangle;$$

$$\vdash \forall_{x \in A}\, \forall_{x' \in A}\, \forall_{y \in B}\, \forall_{y' \in B}\, (\langle x, y \rangle = \langle x', y' \rangle \Rightarrow (x = x' \wedge y = y')).$$

(d) *Peano Axioms*

$$\vdash \forall_{x \in N}\, (Sx = 0 \Rightarrow \bot);$$

$$\vdash \forall_{x \in N}\, \forall_{y \in N}\, (Sx = Sy \Rightarrow x = y);$$

$$\vdash \forall_{u \in PN}\, ((0 \in u \wedge \forall_{x \in N}(x \in u \Rightarrow Sx \in u)) \Rightarrow \forall_{y \in N}\, y \in u).$$

Remarks. The logical axioms are somewhat non-standard (see e.g. [13, p. 98]). Following Lawvere, these axioms were obtained by considering $\wedge$, $\Rightarrow$, etc. as adjoint functors (see e.g. [15]). Logical axioms (2), (4) and (7) are redundant if $\bot$, $\vee$ and $\exists$ are suitably defined. We have considered an alternative system elsewhere in which all logical symbols are defined in terms of equality.

Classical type theory differs from intuitionist type theory by having one additional axiom, the so-called *Boolean axiom:*

$$\vdash \forall_{t \in \Omega}\, (t \vee \neg t).$$

This axiom is rejected by intuitionists, because its presence permits non-constructive existence proofs. That such non-constructive existence proofs are avoided in the absence of the Boolean axiom, is the *fundamental theorem* of intuitionist type theory:

Theorem 1.3. *In* $\mathscr{L}_1$ *we have*

(1) *not* $\vdash \perp$;

(2) *if* $\vdash p \vee q$, *then* $\vdash p$ *or* $\vdash q$;

(3) *if* $\vdash \exists_{x \in A}\, \varphi(x)$, *then* $\vdash \varphi(a)$ *for some closed term a of type A.*

Remark. There is only one closed term of type 1, to wit $*$. The closed terms of type N are *numerals,* namely 0, $S0$, $SS0$, etc. The closed terms of type PA are comprehension terms of the form $\{x \in A \mid \varphi(x)\}$.

The reader who is willing to accept Theorem 1.3 without proof may immediately turn to Section 3, Section 2 being concerned with the proof of Theorem 1.3 using the notion of realizability. Another proof will be given in Section 6.

It should be pointed out that the language $\mathscr{L}_1$ has many variants. Originally we had considered a language $\mathscr{L}_2$ which admits also projection symbols π and π' and requires a change in the statement of the product axioms. In Section 2 we shall meet a language $\mathscr{L}_0$ which lacks comprehension terms and requires a change in the statement of the comprehension axiom. Of course, assertion (3) of the fundamental theorem will not hold for $\mathscr{L}_0$.

Finally, we remark that Theorem 1.3 establishes the consistency of the other versions of type theory as well. Extending Gödel's double negation translation (see [13] or [26]) we also obtain the consistency of classical type theory (with comprehension and extensionality). Now, by Gödel's Incompleteness Theorem, the above consistency proof must use proof-theoretical methods not available in type theory. Already in first-order arithmetic a consistency proof requires transfinite induction on ε_0 (induction on ordinals $<\varepsilon_0$ being in fact derivable). So, to formalize the above consistency proof via the fundamental theorem (whose proof uses realizability) presumably requires transfinite induction on quite large ordinals (see [4]).

Suppose next that we allow a variable z of type C as a "parameter". In other words, we study the language $\mathscr{L}_1(z)$ whose closed formulas are the formulas of $\mathscr{L}_1$ which may contain free occurrences of z but of no other variable. An examination of the proof of the fundamental theorem will show that it remains valid when $\vdash$ is replaced by $\vdash_z$, as long as $C = \Omega$ or PB. (Here, and elsewhere, we write $\vdash_z$ for $\vdash_{\{z\}}$.) In particular, (3) then becomes:

$$(3_z) \qquad \text{if } \vdash_z \exists_{x \in A}\, \varphi(z,x), \text{ then } \vdash_z \varphi(z, \alpha(z))$$

for some term $\alpha(z)$ of type A.

As a consequence of (3_z) we obtain the following, which is also known as the Uniformity Property [27].

Theorem 1.4. *In $\mathscr{L}_1$, if $C=\Omega$ or PB, then $\vdash \forall_{z\in C}\ \exists_{x\in N}\ \varphi(z,x)$ implies $\vdash \exists_{x\in N}\ \forall_{z\in C}\ \varphi(z,x)$.*

Proof. From the hypothesis we infer that $\vdash_z \exists_{x\in N}\ \varphi(z,x)$, hence by (3_z) that $\vdash_z \varphi(z,\alpha(z))$, whence $\vdash \forall_{z\in C}\ \varphi(z,\alpha(z))$, from which the conclusion follows, since $\alpha(z)$ must be a numeral.

We should point out that there is no hope of extending (3_z) to parameters z of arbitrary type C, at least as long as we stick to the language $\mathscr{L}_1$. For example, when $C=N$ and $A=N$ we can take $\varphi(z,x)$ to mean that $z^2=x$, but there is no way of expressing the squaring function by a term $\alpha(z)$ of $\mathscr{L}_1$.

2. Proof of fundamental theorem

The purpose of this section is to prove the fundamental theorem of intuitionist type theory. It is convenient to do this for a language $\mathscr{L}$, which is equivalent to $\mathscr{L}_1$, but has fewer names and avoids nested comprehension terms, yet still has enough names to witness all existence theorems. On the way to introducing $\mathscr{L}_1$, we shall also mention another language $\mathscr{L}_0$, which is still equivalent to $\mathscr{L}_1$, but has no comprehension terms at all.

We shall prove the fundamental theorem for $\mathscr{L}$ using the profound Kleene–Friedman method of realizability, as developed for related languages in [7, 23, 25]. Roughly speaking, the idea is to define a predicate $R(p)$, meaning "formula p is realizable", by induction on the complexity of p, and then to prove a Soundness Theorem: if $\vdash p$ then $R(p)$, from which the fundamental theorem follows.

The difficulty is that the natural definition of $R(a\in\{x\in A \mid \varphi(x)\})$ should be $R(\varphi(a))$; however, $\varphi(a)$ may be more complicated than the original formula! To overcome this difficulty, which is inherent in type theory, we follow Friedman in splitting each comprehension term into many "indexed" comprehension terms. This gives rise to yet another language $\mathscr{L}^+$, and it is in $\mathscr{L}^+$ rather than $\mathscr{L}$ that realizability is defined. Thus we are dealing with four languages:

$$\begin{tikzcd}[arrows=-]
 & & \mathscr{L}_1 \\
\mathscr{L}^+ \arrow[dr] & & \mathscr{L} \arrow[u] \arrow[dl] \\
 & \mathscr{L}_0 &
\end{tikzcd}$$

We believe that our treatment somewhat simplifies the details in the cited papers.

A property of $\mathscr{L}_1$

We begin by proving a property of $\mathscr{L}_1$. We shall write $p\equiv\varphi(X)$, where X is a set of variables, to indicate that all free variables occurring in p are elements of X.

Proposition 2.1. *For any formula $p \equiv \varphi(X)$ in $\mathscr{L}_1$ there is a formula p^τ which contains no comprehension terms and no subformulas $q \in \alpha$ with q of type Ω unless q is a variable. Moreover, $\vdash_X p^\tau \Leftrightarrow p$.*

The reason for the restriction on subformulas is that in the intended interpretations of $\mathscr{L}_1$ any q of type Ω corresponds to $\{x \in 1 \mid q\}$ under the isomorphism $\Omega \cong P1$.

Proof. We shall call "forbidden" any occurrence of a comprehension term $\{x \in A \mid \psi(x)\}$ and any occurrence of a subformula $q \in \alpha$ with q of type Ω not a variable.

Given $p \equiv \varphi(X)$, we can find a formula p' containing fewer forbidden occurrences than p such that $\vdash_X p' \Leftrightarrow p$.

In case p contains $\{x \in A \mid \psi(x)\}$, but $\psi(x)$ contains no forbidden occurrence, say $p \equiv \chi\,(\{x \in A \mid \psi(x)\})$, we take

$$p' \equiv \exists_{u \in PA}\,(\chi(u) \wedge \forall_{x \in A}\,(x \in u \Leftrightarrow \psi(x)).$$

In case p contains $q \in \alpha$, q of type Ω not a variable, where q contains no forbidden occurrence, say $p \equiv \chi(q \in \alpha)$, we take

$$p' \equiv \exists_{t \in \Omega}\,(\chi(t \in \alpha) \wedge t \Leftrightarrow q).$$

It is clear that $\vdash_X p' \Leftrightarrow p$ in both cases. We may obtain p^τ from p by eliminating the forbidden occurrences, one at a time, in some systematic way.

The language $\mathscr{L}_0$

The language $\mathscr{L}_1$ has, in some sense, more names than necessary. We shall construct a language $\mathscr{L}_0$ which has too few names. It is like $\mathscr{L}_1$ except that it lacks comprehension terms $\{x \in A \mid \psi(x)\}$ and $a \in \alpha$ with a of type Ω not a variable. Thus formation rule (5) is deleted and rule (4) is restricted in case $A = \Omega$: $a \in \alpha$ will be a term of type Ω only if a is a variable. The axioms for $\mathscr{L}_0$ are the same as those for $\mathscr{L}_1$, except that the comprehension scheme is replaced by the following:

$$\vdash_X \exists_{u \in PA}\, \forall_{x \in A}\,(x \in u \Leftrightarrow \varphi(x)).$$

Proposition 2.2. *$\mathscr{L}_1$ is a conservative extension of $\mathscr{L}_0$.*

Proof. Suppose p is a formula of $\mathscr{L}_0$, all free variables in p are elements of X, and $\vdash_X p$ in $\mathscr{L}_1$. We claim that $\vdash_X p$ in $\mathscr{L}_0$.

Suppose the proof of p in $\mathscr{L}_1$ makes use of an instance q of the comprehension scheme, say

$$q \equiv \forall_{x \in A}\,(x \in \{x \in A \mid \varphi(x)\} \Leftrightarrow \varphi(x)).$$

Then there is a proof of $q \vdash_X p$ not using the axiom q. Replacing $\{x \in A \mid \varphi(x)\}$ by a variable u of type PA, we have

$$\forall_{x \in A} (x \in u \Leftrightarrow \varphi(x)) \vdash_{X \cup \{u\}} p,$$

hence $q' \vdash_X p$, where

$$q' \equiv \exists_{u \in PA} \forall_{x \in A} (x \in u \Leftrightarrow \varphi(x)).$$

In this manner we may eliminate one instance of the comprehension scheme after another, until we obtain a proof of $\vdash_X p$ in a language whose formation rules are those of $\mathscr{L}_1$, but whose axioms are those of $\mathscr{L}_0$.

The proof of $\vdash_X p$ may still contain comprehension terms such as $\{x \in A \mid \varphi(x)\}$. Replacing these by variables of type PA, we obtain a proof of $\vdash_{X \cup U} p$ in $\mathscr{L}_0$, where U is a set of variables of type PA for some types A. Can we deduce from this that $\vdash_X p$ in $\mathscr{L}_0$? Yes, in view of the following observation:

If $\vdash_{X \cup \{u\}} p$ in $\mathscr{L}_0$, where u is a variable of type PA, then $\vdash_X p$ in $\mathscr{L}_0$.

Indeed, given $\vdash_{X \cup \{u\}} p$, we have a fortiori

$$\forall_{x \in A} (x \in u \Leftrightarrow \varphi(x)) \vdash_{X \cup \{u\}} p,$$

hence

$$\exists_{u \in PA} \forall_{x \in A} (x \in u \Leftrightarrow \varphi(x)) \vdash_X p.$$

Distinguishing provability in $\mathscr{L}_1$ from provability in $\mathscr{L}_0$ by writing $\vdash^1_X$ and $\vdash^0_X$ respectively, we have the following, where p and q are assumed to be formulas whose free variables are in X.

Corollary 2.3. (0) *For p in $\mathscr{L}_0$, $p^\tau \equiv p$;*

(1) *For p in $\mathscr{L}_1$, $\vdash^1_X p^\tau \Leftrightarrow p$;*

(2) *For p in $\mathscr{L}_0$, $\vdash^0_X p$ if and only if $\vdash^1_X p$;*

(3) *For p in $\mathscr{L}_1$, if $\vdash^1_X p$ then $\vdash^0_X p^\tau$.*

Proof. (0) and (1) follow immediately from Proposition 2.1.

The direct implication in (2) holds, because the only new axiom in $\mathscr{L}_0$, the new form of the comprehension axiom, is derivable in $\mathscr{L}_1$. The converse implication in (2) holds by Proposition 2.2.

To prove (3), suppose $\vdash^1_X p$. Then, by (1), $\vdash^1_X p^\tau$, hence, by (2), $\vdash^0_X p^\tau$.

The language $\mathscr{L}$

Of course $\mathscr{L}_0$ does not have enough names to assert the fundamental theorem. However, we shall consider a language $\mathscr{L}$ intermediate between $\mathscr{L}_0$ and $\mathscr{L}_1$. As regards formation rules, $\mathscr{L}$ differs from $\mathscr{L}_1$ in two respects:

First, $\{x \in A \mid \varphi(x)\}$ is only admitted when $\varphi(x)$ is in $\mathscr{L}_0$ and contains no free variables other than x.

Secondly, if α is of type $P\Omega$, the formula $p \in \alpha$ is only admitted when p is a closed formula of $\mathscr{L}_0$ or a variable.

The first restriction is necessary if we don't want $\mathscr{L}$ to contain nested comprehension terms. Even free variables inside a comprehension term may lead to nested comprehension terms after substituting comprehension terms for the variables. The reason for the second restriction is that, in our intended interpretation for $\mathscr{L}$, Ω will be isomorphic to $P1$ and p of type Ω will correspond to $\{x \in 1 \mid p\}$ of type $P1$, which comes under the first restriction.

The axioms for $\mathscr{L}$ are the same as those for $\mathscr{L}_1$ or $\mathscr{L}_0$, except that there are now two comprehension schemes:

$$\vdash_X \exists_{u \in PA} \forall_{x \in A} (x \in u \Leftrightarrow \varphi(x)),$$

with $\varphi(x)$ in $\mathscr{L}_0$, and

$$\vdash \forall_{x \in A} (x \in \{x \in A \mid \varphi(x)\} \Leftrightarrow \varphi(x)),$$

as in $\mathscr{L}_1$ provided $\varphi(x)$ is in $\mathscr{L}_0$ and contains no free variables other than x.

From now on $\vdash$ denotes provability in $\mathscr{L}$, unless otherwise specified.

The language $\mathscr{L}^+$

The language $\mathscr{L}^+$ is an extension of $\mathscr{L}_0$ like $\mathscr{L}$, but obtained by "indexing" the comprehension terms

$$c_\varphi \equiv \{x \in A \mid \varphi(x)\}.$$

Thus, we shall replace c_φ by c_φ^V, where the index denotes a subset V of $[A]$ to be defined presently. In view of the intended isomorphism between Ω and $P1$, we shall also index closed terms p of type Ω in the context $p \in \alpha$.

The sets $[A]$ will turn out to consist of all closed terms of type A in $\mathscr{L}^+$. They are defined by induction on the construction of A as follows.

(i) $[1]$ is the set consisting of the symbol $*$.

(ii) $[N]$ is the set of all numerals $\bar{n} \equiv S^n 0$, which we may as well identify with the set $\mathbf{N}$ of natural numbers.

(iii) $[\Omega]$ is the disjoint union of the set of closed formulas of $\mathscr{L}$ and the set of theorems of $\mathscr{L}$. That is, $[\Omega]$ consists of all p_0 and all q_1, where p is any closed formula of $\mathscr{L}$ and q is any theorem of $\mathscr{L}$.

(iv) $[A \times B]$ consists of all $\langle a, b \rangle$ where $a \in [A]$ and $b \in [B]$.

(v) $[PA]$ is the set of all c_φ^V, where $c_\varphi \equiv \{x \in A \mid \varphi(x)\}$ is a comprehension term in $\mathscr{L}$ and V is a subset of $[A]$ satisfying two conditions:

(a) if $a \in V$, then $\vdash \varphi(a^-)$;

(b) if $a \in V$ and $a \sim a'$, then $a' \in V$.

Here a^- is the closed term of type A in $\mathscr{L}$ obtained by removing all indices. Thus

$$(c_\varphi^V)^- \equiv c_\varphi, \quad p_i^- \equiv p, \quad \langle c_\varphi^V, p_i \rangle^- \equiv \langle c_\varphi, p \rangle,$$

etc. Moreover, $\sim$ is an equivalence relation between elements of $[A]$ defined as follows:

(i) $* \sim *$.

(ii) For $\bar{m}$, $\bar{n} \in [N]$, $\bar{m} \sim \bar{n}$ if and only if $\bar{m} \equiv \bar{n}$, that is, $m = n$.

(iii) For p_i, $q_j \in [\Omega]$, $p_i \sim q_j$ if and only if $i = j$ and $\vdash p \Leftrightarrow q$.

(iv) For a, $a' \in [A]$ and b, $b' \in [B]$, $\langle a, b\rangle \sim \langle a', b'\rangle$ if and only if $a \sim a'$ and $b \sim b'$.

(v) For c_φ^V, $c_\psi^W \in [PA]$, $c_\varphi^V \sim c_\psi^W$ if and only if $V = W$ and $\vdash c_\varphi = c_\psi$.

We define the sets $|A|$ of *terms* of $\mathscr{L}^+$ of type A as follows:

(i) $|1|$ consists of $*$ and all variables of type 1.

(ii) $|N|$ contains 0, all variables of type N, and is closed under S, that is, if $n \in |N|$, then $Sn \in |N|$.

(iii) $|\Omega|$ consists of all elements of $[\Omega]$ and all variables of type Ω.

(iv) $|A \times B|$ consists of all $\langle a, b\rangle$, where $a \in |A|$ and $b \in |B|$, and all variables of type $A \times B$.

(v) $|PA|$ consists of all elements of $[PA]$ and all variables of type PA.

Note that $|A|$ contains $[A]$ and all variables of type A; but $|N|$ is also closed under S and $|A \times B|$ is also closed under pairing. $[A]$ is now the set of closed terms of type A.

Formulas in $\mathscr{L}^+$ will not be identified with terms of type Ω, but are defined as follows:

(i) $\top$ and $\bot$ are formulas;

(ii) if p and q are formulas, then so are $p \wedge q$, $p \vee q$, $p \Rightarrow q$;

(iii) if $\varphi(x)$ is a formula, with x a variable of type A, then $\forall_{x \in A}\, \varphi(x)$ and $\exists_{x \in A}\, \varphi(x)$ are formulas;

(iv) if a and α are terms of types A and PA respectively, then $a \in \alpha$ is a formula;

(v) all terms of type Ω are formulas.

There is an unexpected abundance of formulas. For example, T, T_0, T_1, $\bot$, $\bot_0$ are formulas, although $\bot_1$ is not. Indeed, if $\mathscr{L}$ is to be contained in $\mathscr{L}^+$, $\top$, $\bot$ and all variables of type Ω have to be formulas. Moreover, if we want to allow substitution of terms of type Ω for variables of type Ω, we must admit p_i as a formula in $\mathscr{L}^+$, for any closed formula p of $\mathscr{L}$. We could reduce the number of formulas in $\mathscr{L}^+$ somewhat by identifying p_0 with p, but we shall refrain from doing so.

We may extend the mapping $^-: \mathscr{L}^+ \to \mathscr{L}$ to all terms and formulas of $\mathscr{L}^+$ in an obvious way: erase all indices.

$\mathscr{L}^+$ will not be made into a deductive system. Instead, we introduce the notion of *realizability* in $\mathscr{L}^+$.

Realizability

Definition 2.4. We define $R(p)$ for closed formulas p of $\mathscr{L}^+$ as follows:

$R(\top)$; not $R(\bot)$;

$R(p \wedge q)$ if and only if $R(p)$ and $R(q)$;

$R(p \Rightarrow q)$ if and only if, if $R(p)$ and $\vdash p^-$, then $R(q)$;

$R(p \vee q)$ if and only if, either $R(p)$ and $\vdash p^-$, or $R(q)$ and $\vdash q^-$;

$R(\forall_{x\in A}\ \varphi(x))$ if and only if $R(\varphi(a))$ for all $a\in[A]$;
$R(\exists_{x\in A}\ \varphi(x))$ if and only if $R(\varphi(a))$ and $\vdash\varphi(a)^-$ for some $a\in[A]$;
$R(a\in c_\varphi^V)$ if and only if $a\in V$;
$R(p_1)$; not $R(p_0)$.

For an open formula $\varphi(x_1,\dots,x_n)$, $R(\varphi(x_1,\dots,x_n))$ shall mean the same as $R(\forall_{x_1\in A_1}\cdots\forall_{x_n\in A_n}\varphi(x_1,\dots,x_n))$.

Proposition 2.5. *If $a\sim a'$, then*

(I) $\vdash a^-=a'^-$;
(II) $R(a=a')$;
(III) $\beta(a)\sim\beta(a')$ for any term $\beta(x)$ of $\mathcal{L}^-$;
(IV) $R(\varphi(a)\Leftrightarrow\varphi(a'))$ for any formula $\varphi(x)$ of $\mathcal{L}^+$.

Proof. (I) This is immediate from the inductive definition of $\sim$. For example, when $\langle a,b\rangle\sim\langle a',b'\rangle$, we have $a\sim a'$ and $b\sim b'$. By inductional assumption, we infer that $\vdash a^-=a'^-$ and $\vdash b^-=b'^-$, hence that $\vdash\langle a,b\rangle^-=\langle a',b'\rangle^-$.

(II) We wish to show that $R(\forall_{u\in PA}(a\in u\Leftrightarrow a'\in u))$, that is, for each $c_\varphi^V\in[PA]$, that $R(a\in c_\varphi^V\Rightarrow a'\in c_\varphi^V)$, and the same for the converse implication. This follows immediately from the fact that V is closed under $\sim$.

(III) We proceed by induction on the length of $\beta(x)$. If $\beta(x)$ does not contain x or if $\beta(x)\equiv x$, there is nothing to prove. If $\beta(x)\equiv S\beta'(x)$ or $\beta(x)\equiv\langle\beta'(x),\beta''(x)\rangle$, we use the inductional assumption on $\beta'(x)$ and $\beta''(x)$.

Note that the terms of type Ω in $\mathcal{L}^+$ are elements of $[\Omega]$ or variables of type Ω. For example, $x\in c_\varphi^V$, though a formula in $\mathcal{L}^+$, is not a term of type Ω in $\mathcal{L}^+$, whereas $(x\in c_\varphi^V)^-\equiv x\in c_\varphi$ is a term of type Ω in $\mathcal{L}$.

(IV) We shall prove that $R(\varphi(a)\Rightarrow\varphi(a'))$, that is, given $R(\varphi(a))$ and $\vdash\varphi(a)^-$, we shall show that $R(\varphi(a'))$. We proceed by induction on the complexity of $\varphi(x)$. By this we shall mean the number of occurrences of $\wedge$, $\vee$, $\Rightarrow$, $\forall$ and $\exists$, provided c_ψ^W and p_i are regarded as "opaque": their complexity is zero, even if $\psi(x)$ or p should contain one of the symbols which are being counted.

The proof will consist of an examination of the following nine cases:

(1) If $\varphi(x)$ does not contain x, there is nothing to prove.

(2) If $\varphi(x)\equiv\varphi_1(x)\wedge\varphi_2(x)$, we are given that $R(\varphi_i(a))$ and $\vdash\varphi_i(a)^-$ for $i=1$, 2. By inductional assumption we may infer that $R(\varphi_1(a'))$ and $R(\varphi_2(a'))$, hence that $R(\varphi_1(a')\wedge\varphi_2(a'))$.

(3) If $\varphi(x)\equiv\varphi_1(x)\vee\varphi_2(x)$, we are given that $R(\varphi_1(a))$ and $\vdash\varphi_i(a)^-$ for $i=1$ or $i=2$, say the former. Then, by inductional assumption, we may infer that $R(\varphi_1(a'))$ and, by (I), that $\vdash\varphi_1(a')^-$, hence that $R(\varphi_1(a')\vee\varphi_2(a'))$.

(4) If $\varphi(x)\equiv\varphi_1(x)\Rightarrow\varphi_2(x)$, we distinguish three cases.

Case (i): not $\vdash\varphi_1(a)^-$. Then also not $\vdash\varphi_1(a')^-$, by (I). It follows trivially that $R(\varphi_1(a')\Rightarrow\varphi_2(a'))$.

Case (ii): $\vdash\varphi_1(a)^-$ but not $R(\varphi_1(a))$. Then $\vdash\varphi_1(a')^-$, by (I), and not $R(\varphi_1(a'))$, by inductional assumption. It follows trivially that $R(\varphi_1(a')\Rightarrow\varphi_2(a'))$.

Case (iii): $\vdash \varphi_1(a)^-$ and $R(\varphi_1(a))$. Since we are given that $R(\varphi_1(a) \Rightarrow \varphi_2(a))$, it follows that $R(\varphi_2(a))$. Since we are given that $\vdash \varphi_1(a)^- \Rightarrow \varphi_2(a)^-$, it also follows that $\vdash \varphi_2(a)^-$. By inductional assumption we may infer that $R(\varphi_2(a'))$, hence that $R(\varphi_1(a') \Rightarrow \varphi_2(a'))$.

(5) If $\varphi(x) \equiv \forall_{y \in B}\, \psi(x,y)$, we are given that, for all $b \in [B]$, $R(\psi(a,b))$ and $\vdash \psi(a,b)^-$. Now $\psi(x,b)$ is less complex than $\forall_{y \in B}\, \psi(x,y)$, hence we may apply inductional assumption and obtain $R(\psi(a',b))$. This being so for each $b \in [B]$, we infer that $R(\forall_{y \in B}\, \psi(a',b))$.

(6) If $\varphi(x) \equiv \exists_{y \in B}\, \psi(x,y)$, we are given that, for some $b \in [B]$, $R(\psi(a,b))$ and $\vdash \psi(a,b)^-$. As above, we infer that $R(\psi(a',b))$. But also $\vdash \psi(a',b)^-$, by (I), hence $R(\exists_{y \in B}\, \psi(a',y))$.

(7) If $\varphi(x) \equiv x$ (of type Ω), we make take $a \equiv p_i$ and $a' \equiv q_j$. Since $a \sim b$, we have $i = j$ and $\vdash p \Leftrightarrow q$. Now we are given that $R(p_i)$, whence $i = 1$. Since $j = i = 1$, we surely have $R(q_j)$.

It only remains to discuss $\varphi(x) \equiv \beta(x) \in \gamma(x)$, where $\beta(x)$ is of type B and $\gamma(x)$ of type PB. We have two possibilities for $\gamma(x)$, namely $\gamma(x) \equiv c_\psi^W$ and $\gamma(x) \equiv x$.

(8) If $\varphi(x) \equiv \beta(x) \in c_\psi^W$, we are given that $R(\varphi(a))$, that is, $\beta(a) \in W$, and want to deduce that $R(\varphi(a'))$, that is, $\beta(a') \in W$. This follows from (III) and the fact that W is closed under $\sim$.

(9) If $\varphi(x) \equiv \beta(x) \in x$, x of type PB, we may take $a \equiv c_\psi^W$ and $a' \equiv c_{\psi'}^W$, where $\vdash c_\psi = c_{\psi'}$. We are given that $R(\varphi(a))$, that is, $\beta(a) \in W$, and want to deduce that $R(\varphi(a'))$, that is, $\beta(a') \in W$. This follows as above.

Soundness

We shall now define a mapping $\mathscr{L} \to \mathscr{L}^+$. If q is any formula of $\mathscr{L}$, we define a formula q^+ in $\mathscr{L}^+$ by replacing every occurrence of c_φ by $c_\varphi^{V(\varphi)}$, where

$$V(\varphi) = \text{set of all } a \in [A] \text{ such that } R(\varphi(a)) \text{ and } \vdash \varphi(a^-),$$

and every occurrence of a closed term p of type Ω in $\mathscr{L}_0$ in the context $p \in \alpha$ by $p_{i(p)}$, where

$$i(p) = \begin{cases} 1 & \text{if } \vdash p \text{ and } R(p), \\ 0 & \text{otherwise.} \end{cases}$$

Recall that there are no nested comprehension terms.

Note that $i(p)$ corresponds to $V(\hat{p})$ under the intended isomorphism between Ω and $P1$ which replaces p by $\hat{p} \equiv \{x \in 1 \mid p\}$, except that we have written $i(p) = 0$ or 1 where $V(\hat{p}) = \emptyset$ or $\{*\}$.

It remains to check that $V(\varphi)$ satisfies the conditions on indices:

(a) if $a \in V(\varphi)$ then $\vdash \varphi(a^-)$;

(b) if $a \in V(\varphi)$ and $a \sim a'$ then $a' \in V(\varphi)$.

Here (a) is immediate from the definition of $V(\varphi)$, and (b) asserts:

if $R(\varphi(a))$ and $\vdash \varphi(a^-)$ and $a \sim a'$ then $R(\varphi(a'))$ and $\vdash \varphi(a'^-)$, which follows immediately from Proposition 2.5 (IV) and (I).

Proposition 2.6 (Soundness). *If $p \vdash_X q$ in $\mathscr{L}$, then $R(p^+ \Rightarrow q^+)$. In particular, if $\vdash p$, then $R(p^+)$.*

Proof. We proceed by induction on the proof of $p \vdash_X q$ to show that $R(p^+ \Rightarrow q^+)$. For example, if $p \equiv \varphi(x)$ and $q \equiv \psi(x)$, we want to show that, for all $a \in [A]$, $R(\varphi^+(a) \Rightarrow \psi^+(a))$.

If the last step in the proof involves a structural or logical rule, there is no difficulty. For example, the last step may have been:

$$\frac{\varphi(x,y) \vdash_{\{x,y\}} \psi(x,y)}{\varphi(x,b) \vdash_{\{x\}} \psi(x,b)},$$

where $b \in [B]$. By inductional assumption, we have $R(\varphi^+(a,b) \Rightarrow \psi^+(a,b))$ for all $a \in [A]$ and $b \in [B]$, and this is what we want to show for a particular b.

To look at another example, suppose the last step in the proof was:

$$\frac{\varphi(x) \wedge \psi(x) \vdash_{\{x\}} \chi(x)}{\varphi(x) \vdash_{\{x\}} \psi(x) \Rightarrow \chi(x)}.$$

We are given that $R(\varphi^+(a))$ and $\vdash \varphi(a^-)$ and want to show that $R(\psi^+(a) \Rightarrow \chi^+(a))$. So let $R(\psi^+(a))$ and $\vdash \psi(a^-)$ be given, we want to show that $R(\chi^+(a))$. Now $R(\varphi^+(a) \wedge \psi^+(a))$ and $\vdash \varphi(a^-) \wedge \psi(a^-)$, hence $R(\chi^-(a))$, in view of the inductional assumption that $R((\varphi^+(a) \wedge \psi^+(a)) \Rightarrow \chi^+(a))$.

We shall go through the various nonlogical axioms p to show that $R(p^+)$.

(a) *Comprehension:* to realize $(\forall_{x \in A}(x \in c_\varphi \Leftrightarrow \varphi(x)))^+$, i.e., since $\varphi^+(x) \equiv \varphi(x)$, that $\forall_{x \in A}(x \in c_\varphi^{V(\varphi)} \Leftrightarrow \varphi(x))$, we must realize, for all $a \in [A]$, $a \in c_\varphi^{V(\varphi)} \Leftrightarrow \varphi(a)$. This amounts to showing that $a \in V(\varphi)$ and $\vdash \varphi(a^-)$ implies $R(\varphi(a))$, and also that $R(\varphi(a))$ and $\vdash \varphi(a^-)$ implies $a \in V(\varphi)$. Both implications follow from the definition of $V(\varphi)$.

$\mathscr{L}$ also inherits the comprehension axiom of $\mathscr{L}_0$: for $\varphi(x)$ in $\mathscr{L}_0$, $\vdash_X \exists_{u \in PA} \forall_{x \in A}(x \in u \Leftrightarrow \varphi(x))$. To realize this, for instance in case $X = \{y\}$, we want to realize, for each $b \in [B]$, that $\exists_{u \in PA} \forall_{x \in A}(x \in u \Leftrightarrow \varphi(b,x))$. Writing $\varphi(b^-,x)^\tau \equiv \psi(x)$, it suffices to realize $\forall_{x \in A}(x \in c_\psi^W \Leftrightarrow \varphi(b,x))$ and to prove $\vdash \forall_{x \in A}(x \in c_\psi \Leftrightarrow \varphi(b^-,x))$. The first is easily checked if W is the set of all $a \in [A]$ such that $R(\varphi(b,a))$ and $\vdash \varphi(b^-,a^-)$, and the second follows from Proposition 2.1, which remains valid if $\mathscr{L}_1$ is replaced by $\mathscr{L}$.

(b) *Extensionality:* to realize $\forall_{u \in PA} \forall_{v \in PA} (\forall_{x \in A}(x \in u \Leftrightarrow x \in v) \Rightarrow u = v)$, we must realize, for each c_φ^V, $c_\psi^W \in [PA]$,

$$\forall_{x \in A} (x \in c_\varphi^V \Leftrightarrow x \in c_\psi^W) \Rightarrow c_\varphi^V = c_\psi^W.$$

so we assume

(i) $R(\forall_{x \in A} (x \in c_\varphi^V \Leftrightarrow x \in c_\psi^W))$, and

(ii) $\vdash \forall_{x \in A}(x \in c_\varphi \Leftrightarrow x \in c_\psi)$,

and want to show that $R(c_\varphi^V = c_\psi^W)$. Now, by (ii) and extensionality, $\vdash c_\varphi = c_\psi$. By (i), for each $a \in [A]$, $R(a \in c_\varphi^V \Leftrightarrow a \in c_\psi^W)$. In particular, from $a \in V$ and $\vdash \varphi(a^-)$ follows $a \in W$. Now $a \in V$ implies $\vdash \varphi(a^-)$, by the first condition on indices. Thus $V \subseteq W$, and similarly $W \subseteq V$. Since $\vdash c_\varphi = c_\psi$ and $V = W$, we have $c_\varphi^V \sim c_\psi^W$, hence $R(c_\varphi^V = c_\psi^W)$, by Proposition 2.5 (II).

Under the heading of extensionality we also have to realize $\forall_{s\in\Omega}\ \forall_{t\in\Omega}\ ((s \Leftrightarrow t) \Rightarrow s = t)$. Thus, for each $p_i,\ q_j \in [\Omega]$, we must show that $R((p_i \Leftrightarrow q_j) \Rightarrow p_i = q_j)$. So we assume $R(p_i \Leftrightarrow q_j)$ and $\vdash p \Leftrightarrow q$, and want to show that $R(p_i = q_i)$. We know that $\vdash p = q$ from extensionality. Suppose $i = 1$, then $R(p_i)$ and $\vdash p$, hence, from $R(p_i \Rightarrow q_j)$ we deduce $R(q_j)$, and so $j = 1$. Similarly $j = 1$ implies $i = 1$. Therefore, $i = j$, and so $p_i \sim q_j$, hence $R(p_i = q_j)$, by Proposition 2.5 (II).

(c) *Products:* to realize $\forall_{x\in 1}\ x = *$, it suffices to realize $* = *$, which follows from Proposition 2.5 (II).

To realize $\forall_{z\in A\times B}\ \exists_{x\in A}\ \exists_{y\in B}\ z = \langle x, y\rangle$, we take any $\langle a, b\rangle \in [A \times B]$ and need only realize $\langle a, b\rangle = \langle a, b\rangle$, which also follows from Proposition 2.5 (II), and prove that $\vdash \langle a^-, b^-\rangle = \langle a^-, b^-\rangle$, which is evident.

To realize

$$\forall_{x\in A}\ \forall_{x'\in A}\ \forall_{y\in B}\ \forall_{y'\in B}\ (\langle x, y\rangle = \langle x', y'\rangle \Rightarrow (x = x' \wedge y = y')),$$

we take any $a,\ a' \in [A]$ and any $b,\ b' \in [B]$, and assume that $R(\langle a, b\rangle = \langle a', b'\rangle)$ and that $\vdash \langle a^-, b^-\rangle = \langle a'^-, b'^-\rangle$. We want to show that $R(a = a')$ and $R(b = b')$. Now, for each $\gamma \in [P(A \times B)]$, we can realize $\langle a, b\rangle \in \gamma \Leftrightarrow \langle a', b'\rangle \in \gamma$, and we would like to realize, for each $\alpha \in PA$, that $a \in \alpha \Leftrightarrow a' \in \alpha$. Thus, given $R(a \in \alpha)$ and $\vdash a^- \in \alpha^-$, we want to show that $R(a' \in \alpha)$. We may take $\alpha \equiv c_\varphi^V$, then we are given that $a \in V$ and $\vdash \varphi(a^-)$ and want to show $a' \in V$. To this purpose take $\gamma \equiv c_\psi^W$, where

$$\psi(z) \equiv \exists_{x\in A}\ (\varphi(x) \wedge z = \langle x, b^-\rangle),$$

$$W = \text{set of all } \langle a, b\rangle \in [A \times B] \text{ such that } a \in V.$$

It is easily verified that W satisfies the conditions on indices since V does. Moreover, it is easily seen that the given pair $\langle a, b\rangle \in W$, since $a \in V$, and that $\vdash \langle a^-, b^-\rangle \in \gamma^-$, since $\vdash \varphi(a^-)$. Since we can realize that $\langle a, b\rangle \in c_\psi^W \Rightarrow \langle a', b'\rangle \in c_\psi^W$, it follows that $\langle a', b'\rangle \in W$, hence $a' \in V$.

(d) *Peano axioms:* to realize $\forall_{x\in N}\ (Sx = 0 \Rightarrow \bot)$, we may assume that, for a given numeral $\bar{n} = S^n 0$, $R(S\bar{n} = 0)$ and $\vdash S\bar{n} = 0$, and we want to show that $R(\bot)$, that is, we want to derive a contradiction. Now $R(S\bar{n} = 0)$ means that, for all $c_\varphi^V \in [PN]$, $R(S\bar{n} \in c_\varphi^V \Leftrightarrow 0 \in c_\varphi^V)$. In particular, $0 \in V$ and $\vdash \varphi(0)$ imply $S\bar{n} \in V$. We shall deduce a contradiction from this by a particular choice of c_φ^V. Take

$$\varphi(x) \equiv x = 0, \quad V = \{0\},$$

then V is easily seen to satisfy the conditions on indices. Moreover $S\bar{n} \in V$ if and only if $S\bar{n} \equiv 0$, which is false, while $0 \in V$ and $\vdash \varphi(0)$.

To realize $\forall_{x\in N}\ \forall_{y\in N}\ (Sx = Sy \Rightarrow x = y)$, we take any numerals $\bar{m}$ and $\bar{n}$ and want to realize $S\bar{m} = S\bar{n} \Rightarrow \bar{m} = \bar{n}$. So suppose $R(S\bar{m} = S\bar{n})$ and $\vdash S\bar{m} = S\bar{n}$, we want to infer that $R(\bar{m} = \bar{n})$. Thus we want to show that, for each $c_\varphi^V \in [PN]$, $R(\bar{m} \in c_\varphi^V \Rightarrow \bar{n} \in c_\varphi^V)$, and similarly for the converse. So, given $\bar{m} \in V$ and $\vdash \varphi(\bar{m})$, we want to show $\bar{n} \in V$.

Let

$$\varphi'(x) \equiv \exists_{y \in N}(\varphi(y) \wedge x = Sy),$$

$$V' = \text{set of all } S\bar{m} \text{ such that } \bar{m} \in V.$$

It is easily checked that V' satisfies the two conditions on indices because V does. Now, the given data translate into $S\bar{m} \in V'$ and $\vdash \varphi'(S\bar{m})$. Since $R(S\bar{m} \in c_{\varphi'}^{V'} \Rightarrow S\bar{n} \in c_{\varphi'}^{V'})$, this implies $S\bar{n} \in V'$, that is, $\bar{n} \in V$.

To realize the induction axiom, take $c_\varphi^V \in [PN]$ and assume that $R(0 \in c_\varphi^V)$, that is, $0 \in V$, and that $R(\forall_{x \in N}(x \in c_\varphi^V \Rightarrow Sx \in c_\varphi^V))$. We want to show that $R(\forall_{y \in N}\ y \in c_\varphi^V)$, that is, $\bar{n} \in V$ for all $\bar{n}$. Since $R(\bar{n} \in c_\varphi^V \Rightarrow S\bar{n} \in c_\varphi^V)$, we know that $\bar{n} \in V$ and $\vdash \varphi(\bar{n})$ implies $S\bar{n} \in V$, hence, in view of the first condition on indices, that $\bar{n} \in V$ implies $S\bar{n} \in V$. But also $0 \in V$, and so the desired result follows by induction.

This completes the proof of the Soundness Theorem.

Remark. It should perhaps be pointed out that, in spite of its name, the Soundness Theorèm shows that realizability is a rather paradoxical concept: even contradictions may be realized.

To see this, let p be any closed formula, then $\neg p$ is short for $p \Rightarrow \perp$. Therefore $R(\neg p)$ if and only if $R(p)$ and $\vdash p$ implies $R(\perp)$. Now $R(\perp)$ is false and, by the Soundness Theorem, $\vdash p$ implies $R(p)$. Thus $R(\neg p)$ if and only if not $\vdash p$. Suppose now p is any undecidable proposition, then not $\vdash p$ and not $\vdash \neg p$, hence $R(\neg p)$ and $R(\neg\neg p)$, and therefore $R(\neg p \wedge \neg\neg p)$.

As a corollary to the Soundness Theorem we obtain the fundamental theorem for $\mathscr{L}$.

Corollary 2.7. *In $\mathscr{L}$ we have*

(1) *not* $\vdash \perp$;

(2) *if* $\vdash p \vee q$, *then* $\vdash p$ *or* $\vdash q$;

(3) *if* $\vdash \exists_{x \in A}\, \varphi(x)$, *then* $\vdash \varphi(a^{\urcorner})$ *for some* $a \in [A]$.

Proof. For example, to show (3), suppose $\vdash \exists_{x \in A}\, \varphi(x)$. Then, by the Soundness Theorem, $R(\exists_{x \in A}\, \varphi^+(x))$, that is, $R(\varphi^+(a))$ and $\vdash \varphi(a^{\urcorner})$ for some $a \in [A]$.

Proof of fundamental theorem for $\mathscr{L}_1$

We are now in a position to prove the fundamental theorem for $\mathscr{L}_1$, namely Theorem 1.3. For example, we show:

(3) if $\vdash^1 \exists_{x \in A}\, \varphi(x)$, then $\vdash^1 \varphi(a')$ for some closed term a' of type A.

Suppose $\vdash^1 \exists_{x \in A}\ \varphi(x)$. Now, by Corollary 2.3(1), $\vdash^1_{\{x\}} \varphi(x) \Leftrightarrow \varphi^\tau(x)$, hence $\vdash^1 \exists_{x \in A}\, \varphi^\tau(x)$. Therefore, by Corollary 2.3 (2), $\vdash^0 \exists_{x \in A}\, \varphi^\tau(x)$. Since $\mathscr{L}_0$ is contained in $\mathscr{L}$, $\vdash \exists_{x \in A}\, \varphi^\tau(x)$. Therefore, by Corollary 2.7, $\vdash \varphi^\tau(a^{\urcorner})$ for some $a \in [A]$. Since $\mathscr{L}$ is contained in $\mathscr{L}_1$, $\vdash^1 \varphi^\tau(a^{\urcorner})$, and so $\vdash^1 \exists_{x \in A}(\varphi^\tau(x) \wedge x = a^{\urcorner})$. Recalling once more that $\vdash^1_{\{x\}} \varphi(x) \Leftrightarrow \varphi^\tau(x)$, we obtain $\vdash^1 \exists_{x \in A}(\varphi(x) \wedge x = a^{\urcorner})$, and so $\vdash^1 \varphi(a^{\urcorner})$.

Presence of a parameter

Let $C=\Omega$ or PB and suppose z is a variable of type C. We shall regard z as a *parameter,* that is, it will be constant throughout the present discussion; in particular, it will never become bound. $\mathscr{L}(z)$ will be the language whose closed terms and formulas may contain free occurrences of z but of no other variable. We shall examine what happens to Section 2 if $\mathscr{L}$ is replaced by $\mathscr{L}(z)$.

$\mathscr{L}(z)$ contains comprehension terms $\{x\in A\,|\,\varphi(z,x)\}$; but we must remember never to replace z by another comprehension term. The "closed" terms in $[A]$ may now contain occurrences of z. In particular, $[PA]$ will contain c_φ^V where $c_\varphi\equiv\{x\in A\,|\,\varphi(z,x)\}$. When $C=\Omega$, $[\Omega]$ will contain z_i; but, since not $\vdash_z z$, we must have $i=0$, and so $[\Omega]$ will contain only z_0. The proof of Proposition 2.5 remains valid.

In defining the mapping $\mathscr{L}(z)\to\mathscr{L}(z)^+$, we replace z by z_0 if $C=\Omega$. When $C=PB$, we shall replace z by $z^\emptyset$, which we define to be $c_{y\in z}^\emptyset$, where $c_{y\in z}\equiv\{y\in B\,|\,y\in z\}$. (The reader will check that in $c_{y\in z}^V$ the first condition on indices forces $V=\emptyset$.) The proof of the Soundness Theorem remains valid.

In view of these considerations, we have established (3_z) of Section 1, hence also the Uniformity Property for $\mathscr{L}_1$, that is, Theorem 1.4.

It is instructive to realize that the proof of the Soundness Theorem would not remain valid in the presence of a parameter of type $C=N$. Indeed, in realizing the induction axiom, we had to show that $R(\forall_{y\in N}\, y\in c_\varphi^V)$, which amounted to showing that V contains all closed terms $\bar{n}$ of type N. This was proved by induction on n; but there is no way of showing that the parameter z of type N is in V.

3. Representability

In this section we discuss how to represent recursive functions in type theory. Although most of the results are well-known, we emphasize those aspects useful in category-theoretical applications (see Section 4).

All the languages we are dealing with contain numerals $\bar{0},\bar{1},\bar{2},\ldots$. In our languages $\mathscr{L}_0$, $\mathscr{L}_1$ etc. $\bar{n}\equiv S^n0$.

A formula $\varphi(x_1,\ldots,x_k,y)$ *represents* a function $f:\mathbf{N}^k\to\mathbf{N}$ provided

(0) $f(m_1,\ldots,m_k)=n$ if and only if $\vdash\varphi(\bar{m}_1,\ldots,\bar{m}_k,\bar{n})$.

It follows from this that

(1) for each $m_1,\ldots,m_k\in\mathbf{N}$ there exists a unique n such that $\vdash\varphi(\bar{m}_1,\ldots,\bar{m}_k,\bar{n})$.

One usually imposes a condition somewhat related to (1) in addition to (0):

(2) for each $m_1,\ldots,m_k\in\mathbf{N}$, $\vdash\exists!_{y\in N}\,\varphi(\bar{m}_1,\ldots,\bar{m}_k,y)$.

For example, in [13] φ is then said to represent f "numeralwise". Sometimes one even imposes the following in addition to (0):

(3) $\vdash\forall_{x_1\in N}\cdots\forall_{x_k\in N}\,\exists!_{y\in N}\,\varphi(x_1,\ldots,x_k,y)$.

In the literature f is then said to be "strongly" representable.

For our present purposes, "intuitionist type theory" will mean $\mathscr{L}_1$ and "classical type theory" will mean $\mathscr{L}_1$ with the Boolean axiom added.

Remark 3.1. In intuitionist type theory, (2) implies (1).

Proof. For simplicity take $k=1$ and assume that $\vdash \exists!_{y\in N}\, \varphi(\bar{m},y)$. It follows from the fundamental theorem that $\vdash \varphi(\bar{m},\bar{n})$ for some $n \in \mathbf{N}$. Moreover, if also $\vdash \varphi(\bar{m},\overline{n'})$, then $\vdash \bar{n}=\overline{n'}$, hence $n=n'$ by consistency, which is also contained in the fundamental theorem.

The following was discovered by Verena Huber-Dyson [11,22].

Proposition 3.2. *In classical type theory, if f is represented by a formula φ satisfying* (2), *then it is also represented by a formula ψ satisfying* (3), *that is, numeralwise representability implies strong representability.*

Proof. For simplicity we take $k=1$. Suppose $\varphi(x,y)$ satisfies (2) and represents the function $f: \mathbf{N} \to \mathbf{N}$. Consider the formula $\varphi'(x,y)$ given by

$$\varphi'(x,y) \equiv \exists_{z\in N}\, \varphi(x,z) \Rightarrow \varphi(x,y).$$

Since $\vdash_x \exists_{z\in N}\ \varphi(x,z) \Rightarrow \exists_{y\in N}\ \varphi(x,y)$, we may infer by classical logic that $\vdash_x \exists_{y\in N}\, \varphi'(x,y)$, hence that $\vdash \forall_{x\in N}\, \exists_{y\in N}\, \varphi'(x,y)$. Now let

$$\psi(x,y) \equiv \varphi'(x,y) \wedge \forall_{z\in N}\, (\varphi'(x,z) \Rightarrow z \geq y).$$

Applying the least number principle to φ', we deduce that $\vdash \forall_{x\in N}\, \exists!_{y\in N}\, \psi(x,y)$. Thus ψ satisfies (3).

We claim that ψ still represents f. Indeed, $f(m)=n$ if and only if $\vdash \varphi(\bar{m},\bar{n})$. Since $\vdash \exists_{z\in N}\, \varphi(\bar{m},z)$ by (2), this is so if and only if $\vdash \varphi'(\bar{m},\bar{n})$. Finally, this is easily seen to be equivalent to $\vdash \psi(\bar{m},\bar{n})$. Indeed, suppose $\vdash \varphi'(\bar{m},\bar{n})$. From (2) we have $\vdash \forall_{z\in N}\, (\varphi(\bar{m},z) \Rightarrow z=\bar{n})$, hence $\vdash \forall_{z\in N}\, (\varphi'(\bar{m},z) \Rightarrow z \geq \bar{n})$, therefore $\vdash \psi(\bar{m},\bar{n})$.

In his famous paper of 1931, Gödel characterized the representable functions of classical type theory as follows:

Theorem 3.3. *In classical type theory:*

(i) *every recursive function is representable,*

(ii) *every representable function is recursive.*

For the proof see [8,13,24]. It appears in the proof that the formula φ representing a recursive function may be assumed to satisfy (2), hence (3).

Given a formula $\varphi(x,y)$ with x and y of type N, clearly (1) is a necessary and sufficient condition for the existence of a function $f: \mathbf{N} \to \mathbf{N}$ which is represented by φ. In intuitionist type theory (1) is implied by (2), in view of Remark 3.1, hence by (3). Unfortunately, in classical type theory neither (3) nor (2) will assure that $\varphi(x,y)$ represents a "total" function.

Example 3.4. Let p be an undecidable sentence and consider the formula $\varphi(x,y)$ given by

$$\varphi(x,y) \equiv (p \wedge y=0) \vee (\neg p \wedge y=1).$$

Clearly, classically we have $\vdash \forall_{x \in N^k} \exists !_{y \in N}\, \varphi(x,y)$. Since φ does not contain x, any function represented by φ would have to have constant value 0 or 1. In either case, we would be able to decide p. Thus $\varphi(x,y)$ satisfies (3) classically, hence (2), but does not represent a function.

The following is found in [13].

Proposition 3.5. *In intuitionist type theory, suppose for each* $m_1, \ldots, m_k \in \mathbf{N}$ *there exists* $n \in \mathbf{N}$ *such that* $\vdash \varphi(\bar{m}_1, \ldots, \bar{m}_k, \bar{n})$, *then there is a recursive function* $f(m_1, \ldots, m_k)$ *such that* $\vdash \varphi(\overline{m_1}, \ldots, \overline{m_k}, \overline{f(m_1, \ldots, m_k)})$ *for all* $m_1, \ldots, m_k \in \mathbf{N}$.

Proof. For simplicity we take $k=1$. We assume that for each $m \in \mathbf{N}$ we can find $n \in \mathbf{N}$ so that $\varphi(\bar{m}, \bar{n})$ has a proof, let us say with Gödel number p. *Write* $\ulcorner A \urcorner$ for the Gödel number of A and let $\mathrm{Proof}(q,p)$ assert that p is the Gödel number of a proof of a closed formula with Gödel number q. Thus, for each $m \in \mathbf{N}$, we can find n and $p \in \mathbf{N}$ so that $\mathrm{Proof}(\ulcorner \varphi(\bar{m}, \bar{n}) \urcorner, p)$. Now Cantor discovered a primitive recursive "pairing" function $\mathbf{N} \times \mathbf{N} \to \mathbf{N}$ whose converse is given by primitive recursive functions $(\ )_0$ and $(\ )_1 : \mathbf{N} \to \mathbf{N}$. Thus we can find $k \in \mathbf{N}$ such that $\mathrm{Proof}(\ulcorner \varphi(\bar{m}, \overline{(k)_0}) \urcorner, (k)_1)$. If "$\mu_k \cdots$" means "the least k such that ...", we may put

$$f(m) = (\mu_k\, \mathrm{Proof}(\ulcorner \varphi(\bar{m}, \overline{(k)_0}) \urcorner, (k)_1))_0.$$

f is easily seen to be a recursive function. Moreover, it follows that $\vdash \varphi(\bar{m}, \overline{f(m)})$.

Corollary 3.6. *In intuitionist type theory, suppose* $\varphi(x_1, \ldots, x_k, y)$ *satisfies* (1) (*or* (2) *or* (3)), *then* φ *represents a recursive function* f.

Proof. From Proposition 3.5 we have $\vdash \varphi(\bar{m}, \overline{f(m)})$. Therefore, $f(m)=n$ implies $\vdash \varphi(\bar{m}, \bar{n})$. Conversely, if $\vdash \varphi(\bar{m}, \bar{n})$, it follows from (1) that $\vdash \overline{f(m)} = \bar{n}$. Since intuitionist type theory is consistent by the fundamental theorem, $f(m)=n$.

On the other hand we have the following.

Proposition 3.7. *In intuitionist type theory, not every recursive function is representable by a formula satisfying* (3), *that is, strongly representable.*

Proof. Let E be the set of Gödel numbers of proofs of formulas of the form $\forall_{x \in N} \exists !_{y \in N}\, \varphi(x,y)$. For any $e \in E$ we thus have a formula $\varphi_e(x,y)$ such that $\vdash \forall_{x \in N} \exists !_{y \in N}\, \varphi_e(x,y)$. By Proposition 3.6, there is a recursive function f_e represented by φ_e.

Now E is recursively enumerable, so let h enumerate it. Consider the function g

such that $g(m) = f_{h(m)}(m) + 1$. Clearly g is computable, hence recursive. However, g is not representable by a formula φ such that $\vdash \forall_{x \in N} \exists !_{y \in N} \varphi(x, y)$. For, if it were, let $h(k)$ be the Gödel number of this theorem. Then $g(m) = f_{h(k)}(m)$ for all m, hence $f_{h(k)}(k) + 1 = g(k) = f_{h(k)}(k)$, a contradiction.

For the idea of the above proof see [13, Chapter 14, Example 10]. Unfortunately, we do not know an intrinsic characterization of the recursive functions represented by a formula $\varphi(x_1, \ldots, x_k, y)$ satisfying (3) in intuitionist type theory. It may be shown that these functions properly include a version of Gödel's Dialectica functionals of type N^N.

The question remains: which recursive functions are strongly representable in intuitionist type theory? For example, addition is strongly representable by the formula $\alpha(x, y, z)$ which asserts that $\langle x, y, z \rangle$ belongs to all $u \in P(N \times N \times N)$ such that

(i) $\forall_{x \in N} \langle x, 0, x \rangle \in u$,
(ii) $\forall_{x \in N} \forall_{y \in N} \forall_{z \in N} (\langle x, y, z \rangle \in u \Rightarrow \langle x, Sy, Sz \rangle \in u)$.

A bit of work has to be done to show that $\alpha(x, y, z)$ satisfies (3), that is, $\vdash \forall_{x \in N} \forall_{y \in N} \exists !_{z \in N} \alpha(x, y, z)$.

In the same way one can show that all primitive recursive functions are strongly representable in intuitionist type theory. Now every recursive function $f(m_1, \ldots, m_k)$ can be expressed in the form

$$f(m_1, \ldots, m_k) = h(\mu_n(g(m_1, \ldots, m_k, n) = 0))$$

in terms of primitive recursive functions $h(n)$ and $g(m_1, \ldots, m_k, n)$, provided

$$\forall_{m_1} \cdots \forall_{m_k} \exists_n \, g(m_1, \ldots, m_k, n) = 0.$$

We shall call f a *provably recursive* function if moreover g is strongly representable by a formula $\varphi(x_1, \ldots, x_k, y, z)$ so that

$$\vdash \forall_{x_1 \in N} \cdots \forall_{x_k \in N} \exists_{y \in N} \varphi(x_1, \ldots, x_k, y, 0).$$

Proposition 3.8. *If a function* $\mathbf{N}^k \to \mathbf{N}$ *is provably recursive in intuitionist type theory, then it is strongly representable.*

Proof. Assume f is provably recursive. For example, take $k = 1$ and let h be the identity function. Then $f(m)$ is the least n for which $g(m, n) = 0$, that is, $\vdash \varphi(\bar{m}, \bar{n}, 0)$, where $\vdash \forall_{x \in N} \forall_{y \in N} \exists !_{z \in N} \varphi(x, y, z)$ and $\vdash \forall_{x \in N} \exists_{y \in N} \varphi(x, y, 0)$. We claim that f is strongly representable by $\psi(x, y)$ where

$$\psi(x, y) \equiv \varphi(x, y, 0) \wedge \forall_{z \in N} (z < y \Rightarrow \neg \varphi(x, z, 0)).$$

We first verify that φ is *decidable*, that is,

$$\vdash_{\{x, y, z\}} \varphi(x, y, z) \vee \neg \varphi(x, y, z).$$

Indeed, given x and y, let z_0 be the unique z such that $\varphi(x, y, z)$. Now $(z = z_0) \vee$

$\neg(z=z_0)$ and $((z=z_0)\Rightarrow\varphi(x,y,z))\wedge(\neg(z=z_0)\Rightarrow\neg\varphi(x,y,z))$, hence $\varphi(x,y,z)\vee\neg\varphi(x,y,z)$.

Since φ is decidable, we may apply the *least number principle* (see [13]), and deduce that $\vdash\forall_{x\in N}\ \exists!_{y\in N}\ \psi(x,y)$.

Why does ψ represent f? In view of (3), it suffices to show that $f(m)=n$ implies $\vdash\psi(\bar{m},\bar{n})$. Suppose $f(m)=n$, then $g(m,n)=0$ and, for all $k<n$, $g(m,n)\neq 0$. Thus $\vdash\varphi(\bar{m},\bar{n},0)$ and, for all $k<n$, not $\vdash\varphi(\bar{m},\bar{k},0)$. Because φ is decidable, we deduce by the fundamental theorem that, for all $k<n$, $\vdash\neg\varphi(\bar{m},\bar{k},0)$, whence $\vdash\forall_{z\in N}\,(z<\bar{n}\Rightarrow\neg\varphi(\bar{m},z,0))$, hence $\vdash\psi(\bar{m},\bar{n})$ as required. This completes the proof.

It seems quite reasonable to expect the converse of Proposition 3.8 to hold too. Indeed, suppose f is strongly representable. We saw in the proof of Proposition 3.5 that

$$f(m)=(\mu_k(h(m,(k)_0,(k)_1)=0))_0,$$

where

$$h(m,n,p)=\begin{cases}0 & \text{if Proof}(\ulcorner\varphi(\bar{m},\bar{n})\urcorner,p),\\ 1 & \text{otherwise.}\end{cases}$$

It may be shown that h is primitive recursive, so it may be strongly represented by a formula $\chi(x,y,z,t)$ and we need only verify that $\vdash\forall_{x\in N}\ \exists_{y\in N}\ \exists_{z\in N}\ \chi(x,y,z,0)$. In fact, it suffices to represent the primitive recursive predicate $\text{Proof}(\ulcorner\varphi(\bar{m},\bar{n})\urcorner,p)$ by a formula $\xi(x,y,z)$ so that $\vdash\forall_{x\in N}\ \exists_{y\in N}\ \exists_{z\in N}\ \xi(x,y,z)$. Surely, the representation can be carried out, hence, for each $m\in N$, $\vdash\exists_{y\in N}\ \exists_{z\in N}\ \xi(\bar{m},y,z)$. All we require then is to show that one proof will do for all m, that is, $\vdash_x\exists_{y\in N}\ \exists_{z\in N}\ \xi(x,y,z)$. If this could be shown, the proof of the converse of Proposition 3.8 would be complete.[3]

4. The free topos

In this section we study the so-called "free topos" $T(\Gamma)$ generated by a graph Γ (at least when Γ is the empty graph $\emptyset$) with the help of the language $\mathscr{L}_1$. Constructions of $T(\emptyset)$ using languages somewhat like $\mathscr{L}_1$ were carried out by Coste, Fourman and Boileau, although the essential idea in them goes back to an earlier, more circuitous construction of the free topos by Volger. We shall postpone discussion of the universal property of $T(\Gamma)$ until Section 5. Instead we shall concentrate on an investigation of those arrows in $T(\emptyset)$ whose source and target are determined by types of $\mathscr{L}_1$.

We shall require a definition of "topos" which is a little tighter than usual, in as much as products and exponents are not just required to exist but are posited as part of the structure of a topos.

[3] See Appendix 1 for a proof of the converse of Proposition 3.8 along somewhat different lines.

Predogmas

Definition 4.1. A *predogma* is essentially a category $\mathscr{A}$ with finite products and exponentiation Ω^A, for a given object Ω and any object A of $\mathscr{A}$, with a natural isomorphism $\mathscr{A}(A\times B,\Omega)\cong\mathscr{A}(A,\Omega^B)$. More precisely, it is a category $\mathscr{A}$ with distinguished data $(1, *, \times, \pi, \pi', \langle\ \rangle, \Omega, P, \in, {}^*)$, where

(i) 1 and Ω are distinguished objects;

(ii) PA and $A\times B$ are objects when A and B are;

(iii) $*_A : A\to 1$, $\pi_{A,B} : A\times B\to A$, $\pi'_{A,B} : A\times B\to B$ and $\in_A : PA\times A\to\Omega$ are distinguished arrows;

(iv) the following are rules for generating arrows:

$$\frac{f : C\to A \quad g : C\to B}{\langle f,g\rangle : C\to A\times B}, \qquad \frac{h : A\times B\to\Omega}{h^* : A\to PB}.$$

Moreover, these data are subject to the following equations, where $\bullet\!=\!\bullet$ is written for equality between arrows in $\mathscr{A}$.

Terminal object: $f \bullet\!=\!\bullet *_A$, for all $f : A\to 1$.

Product: $\pi_{A,B}\langle f,g\rangle \bullet\!=\!\bullet f$,

$$\pi'_{A,B}\langle f,g\rangle \bullet\!=\!\bullet g,$$

$$\langle \pi_{A,B}h, \pi'_{A,B}h\rangle \bullet\!=\!\bullet h,$$

for all $f : C\to A$, $g : C\to B$ and $h : C\to A\times B$.

Exponentiation: $\in_B\langle h^*\pi_{A,B}, \pi'_{A,B}\rangle \bullet\!=\!\bullet h$,

$$(\in_B\langle g\pi_{A,B}, \pi'_{A,B}\rangle)^* \bullet\!=\!\bullet g,$$

for all $h : A\times B\to\Omega$ and $g : A\to PB$.

Toposes

Definition 4.2. An *elementary topos,* or just a *topos,* is a predogma with a morphism $\top : 1\to\Omega$ such that

(I) every monomorphism $B\rightarrowtail A$ has a unique *characteristic morphism* $h : A\to\Omega$, which means that

$$\begin{tikzcd}
A \arrow[r, "h"] & \Omega \\
B \arrow[u, tail] \arrow[r, "*_B"'] & 1 \arrow[u, "\top"']
\end{tikzcd}$$

is a pullback;

(II) every arrow $h : A\to\Omega$ is a characteristic morphism, which means that the above pullback exists. Ω is called the *subobject classifier.*

Furthermore, it is here understood that a topos also contains a *natural number object* N equipped with arrows $0:1\to N$ and $S:N\to N$ such that

(III) $1 \xrightarrow{0} N \xrightarrow{S} N$ is initial in the category of diagrams $1 \xrightarrow{a} A \xrightarrow{f} A$.

The last condition is known as the *Peano–Lawvere axiom.*

The free topos generated by the empty graph

To any graph Γ we shall associate a topos $T(\Gamma)$ called the "free topos" generated by Γ. First we look at the special case when Γ is the empty graph $\emptyset$.

Definition 4.3. The *objects* of $T(\emptyset)$ are closed terms α in $\mathcal{L}_1$ of type PA for some type A. We call α and α' *equal* if $\vdash \alpha=\alpha'$.

The *arrows* $f:\alpha\to\beta$ of $T(\emptyset)$ are triples $(\alpha, |f|, \beta)$, where $|f|$ is a closed formula of type $P(A\times B)$ subject to two conditions:

(1) $\vdash \forall_{x\in A}\ \forall_{y\in B}\,(\langle x,y\rangle\in|f|\Rightarrow(x\in\alpha\wedge t\in\beta))$,

(2) $\vdash \forall_{x\in A}\,(x\in\alpha\Rightarrow \exists!_{y\in B}\ \langle x,y\rangle\in|f|)$.

We call $f\equiv(\alpha,|f|,\beta)$ and $f'\equiv(\alpha',|f'|,\beta')$ *equal* if $\vdash\alpha=\alpha'$, $\vdash\beta=\beta'$ and $\vdash|f|=|f'|$. We then write $f\bullet\!=\!\bullet f'$.

The *identity morphism* $1_\alpha:\alpha\to\alpha$ is given by

$$|1_\alpha|\equiv\{z\in A\times A\mid \exists_{x\in A}\,(\langle x,x\rangle=z\wedge x\in\alpha)\},$$

which may also be written as

$$\{\langle x,y\rangle\in A\times A\mid x=y\wedge x\in\alpha\}.$$

The *composition* of $f:\alpha\to\beta$ and $g:\beta\to\gamma$ is given by $gf:\alpha\to\gamma$ where

$$|gf|\equiv\{\langle x,z\rangle\in A\times C\mid \exists_{y\in B}\,(\langle x,y\rangle\in|f|\wedge\langle y,z\rangle\in|g|)\}.$$

It is easily seen that $T(\emptyset)$ is a category.

To give $T(\emptyset)$ the structure of a predogma we make the following definitions. To avoid confusion with type symbols, $\underline{1}$ and $\underline{\Omega}$ are underlined.

$$\underline{1}\equiv\{*\};$$

$$\underline{\Omega}\equiv\{t\in\Omega\mid t=t\};$$

$$P\alpha\equiv\{u\in PA\mid\forall_{x\in A}\,(x\in u\Rightarrow x\in\alpha)\};$$

$$\alpha\times\beta\equiv\{\langle x,y\rangle\in A\times B\mid x\in\alpha\wedge y\in\beta\};$$

$$|*_\alpha|\equiv\{\langle x,y\rangle\in A\times 1\mid x\in\alpha\wedge y=*\};$$

$$|\pi_{\alpha,\beta}|\equiv\{\langle\langle x,y\rangle,x'\rangle\in(A\times B)\times A\mid x\in\alpha\wedge y\in\beta\wedge x=x'\};$$

$$|\pi'_{\alpha,\beta}|\equiv\text{similarly};$$

$$|\in_\alpha|\equiv\{\langle\langle u,x\rangle,t\rangle\in(PA\times A)\times\Omega\mid(x\in u)=t\wedge x\in\alpha\wedge u\in P\alpha\};$$

$$|\langle f,g\rangle|\equiv\{\langle z,\langle x,y\rangle\rangle\in C\times(A\times B)\mid\langle z,x\rangle\in|f|\wedge\langle z,y\rangle\in|g|\},$$

where $f : \gamma \to \alpha$, $g : \gamma \to \beta$ and $\langle f, g \rangle : \gamma \to \alpha \times \beta$;

$$|h^*| \equiv \{\langle x, v \rangle \in A \times PB \mid \forall_{y \in B} (\langle\langle x, y \rangle, y \in v \rangle \in |h|) \wedge x \in \alpha \wedge v \in P\beta\},$$

where $h : \alpha \times \beta \to \underline{\Omega}$ and $h^* : \alpha \to P\beta$. We omit the calculations which show that $T(\emptyset)$ is a predogma. Similar calculations may be found in [16].

To give $T(\emptyset)$ the structure of a topos, we first define $\underline{\top} : \underline{1} \to \underline{\Omega}$ by

$$|\underline{\top}| \equiv [\langle *, \top \rangle\},$$

then proceed as follows.

(I) Given a monomorphism $m : \beta \rightarrowtail \alpha$, we define its characteristic morphism char $m : \alpha \to \Omega$ by

$$|\text{char } m| \equiv \{\langle x, t \rangle \in A \times \Omega \mid t = \exists_{y \in B} (\langle x, y \rangle \in |m|)\}$$

and check that the square

$$\begin{tikzcd}
\alpha \arrow[r] & \underline{\Omega} \\
\beta \arrow[u, tail] \arrow[r] & \underline{1} \arrow[u]
\end{tikzcd}$$

is a pullback.

(II) Given an arrow $h : \alpha \to \underline{\Omega}$ we obtain a monomorphism ker $h : b \rightarrowtail \alpha$ where

$$\beta \equiv \{x \in A \mid \langle x, \top \rangle \in |h|\},$$

$$|\text{ker } h| \equiv \{\langle x, x \rangle \in A \times A \mid \langle x, \top \rangle \in |h|\}$$

and again check that the above square is a pullback.

(III) We put

$$\underline{N} \equiv \{x \in N \mid x = x\}$$

and define $\underline{0} : \underline{1} \to \underline{N}$ and $\underline{S} : \underline{N} \to \underline{N}$ by

$$|\underline{0}| \equiv \{\langle *, 0 \rangle\},$$

$$|\underline{S}| \equiv \{\langle x, Sx \rangle \in N \times N \mid x = x\}$$

and check the Peano–Lawvere axiom.

Note that $\underline{\top}$, $\underline{N}$, $\underline{0}$ and $\underline{S}$ are underlined to distinguish them from the corresponding symbols in the language $\mathscr{L}_1$.

The free topos generated by any graph

We recall that a *graph*, that is, an *oriented graph*, consists of two classes, the class of *arrows* and the class of *objects*, and two mappings from the former to the latter, called *source* and *target*. One writes $f : A \to B$ for source$(f) = A$ and target$(f) = B$. A category is thus a graph with additional structure. A *morphism* $F : \Gamma \to \Gamma'$ of graphs sends arrows and objects of Γ to arrows and objects of Γ' so that $f : A \to B$ implies $F(f) : F(A) \to F(B)$.

When Γ is any graph, we may construct $T(\Gamma)$ from a language $\mathscr{L}_1(\Gamma)$ in a similar fashion to the construction of $T(\emptyset)$ from $\mathscr{L}_1$. Primitive types of $\mathscr{L}_1(\Gamma)$ are not only 1, Ω and N, but also all objects of Γ. Moreover, for any arrow $f : X \to Y$ of Γ and any term ξ of type X, we stipulate that $f\xi$ is a term of type Y. $T(\Gamma)$ comes equipped with a morphism from the graph Γ into the underlying graph of $T(\Gamma)$.

Arrows between types

With any type A of $\mathscr{L}_1$ there is associated an object $\underset{\sim}{A}$ of $T(\emptyset)$, where

$$\underset{\sim}{A} \equiv \{x \in A \mid x = x\}$$

is the *universal set* of entities of type A. An arrow $\underset{\sim}{A} \to \underset{\sim}{B}$ is determined by a formula $\varphi(x,y)$ such that $\vdash \forall_{x\in A}\ \exists!_{y\in B}\ \varphi(x,y)$.

For example, an arrow $\underset{\sim}{1} \to \underset{\sim}{B}$ is given by a formula $\psi(y) \equiv \varphi(*,y)$, since any variable of type 1 is provable equal to $*$, such that $\vdash \exists!_{y\in B}\ \psi(y)$. By the fundamental theorem, there is a closed term b of type B such that $\vdash \psi(b)$, hence $\vdash_{\{x,y\}} \varphi(x,y) \Leftrightarrow y = b$. Thus the arrow is given by the explicit equation $y = b$.

To obtain a survey of all arrows $\underset{\sim}{A} \to \underset{\sim}{B}$ in $T(\emptyset)$ we observe that $\underset{\sim}{A}$ is isomorphic to

$$\underset{\sim}{N}^k \times \underline{PC_1} \times \cdots \times \underline{PC_m}$$

and that $\underset{\sim}{B}$ has the same form. Therefore we need only determine the arrows $\underset{\sim}{A} \to \underset{\sim}{N}$ and $\underset{\sim}{A} \to \underline{PC}$. (Note that $\underline{PC} = P\underset{\sim}{C}$ and $\underline{C \times D} = \underset{\sim}{C} \times \underset{\sim}{D}$.)

The arrows $\underset{\sim}{A} \to \underline{PC}$ are dealt with most easily. Suppose $\vdash \forall_{x\in A}\ \exists!_{w\in PC}\ \varphi(x,w)$, then we have

$$\vdash_{\{x,w\}} \varphi(x,w) \ \Leftrightarrow\ w = \{z \in C \mid \exists_{w'\in PC}(\varphi(x,w') \wedge z \in w')\}.$$

Hence we may replace $\varphi(x,w)$ by the explicit equation $w = t(x)$, where $t(x)$ is a term of type PC.

Since $\Omega \equiv P1$ is any predogma, we can also describe arrows $\underset{\sim}{A} \to \underset{\sim}{\Omega}$ explicitly. Indeed, from $\vdash \forall_{x\in A}\ \exists!_{t\in\Omega}\varphi(x,t)$ one deduces, as a special case of the above, that

$$\vdash_{\{x,t\}} \varphi(x,t) \ \Leftrightarrow\ t = \varphi(x,\top).$$

To study the arrows $\underset{\sim}{A} \to \underset{\sim}{N}$, we shall first look at two special cases: $A = N^k$ and $A = PC$. We saw in Section 3 that the arrows $\underset{\sim}{N}^k \to \underset{\sim}{N}$ are determined by formulas which represent certain recursive functions.

Arrows $\underline{PC} \to \underset{\sim}{N}$ are determined by formulas $\varphi(w,x)$, where $\vdash \forall_{w\in PC}\exists!_{x\in N}\varphi(w,x)$. Now the uniformity property allows us to infer $\vdash \exists_{x\in N}\ \forall_{w\in PC}\varphi(w,x)$. Hence, by the fundamental theorem, there is a numeral $\bar{n}$ such that $\vdash \forall_{w\in PC}\varphi(w,\bar{n})$. It follows that $\vdash_{\{w,x\}} \varphi(w,x) \Leftrightarrow x = \bar{n}$, hence we may factor $\underline{PC} \to \underset{\sim}{N}$ as $\underline{PC} \to \underset{\sim}{1} \to \underset{\sim}{N}$. In other words, all arrows $\underline{PC} \to \underset{\sim}{N}$, hence also $\underset{\sim}{\Omega} \to \underset{\sim}{N}$, are constants.

Finally, let us look at the general case $\underset{\sim}{A}\to\underset{\sim}{N}$. For argument's sake, we take $m=1$, so we consider an arrow $\underset{\sim}{N}^k\times\underline{PC}\to\underset{\sim}{N}$ given by a formula $\varphi(x,w,y)$ such that

$$\vdash\forall_{x\in N^k}\,\forall_{w\in PC}\,\exists!_{y\in N}\varphi(x,w,y). \tag{1}$$

Take any $m\in\mathbf{N}^k$, then

$$\vdash\forall_{w\in PC}\,\exists!_{y\in N}\varphi(\bar{m},w,y).$$

Therefore, by the uniformity property or the fundamental theorem for $\mathscr{L}_1(w)$,

$$\vdash\forall_{w\in PC}\varphi(\bar{m},w,\overline{f(m)}), \tag{2}$$

where f is some function $\mathbf{N}^k\to\mathbf{N}$. On the other hand, substituting $\underset{\sim}{C}$ for w in (1), we obtain

$$\vdash\forall_{x\in N^k}\,\exists!_{y\in N}\varphi(x,\underset{\sim}{C},y).$$

Therefore, there is a representable recursive function $g:\mathbf{N}^k\to\mathbf{N}$ such that, for all $m\in\mathbf{N}^k$,

$$\vdash\varphi(\bar{m},\underset{\sim}{C},\overline{g(m)}).$$

Comparing this with (2), we see that $f=g$ is a representable recursive function.

Applying the functor $\Gamma=\mathscr{A}(\underset{\sim}{1},-)$ to the arrow $\underset{\sim}{A}\to\underset{\sim}{N}$, we see that $\Gamma(\underset{\sim}{A})\to\Gamma(\underset{\sim}{N})$ factors as follows:

$$\Gamma(\underset{\sim}{A})\xrightarrow{\Gamma(\text{projection})}\Gamma(\underset{\sim}{N}^k)\xrightarrow{\sim}\mathbf{N}^k\xrightarrow{f}\mathbf{N}\xrightarrow{\sim}\Gamma(\underset{\sim}{N}),$$

where f is a representable recursive function. Perhaps a more refined argument will show that already the arrow $\underset{\sim}{A}\to\underset{\sim}{N}$ factors as

$$\underset{\sim}{A}\xrightarrow{\text{projection}}\underset{\sim}{N}^k\longrightarrow\underset{\sim}{N}.$$

We summarize the above results in the following proposition.

Proposition 4.4. *The following hold in* $T(\emptyset)$:

(1) *Every arrow* $\underset{\sim}{1}\to\underset{\sim}{B}$ *is given by an equation* $y=b$, b *a term of type* $\underset{\sim}{B}$.

(2) *Every arrow* $\underset{\sim}{A}\to\underline{PC}$ *or* $\underset{\sim}{A}\to\underset{\sim}{\Omega}$ *is given by an equation* $w=t(x)$, *where* $t(x)$ *is a term of type PC or* Ω *respectively.*

(3) *Every arrow* $\underset{\sim}{N}^k\to\underset{\sim}{N}$ *represents a recursive function* $\mathbf{N}^k\to\mathbf{N}$.

(4) *Every arrow* $\underline{PC}\to\underset{\sim}{N}$ *or* $\underset{\sim}{\Omega}\to\underset{\sim}{N}$ *factors as* $\underline{PC}\to\underset{\sim}{1}\to\underset{\sim}{N}$ *or* $\underset{\sim}{\Omega}\to\underset{\sim}{1}\to\underset{\sim}{N}$ *respectively.*

(5) *Every arrow* $\underset{\sim}{A}=\underset{\sim}{N}^k\times\underline{PC_1}\times\cdots\times\underline{PC_m}\to\underset{\sim}{N}$ *is sent by* $\Gamma=\mathscr{A}(\underset{\sim}{1},-)$ *onto*

$$\Gamma(\underset{\sim}{A})\xrightarrow{\Gamma(\text{projection})}\Gamma(\underset{\sim}{N}^k)\xrightarrow{\sim}\mathbf{N}^k\xrightarrow{f}\mathbf{N}\xrightarrow{\sim}\Gamma(\underset{\sim}{N}),$$

where f *is a representable recursive function.*

(1) and (3) of the above theorem were first explicitly obtained by Boileau, essentially by the Kleene–Friedman method of Section 2. Related results were also asserted by the Costes, using the cut-elimination method of Gentzen–Girard. (2) is suggested by the work of Fourman. (4) was conjectured by André Joyal; we originally had the weaker version that $\Gamma(\underline{PC})\to\Gamma(\underset{\sim}{N})$ is constant, which we proved with the help of the Gödel–Rosser incompleteness theorem.

Arrows in the free Boolean topos

While it follows from (1) that every arrow $\underset{\sim}{1} \to \underset{\sim}{N}$ in the free topos $T(\emptyset)$ is given by a standard numeral, this is not so in the free Boolean topos (see Example 3.4 with $k=0$).

Not every recursive function $f : \mathbf{N}^k \to \mathbf{N}$ comes from an arrow $\underset{\sim}{N}^k \to \underset{\sim}{N}$ in $T(\emptyset)$ (see Proposition 3.7); however, it does come from such an arrow in the free Boolean topos (see Theorem 3.3(i)).

In the free Boolean topos, not every arrow $\underset{\sim}{N}^k \to \underset{\sim}{N}$ gives rise to a total function $\mathbf{N}^k \to \mathbf{N}$ (see Example 3.4); but if it does, then this function is recursive (see Theorem 3.3(ii)).

5. The universal property of the free topos

The free topos $T(\Gamma)$ generated by the graph Γ comes equipped with a morphism $H : \Gamma \to T(\Gamma)$ in the category of graphs and has the following universal property: given any graph Γ and any morphism G from Γ to the underlying graph of a topos $\mathcal{T}$, there is a unique arrow $F : T(\Gamma) \to \mathcal{T}$ in a suitable category of toposes such that $FH = G$. Of course, this means that T is the left adjoint of the forgetful functor from toposes to graphs. It asserts, in the special case when Γ is the empty graph $\emptyset$, that $T(\emptyset)$ is an initial object in the category of toposes.

The universal property was first obtained by Volger, for another construction of $T(\Gamma)$, with some handwaving: for him G was not a functor but only a pseudo-functor and its uniqueness held only up to isomorphism. All this was straightened out in [16], by confining attention to toposes with canonical subobjects.

In this section we shall establish the universal property of $T(\emptyset)$, as constructed from $\mathcal{L}_1$, by showing that it is an initial object in the category of toposes with canonical subobjects.

Indeterminates

One may adjoin an *indeterminate* arrow $x : 1 \to A$ to a predogma $\mathcal{A}$, when A is an object of $\mathcal{A}$. The resulting predogma $\mathcal{A}[x]$ has the expected universal property. Moreover, each morphism $\varphi(x) : 1 \to B$ in $\mathcal{A}[x]$ has the form $\varphi(x) \bullet\underset{x}{=}\bullet fx$, where $f : A \to B$ is a uniquely determined arrow of $\mathcal{A}$. Here $\bullet\underset{x}{=}\bullet$ denotes equality in $\mathcal{A}[x]$. For details of this construction see [16].

If $X = \{x_1, \ldots, x_n\}$ is any finite set of indeterminates $x_i : 1 \to A_i$, we similarly can form $\mathcal{A}[X]$. For example, if $X = \{x, y\}$, we have $\mathcal{A}[X] = \mathcal{A}[x,y] \cong \mathcal{A}[x][y]$. It does not matter whether we adjoin indeterminates simultaneously or one at a time. In fact, we could replace $x : 1 \to A$ and $y : 1 \to B$ by a single indeterminate $z : 1 \to A \times B$ so that $\mathcal{A}[x,y] \cong \mathcal{A}[z]$.

It already follows that certain expressions $\varphi(x_1, \ldots, x_n)$ of the language $\mathcal{L}_1$ of type theory may be interpreted as arrows in $\mathcal{A}[x_1, \ldots, x_n]$, provided we regard variables of type A as indeterminates $1 \to A$. Thus we interpret

(1) terms t of type A as arrows $t : 1 \to A$;

(2) a formula $a \in \alpha$ as the arrow $\in_A \langle \alpha, a \rangle : 1 \to \Omega$;

(3) a term $\{x \in A \mid \varphi(x)\}$ as the unique arrow $1 \to PA$ such that the corresponding arrow $f : A \to \Omega$ satisfies $fx \bullet\!\underset{x}{=}\!\bullet\ \varphi(x)$, where $\varphi(x)$ has already been interpreted.

The equation in (3) is easily seen to be equivalent to

$$x \in \{x \in A \mid \varphi(x)\} \bullet\!\underset{x}{=}\!\bullet\ \varphi(x).$$

The presence of other free variables does not essentially change anything.

Furthermore we interpret

(4) the term $*$ of type 1 as the arrow $*_1 : 1 \to 1$, which is of course the same as the identity arrow 1_1;

(5) the term $\langle a, b \rangle$ of type $A \times B$ as the arrow $\langle a, b \rangle : 1 \to A \times B$, where a and b have already been interpreted as arrows $1 \to A$ and $1 \to B$ respectively.

This interpretation can be extended to all terms of the language $\mathscr{L}_1$ provided $\mathscr{A}$ is a topos or, more generally, a "dogma", which we shall not define here. It should be pointed out though that, even when $\mathscr{A}$ is a topos, $\mathscr{A}[x]$ is only a dogma and not a topos.

Interpretation in a topos

Let $\delta_C : C \times C \to \Omega$ be the characteristic morphism of $\langle 1_C, 1_C \rangle : C \to C \times C$ in a topos $\mathscr{A}$. If terms c and c' of type C are interpreted as arrows $1 \to C$, we write

$$c = c' \quad \text{for } \delta_C \langle c, c' \rangle.$$

If formulas p and q are interpreted as arrows $1 \to \Omega$, we write

$$p \wedge q \quad \text{for } \langle p, q \rangle = \langle \top, \top \rangle,$$

$$p \Rightarrow q \quad \text{for } p \wedge q = p,$$

$$\forall_{x \in A}\, \varphi(x) \quad \text{for } \{x \in A \mid \varphi(x)\} = \{x \in A \mid \top\}.$$

As it is well-known that $\perp$, $\neg p$, $p \vee q$ and $\exists_{x \in A}\, \varphi(x)$ may be defined in terms of the above, we have an interpretation of all closed formulas of $\mathscr{L}_1$ by arrows $1 \to \Omega$ in the topos $\mathscr{A}$, in fact, of all closed terms of type A in $\mathscr{L}_1$ by arrows $1 \to A$ in $\mathscr{A}$. More generally, all terms involving variables $x_1, \dots, x_n$ are interpreted as arrows in the predogma (actually dogma) $\mathscr{A}[x_1, \dots, x_n]$.

We shall also interpret

$$p_1, p_2, \dots, p_n \vdash q$$

by saying that the intersection of the subobjects of 1 in $\mathscr{A}$ corresponding to the p_i is contained in the subobject corresponding to q. Similarly we interpret $\vdash_X$ in terms of subobjects of 1 in $\mathscr{A}[X]$. It is well-known that the validity of

$$p_1, p_2, \dots, p_n \vdash_X q$$

in $\mathscr{L}_1$ implies its validity in $\mathscr{A}[X]$ for every topos $\mathscr{A}$.

We note that the interpretation of $\vdash_X$ in a topos can also be explained without mentioning subobjects. For example, if $X=\{x\}$, x of type A,

$$f_1x, f_2x, \ldots, f_nx \vdash_X gx$$

means this: for all objects C of $\mathscr{A}$ and all arrows $h: C \to A$ in $\mathscr{A}$, if $f_1h \bullet\!=\!\bullet \top *_C$ and $f_2h \bullet\!=\!\bullet \top *_C \cdots$ and $f_nh \bullet\!=\!\bullet \top *_C$, then $gh \bullet\!=\!\bullet \top *_C$.

The internal language of a topos

The language $\mathscr{L}_1$ studied so far is *pure* type theory. It contains no types other than those implied by Definition 1.1 and no terms other than those implied by Definition 1.2. Moreover, it is subject to no axioms other than those listed in Section 1. One may also consider various *applied* type theories extending $\mathscr{L}_1$ by permitting additional types, terms or axioms. In particular, the interpretation of the language $\mathscr{L}_1$ in a topos $\mathscr{A}$ may be extended to the so-called *internal* language of that topos. This admits all objects of $\mathscr{A}$ as types and all arrows $1 \to C$ in $\mathscr{A}$ as terms. It follows that each arrow $f: A \to B$ in $\mathscr{A}$ allows one to form a term fa of type B for each term a of type A. For example, if C is any object of $\mathscr{A}$ and c and c' are arrows $1 \to C$, we may regard $\delta_C\langle c, c'\rangle$ as a term of type Ω and write it as $c=c'$.

Canonical subobjects

Definition 5.1. We say that a topos has *canonical subobjects* if to each object A there is associated a representative set Sub A of monomorphisms (or subobjects) $B \rightarrowtail A$ with the following properties:

(i) Every monomorphism $B \rightarrowtail A$ is isomorphic to exactly one element of Sub A.

(ii) $1_A : A \to A$ is in Sub A.

(iii) If $f: B \to A$ is in Sub A and $g: D \to C$ is in Sub C, then $f \times g : B \times D \to A \times C$ is in Sub$(A \times C)$.

(iv) If $f: B \to A$ is in Sub A, then $Pf : PB \to PA$ is in Sub PA.

(v) If $f: B \to A$ is in Sub A and $g: C \to B$ is in Sub B, then $gf: C \to A$ is in Sub A.

Already in a Cartesian category one may define $f \times g$ as $\langle f\pi_{B,D}, g\pi'_{B,D}\rangle$. To define Pf in a topos, we stipulate that, for an indeterminate v of type PB,

$$(Pf)v \bullet\!\underset{v}{=}\!\bullet \{x \in A \mid \exists_{y \in B}(fy = x \wedge y \in v)\}.$$

We remark that all toposes occurring in nature have canonical subobjects and that every topos is equivalent to one with canonical subobjects. Moreover, the free topos $T(\Gamma)$ constructed in Section 4 has canonical subobjects: with any monomorphism $m: \beta \rightarrowtail \alpha$ we associate the isomorphic canonical subobject $m' : \beta' \rightarrowtail \alpha$, where

$$\beta' \equiv \{x \in A \mid \exists_{y \in B}(\langle y, x\rangle \in |m|)\},$$

$$|m'| \equiv \{\langle x, x\rangle \in A \times A \mid x \in \beta'\}.$$

We recall [16] that in a topos with canonical subobjects there is a bijection

$$\text{Sub } A \underset{\text{ker}}{\overset{\text{char}}{\rightleftarrows}} \mathscr{T}(A, \Omega),$$

where char m is the characteristic morphism of m and ker h, the *kernel* of h, is the unique element m of Sub A whose characteristic morphism is h. We also recall [16, Lemma 9.1] that in a topos with canonical subobjects the following equations hold, where we have written

$$P\beta \bullet = \bullet \{v \in PB \mid \forall_{y \in B}(y \in v \Rightarrow y \in \beta)\},$$

$$\beta' \bullet = \bullet \in_B \langle \beta *_B, 1_B \rangle.$$

Lemma 5.2. *In a topos with canonical subobjects,*

$$\ker(T*_A) \bullet = \bullet 1_A,$$

$$\ker \wedge (f \times g) \bullet = \bullet \ker f \times \ker g,$$

$$\ker((P\beta)') \bullet = \bullet P(\ker(\beta')),$$

for all $f : A \to \Omega$, $g : B \to \Omega$ *and* $\beta : 1 \to PB$.

Universal property

For expository purposes we shall only establish the universal property of $T(\Gamma)$ when Γ is the empty category. In other words, we shall show that $T(\emptyset)$ is an initial object in the category Top whose objects are toposes with canonical subobjects and whose arrows are functors which preserve the predogma structure, the natural number object and canonical kernels exactly. This is easily shown [16, Lemma 9.2] to be equivalent to saying that the functors preserve the predogma structure, the natural number object, internal equality (hence all logical symbols) and canonical subobjects. Very roughly speaking, the arrows in Top are the "logical functors" of the topos literature, but they are defined more tightly. The universal property of $T(\Gamma)$ for an arbitrary graph Γ may be proved in the same way.

Theorem 5.3. $T(\emptyset)$ *is an initial object in* Top.

Proof. Given any topos $\mathscr{T}$ with canonical subobjects, we shall show that there is a unique arrow $F : T(\emptyset) \to \mathscr{T}$ in Top. The uniqueness of F will be made clear by showing that its construction is forced at each stage.

First we define $F(\alpha)$ for any object α of $T(\emptyset)$. Now α is a term of type PA in $\mathscr{L}_1$, which may also be interpreted as an arrow $1 \to PA$ in $\mathscr{T}$. As in any predogma, this gives rise to a unique arrow $\alpha' : A \to \Omega$ in $\mathscr{T}$. We define

$$F(\alpha) = \text{Ker}(\alpha'),$$

by which we mean the source of $\ker(\alpha')$.

That this definition is forced upon us is seen as follows. Consider the canonical monomorphism $m_\alpha : \alpha \to \underset{\sim}{A}$ in $T(\emptyset)$ given by

$$|m_\alpha| \equiv \{\langle x,x\rangle \in A \times A \mid x \in \alpha\}.$$

Its characteristic morphism may be calculated[4] to be the arrow $\alpha' : \underset{\sim}{A} \to \underset{\sim}{\Omega}$ which corresponds to $\alpha : \underset{\sim}{1} \to P\underset{\sim}{A}$. Thus $m_\alpha \bullet\!=\!\bullet \ker(\alpha')$, hence $\alpha = \mathrm{Ker}(\alpha')$ in $T(\emptyset)$. Note that the first α in this equation refers to an object of $T(\emptyset)$, while the second α refers to the arrow $\underset{\sim}{1} \to P\underset{\sim}{A}$ in $T(\emptyset)$. Applying the functor F to this, which preserves kernels, the predogma structure (hence the symbol $'$), the natural number object and all logic symbols (hence the term α), we obtain $F(\alpha) = \mathrm{Ker}(\alpha')$ in $\mathscr{T}$.

Next, we wish to define $F(f)$ for any arrow $f : \alpha \to \beta$ in $T(\emptyset)$. We recall that $|f|$ is a term of type $P(A \times B)$ such that

(i) $\vdash \forall_{x\in A}\ \forall_{y\in B}(\langle x,y\rangle \in |f| \Rightarrow (x \in \alpha \wedge y \in \beta))$,

(ii) $\vdash \forall_{x\in A}(x \in \alpha \Rightarrow \exists!_{y\in B}\langle x,y\rangle \in |f|)$.

Now let $x : 1 \to F(\alpha)$ be an indeterminate of type $F(\alpha) = \mathrm{Ker}(\alpha')$ over $\mathscr{T}$. Put

$$n_\alpha \bullet\!=\!\bullet \ker(\alpha') \quad \text{in } \mathscr{T},$$

then

$$\alpha' n_\alpha x \bullet\!\underset{x}{=}\!\bullet T,$$

that is,

$$\vdash_x n_\alpha x \in \alpha \quad \text{in } \mathscr{T}[x].$$

Now (ii) holds in any topos, hence in $\mathscr{T}$. Therefore

$$\vdash_x \exists!_{y\in B}\langle n_\alpha x, y\rangle \in |f| \quad \text{in } \mathscr{T}[x].$$

By [16, Theorem 8.3], there is a unique arrow $h : F(\alpha) \to B$ in $\mathscr{T}$ so that

$$\vdash_x \langle n_\alpha x, hx\rangle \in |f| \quad \text{in } \mathscr{T}[x].$$

Also (i) holds in any topos, hence in $\mathscr{T}$. Therefore

$$\vdash_x hx \in \beta \quad \text{in } \mathscr{T}[x],$$

that is, $\beta' hx \bullet\!\underset{x}{=}\!\bullet T$, hence

$$\beta' h \bullet\!=\!\bullet T_{*F(\alpha)}.$$

But $\mathrm{Ker}(\beta') = F(\beta)$; hence there exists a unique arrow $F(f) : F(\alpha) \to F(\beta)$ such that

$$n_\beta F(f) \bullet\!=\!\bullet h.$$

It follows that $F(f)$ is the unique arrow such that

$$\vdash_x \langle n_\alpha x, n_\beta F(f)x\rangle \in |f| \quad \text{in } \mathscr{T}[x].$$

It is tedious but routine to verify that F thus constructed is a functor which preserves the predogma structure. For example, let us show that

$$F(\alpha \times \beta) = F(\alpha) \times F(\beta).$$

[4] See Appendix 2.

One easily calculates that

$$(\alpha \times \beta)' \bullet = \bullet \wedge(\alpha' \times \beta'),$$

hence, by Lemma 5.2, that

$$\ker((\alpha \times \beta)') \bullet = \bullet \ker(\wedge(\alpha' \times \beta')) \bullet = \bullet \ker(\alpha') \times \ker(\beta').$$

Passing from the arrows to their sources, one obtains $F(\alpha \times \beta) = F(\alpha) \times F(\beta)$ as claimed.

As a special case of the above definition, let us calculate $F(m_\alpha)$, where $m_\alpha : \alpha \to \underline{A}$ is $\ker(\alpha')$. We see that $F(m_\alpha)$ is the unique arrow $F(\alpha) \to A$ in $\mathscr{T}$ so that

$$\vdash_x \langle n_\alpha x, F(m_\alpha)x \rangle \in |m_\alpha|,$$

that is, $n_\alpha x \bullet \underset{x}{=} \bullet F(m_\alpha)x$. It follows that

$$F(m_\alpha) \bullet = \bullet n_\alpha. \tag{1}$$

We shall now prove that

$$F(\ker h) \bullet = \bullet \ker F(h)$$

for any $h : \underline{A} \to \underline{\Omega}$. Let

$$\beta \equiv \mathrm{Ker}\ h = \{x \in A \mid \langle x, T \rangle \in |h|\}$$

according to Section 4. Straightforward calculations show that

$$|m_\alpha \ker h| \bullet = \bullet |\ker h| \bullet = \bullet |m_\beta|$$

and

$$|\beta' m_\alpha| \bullet = \bullet |\beta'| \bullet = \bullet |h|.$$

Since sources and targets agree, we may therefore conclude that in $T(\emptyset)$

$$m_\alpha \ker h \bullet = \bullet m_\beta, \qquad \beta' m_\alpha \bullet = \bullet h. \tag{2}$$

Applying the functor F to this, we obtain,

$$n_\alpha F(\ker h) \bullet = \bullet n_\beta, \qquad \beta' n_\alpha \bullet = \bullet F(h).$$

These equations are utilized in checking that β' is the characteristic morphism of $n_\alpha \ker F(h)$, but we omit the routine verification here. Since $n_\alpha \bullet = \bullet \ker F(\alpha')$ and $\ker F(h)$ are both canonical subobjects, so is their composition. Therefore, in view of (2),

$$n_\alpha \ker F(h) \bullet = \bullet \ker(\beta') \bullet = \bullet n_\beta \bullet = \bullet n_\alpha F(\ker h).$$

Now n_α, being an equalizer, is a monomorphism, and so the result follows.

Finally, to prove the uniqueness of F, suppose that $F : T(\emptyset) \to \mathscr{T}$ is any functor which preserves the predogma structure, the natural number object and kernels, then we claim that $F(f)$ must be as defined, that is,

$$\vdash_x \langle n_\alpha x, n_\beta F(f)x \rangle \in |f|$$

must hold in $\mathscr{T}[x]$ for any indeterminate $x : 1 \to F(\alpha)$. Eliminating the symbols $\in$ and $\vdash_x$ from this, we thus want to show that

$$\in_{A\times B}\langle |f|, \langle n_\alpha, n_\beta F(f)\rangle x\rangle \bullet \underset{x}{=} \bullet \top,$$

that is,

$$\in_{A\times B}\langle |f| *_{F(\alpha)}, \langle n_\alpha, n_\beta F(f)\rangle\rangle \bullet = \bullet \top *_{F(\alpha)}.$$

Now this equation in $\mathscr{T}$ is obtained by applying the functor F to the following equation in $T(\emptyset)$:

$$\in_{\underset{\cdot}{A}\times \underset{\cdot}{B}}\langle |f| *_\alpha, \langle m_\alpha, m_\beta f\rangle\rangle \bullet = \bullet \underset{\cdot}{\top} *_\alpha.$$

A tedious but routine calculation shows that this equation does indeed hold in $T(\emptyset)$.

6. Postscript on the Freyd cover

When we presented the above results at a conference in the fall of 1978, Peter Freyd immediately realized that the $\exists$-property asserts the projectivity of 1 and that the $\vee$-property asserts the indecomposability of 1 in the free topos. He then went ahead and proved the projectivity and indecomposability of 1 directly. We shall give a brief sketch of his ideas and show how the $\exists$-property and $\vee$-property may be deduced, then generalize his method to obtain also the Uniformity Property.

Definition 6.1. The *Freyd cover* of a topos (more generally of a category with terminal object) is the comma category $\hat{\mathscr{A}} = (\text{Sets}, \Gamma)$, where $\Gamma = \mathscr{A}(1, -)$. Its objects are triplets (X, ξ, A), where X is a set, A an object of $\mathscr{A}$ and $\xi : X \to \Gamma(A)$ a mapping. Its arrows from (X, ξ, A) to (Y, η, B) are pairs of arrows $(\varphi : X \to Y, f : A \to B)$ so that the following square commutes:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\xi"'] & Y \arrow[d, "\eta"] \\ \Gamma(A) \arrow[r, "\Gamma(f)"'] & \Gamma(B) \end{tikzcd}$$

$\hat{\mathscr{A}}$ comes equipped with a functor $G : \hat{\mathscr{A}} \to \mathscr{A}$ defined by

$$G(X, \xi, A) = A, \qquad G(\varphi, f) \bullet = \bullet f.$$

$\hat{\mathscr{A}}$ is also a topos (if $\mathscr{A}$ is) and G is then a logical functor.

Here is the crux of Freyd's argument. He observed that the terminal object $\hat{1}$ of $\hat{\mathscr{A}}$ is trivially projective (and indecomposable). Now, if $\mathscr{A} = T(\emptyset)$ is the free topos, then there is also a unique logical functor $F : \mathscr{A} \to \hat{\mathscr{A}}$ and GF is the identity functor on $\mathscr{A}$. It follows that 1 is also projective (and indecomposable) in $T(\emptyset)$.

We wish to check that this argument remains valid if we operate entirely in the category Top, whose arrows are strict functors that preserve everything on the nose. To this purpose we must present some technical details.

With each type A of $\mathscr{L}_1$ there is associated an object A of any topos, hence of $\mathscr{A}$ and $\hat{\mathscr{A}}$. To avoid confusion, we shall denote the corresponding object of $\hat{\mathscr{A}}$ by $\hat{A}$. In particular, $\hat{1}$, $\hat{\Omega}$ and $\hat{N}$ are the terminal object, subobject classifier and natural number object of $\hat{\mathscr{A}}$ respectively. It turns out that $\hat{A} = (S_A, \lambda_A, A)$, where S_A may be defined by induction on A:

$$S_1 = \{*\}, \qquad S_N = \mathbf{N}, \qquad S_\Omega = \Gamma(\Omega) \uplus \{\top\},$$
$$S_{A\times B} = S_A \times S_B, \qquad S_{PA} = \hat{\mathscr{A}}(\hat{A}, \hat{\Omega}).$$

Here $X \uplus Y$ denotes the disjoint union of X and Y and may be identified with $(X\times\{0\} \cup (Y\times\{1\})$. The mappings $\lambda_A : S_A \to \Gamma(A)$ are the obvious ones. In particular, note the striking similarity between $\lambda_\Omega : S_\Omega \to \Gamma(\Omega)$ and the mapping $^- : \mathscr{L}^+ \to \mathscr{L}$ in Section 2.

More generally, one defines

$$(X, \xi, A) \times (Y, \eta, B) = (X\times Y, \zeta, A\times B),$$

where ζ is the compound mapping

$$X \times Y \xrightarrow{\xi\times\eta} \Gamma(A)\times\Gamma(B) \xrightarrow{\sim} \Gamma(A\times B),$$

and

$$P(X, \xi, A) = (\mathscr{A}((X, \xi, A), \hat{\Omega}), \theta, PA),$$

where, for any arrow $(\lambda, h) : (X, \xi, A) \to \hat{\Omega}$,

$$\theta((\lambda, h)) \mathrel{\bullet\!=\!\bullet} \ulcorner h \urcorner$$

is the arrow $1 \to PA$ corresponding to $h : A \to \Omega$ in $\mathscr{A}$.

We wish to verify that $\hat{\mathscr{A}}$ is a predogma in the strict sense of products and exponentiation being part of the structure and that G preserves the predogma structure exactly. As regards objects, this follows from what has been said above. As regards arrows, it may also be readily checked.

Next, we wish to verify that $\hat{\mathscr{A}}$ has canonical subobjects and that G preserves kernels. Now it is easily seen that an arrow

$$(\mu, m) : (Y, \eta, B) \to (X, \xi, A)$$

in $\hat{\mathscr{A}}$ is a monomorphism if and only if μ is a monomorphism in Sets and m is one in $\mathscr{A}$. Thus we are led to call (μ, m) a *canonical subobject* in $\hat{\mathscr{A}}$ precisely when μ is set-inclusion and m is a canonical subobject in $\mathscr{A}$. Properties (i) to (v) of Definition 5.1 are easily verified.

Before discussing kernals, we must identify the arrow $\hat{\top} : \hat{1} \to \hat{\Omega}$. We take $\hat{\top} \mathrel{\bullet\!=\!\bullet} (\tau, \top)$, where $\tau(*) = (\top, 1)$ is the element of $\{\top\}\times\{1\}$. Suppose now $(\lambda, h) : (X, \xi, A) \to \hat{\Omega}$. We construct its *kernel* as $(\mu, m) : (Y, \eta, B) \to (X, \xi, A)$, where $m : B \to A$ is the kernel of $h : A \to \Omega$ in $\mathscr{A}$,

$$Y = \{x \in X \mid \lambda(x) = (T, 1) \wedge \xi(x) \in \operatorname{Im} \Gamma(m)\},$$

μ is the inclusion of Y into X and $\eta : Y \to \Gamma(B)$ is given by

$$\eta(y) = \Gamma(m)^{-1}(\xi(y))$$

for all $y \in Y$, noting that $\Gamma(m)$ is an injection. It is fairly routine to check that the characteristic morphism of (μ, m) is (λ, h).

Conversely, given any canonical subobject (μ, m), one may obtain its characteristic morphism (λ, h) in $\hat{\mathscr{A}}$ by taking $h = \text{char } m$ in $\mathscr{A}$ and defining $\lambda : X \to S_\Omega$ by

$$\lambda(x) = \begin{cases} (h\xi(x), 0) & \text{if } x \in X - Y, \\ (T, 1) & \text{if } x \in Y. \end{cases}$$

This allows us again to check that $\hat{\mathscr{A}}$ is a topos.

Since F preserves the predogma structure and kernels, it also preserves internal equality and therefore all logical symbols, as has already been pointed out. Since $e : A \to B$ is an epimorphism in a topos $\mathscr{A}$ if and only if $\vdash \forall_{y \in B} \exists_{x \in A}\, y = ex$ holds in $\mathscr{A}$ [e.g. 16, Lemma 13.4], it follows that F preserves epimorphisms. This is important in deducing the projectivity of 1 in $\mathscr{A} = T(\emptyset)$.

Lemma 6.2. *If $e : \alpha \to \beta$ is an arrow in $T(\emptyset)$ such that*

$$\vdash \forall_{y \in B}(y \in \beta \Rightarrow \exists_{x \in A}(\langle x, y \rangle \in |e|)$$

in $\mathscr{L}_1$, then e is an epimorphism.

Proof. Suppose $f, g : \beta \to \gamma$ are such that $fe \bullet\!=\!\bullet\, ge$. We shall prove that $\vdash |f| = |g|$. We argue informally thus: suppose $\langle y, z \rangle \in |f|$. By assumption on e, there is an $x \in A$ such that $\langle x, y \rangle \in |e|$, hence $\langle x, z \rangle \in |fe| = |ge|$. Therefore, there exists $y' \in B$ such that $\langle y', z \rangle \in |g|$ and $\langle x, y' \rangle \in |e|$. But, since e is an arrow in $T(\emptyset)$, $y' = y$, hence $\langle y, z \rangle \in |g|$. Therefore $|f| \subseteq |g|$, and similarly $|g| \subseteq |f|$.

New proof of the $\exists$-property

Suppose $\varphi(x)$ is a formula in $\mathscr{L}_1$, x being a variable of type A, and suppose $\vdash \exists_{x \in A} \varphi(x)$. Let $\alpha \equiv \{x \in A \mid \varphi(x)\}$ and define $e : \alpha \to \underline{1}$ in $T(\emptyset)$ by

$$|e| \equiv \{\langle x, y \rangle \in A \times 1 \mid \varphi(x)\}.$$

By Lemma 6.2, e is an epimorphism.

Now $\underline{1}$ is projective by Freyd's argument, hence, e splits, that is, there is an arrow $m : \underline{1} \to \alpha$ such that $em \bullet\!=\!\bullet\, 1$. Now $|m|$ satisfies

$$\vdash \exists !_{x \in A} \langle *, x \rangle \in |m|,$$

$$\vdash \forall_{x \in A}\, (\langle *, x \rangle \in |m| \Rightarrow x \in \alpha).$$

Writing $\psi(x)$ for $\langle *, x \rangle \in |m|$, we thus have $\vdash \exists !_{x \in A} \psi(x)$ and $\vdash \forall_{x \in A}(\psi(x) \Rightarrow \varphi(x))$. By the lemma below, we can find a term a of type A so that $\vdash \psi(a)$, hence also $\vdash \varphi(a)$.

Lemma 6.3 ($\exists!$-property). *If* $\vdash \exists!_{x\in A}\psi(x)$ *in* $\mathscr{L}_1$, *then* $\vdash\psi(a)$ *for some term* a *of type* A.

Proof. Suppose $\vdash\exists!_{x\in A}\psi(x)$. We shall prove the existence of a term a of type A such that $\vdash\psi(a)$ by induction on A.

If $A=1$, take $a\equiv *$.

If $A=\Omega$, take $a\equiv\psi(\top)$. (See the discussion of the arrows $\underset{\sim}{A}\to\underset{\sim}{\Omega}$ in Section 4.)

If $A=PB$, take $a\equiv\{y\in B\mid \exists_{v\in PB}(\psi(v)\wedge y\in v)\}$. (See the discussion of the arrows $\underset{\sim}{A}\to\underset{\sim}{PB}$ in Section 4.)

If $A=B\times C$, we are given that $\vdash\exists!_{x\in B\times C}\psi(x)$. Therefore $\vdash\exists!_{y\in B}\,\exists_{z\in C}\psi(\langle y,z\rangle)$. By inductional assumption, there is a term b of type B such that $\vdash\exists_{z\in C}\psi(\langle b,z\rangle)$. Now actually $\vdash\exists!_{z\in C}\psi(\langle b,z\rangle)$. Again by inductional assumption, there is a term c of type C such that $\vdash\psi(\langle b,c\rangle)$.

If $A=N$, we are given that $\vdash\exists!_{x\in N}\psi(x)$. Then ψ determines an arrow $f:\underset{\sim}{1}\to\underset{\sim}{N}$ in $\mathscr{A}=T(\emptyset)$, where $|f|\equiv\{\langle *,x\rangle\in 1\times N\mid\psi(x)\}$. Now $F(f):\hat{1}\to\hat{N}$ is a standard numeral in $\mathscr{A}$, hence $f\bullet\!=\!\bullet\, GF(f):\underset{\sim}{1}\to\underset{\sim}{N}$ is a standard numeral in $\mathscr{A}=T(\emptyset)$, say $f\bullet\!=\!\bullet\,\underset{\sim}{S}^n\underset{\sim}{0}$. Using the definitions of $\underset{\sim}{S}$ and $\underset{\sim}{0}$ in Section 4, we see that

$$\vdash_x\psi(x)\ \Leftrightarrow\ \langle *,x\rangle\in|f|\ \Leftrightarrow\ x=S^n0.$$

Therefore, we may take $a\equiv S^n0$.

We have now completed the proof of the $\exists!$-property, hence of the $\exists$-property. The $\vee$-property is an easy consequence of this.

Indeed, let

$$\varphi(x)\equiv(x=0\Rightarrow p)\wedge(\neg(x=0)\Rightarrow q).$$

Clearly, $p\vdash\varphi(0)$ and $q\vdash\varphi(1)$, hence $p\vdash\exists_{x\in N}\varphi(x)$ and $q\vdash\exists_{x\in N}\varphi(x)$, hence $p\vee q\vdash\exists_{x\in N}\varphi(x)$. (Here $1\equiv S0$.)

Now suppose $\vdash p\vee q$, then $\vdash\exists_{x\in N}\varphi(x)$. Therefore, by the $\exists$-property, $\vdash\varphi(\bar{n})$ for some numeral $\bar{n}$. Now either $\vdash\bar{n}=0$ or $\vdash\neg(\bar{n}=0)$. In the first case $\vdash p$, in the second case $\vdash q$.

We turn now to another proof of the Uniformity Property by looking at the Freyd cover of a topos obtained from the free topos $T(\emptyset)$ by adjoining an indeterminate arrow $\underset{\sim}{z}:\underset{\sim}{1}\to\underset{\sim}{C}$, where $C=\Omega$ or PB.

It is known [10, 5.11.2] that, if C is an object of the topos $\mathscr{A}$, then the topos $\mathscr{A}/C$ may be regarded in some sense as the topos $\mathscr{A}(z)$ obtained from $\mathscr{A}$ by adjoining an indeterminate $z:1\to C$. This should be distinguished from the predogma (or dogma) $\mathscr{A}[z]$ obtained in the same way. More precisely, we require a logical functor $H:\mathscr{A}\to\mathscr{A}(z)$ and an arrow $z:1\to H(C)$ so that the pair (H,z) is initial in the category of all pairs $(F:\mathscr{A}\to\mathscr{B},c:1\to F(C))$. Unfortunately, when we work with $\mathscr{A}/C$, this seems to be true only "up to isomorphism". Be that as it may, in the case $\mathscr{A}=T(\emptyset)$ of interest to us, when C is a type in $\mathscr{L}_1$ and $\underset{\sim}{z}:\underset{\sim}{1}\to\underset{\sim}{C}$ an indeterminate arrow, there is another construction of $\mathscr{A}(\underset{\sim}{z})$ for which the above universal property holds in the strictest sense.

Recall the language $\mathcal{L}_1(z)$ whose closed terms are the terms of $\mathcal{L}_1$ containing no free variables other than z, which we shall think of as a parameter. (See the end of Section 2.) We now construct $T(\emptyset)(\underline{z})$ from $\mathcal{L}_1(z)$ in the same way as we did $T(\emptyset)$ from $\mathcal{L}_1$: its objects are closed terms $\alpha(z)$ of type A in $\mathcal{L}_1(z)$ and its arrows $f(z) : \alpha(z) \to \beta(z)$ are triplets $(\alpha(z), |f(z)|, \beta(z))$, where $|f(z)|$ is a closed formula in $\mathcal{L}_1(z)$ of type $P(A \times B)$, B being the type of $\beta(z)$, such that

$$(1_z) \qquad \vdash_z \forall_{x \in A}\ \forall_{y \in B}(\langle x, y\rangle \in |f(z)| \Rightarrow (x \in \alpha(z) \wedge y \in \beta(z))),$$

$$(2_z) \qquad \vdash_z \forall_{x \in A}(x \in \alpha(z) \Rightarrow \exists!_{y \in B}\langle x, y\rangle \in |f(z)|).$$

The rest of the construction proceeds exactly as in Section 4.

In what follows, we write $\mathcal{A} = T(\emptyset)$. Clearly, there is an arrow $H : \mathcal{A} \to \mathcal{A}(\underline{z})$, where $H(\alpha) = \alpha$ and $H(f) \bullet\!=\!\bullet f$. Moreover, there is an arrow $\underline{z} : \underline{1} \to \underline{C}$ in $\mathcal{A}(\underline{z})$ given by $|\underline{z}| \equiv \{\langle *, z\rangle\}$. Now suppose $F : \mathcal{A} \to \mathcal{B}$ is an arrow in Top and $c : 1 \to F(\underline{C}) = C$ an arrow in $\mathcal{B}$. We claim that there is a unique arrow $F' : \mathcal{A}(\underline{z}) \to \mathcal{B}$ such that

$$F'H = F, \qquad F'(\underline{z}) \bullet\!=\!\bullet c.$$

Indeed, take $F'(\alpha(z) \equiv \mathrm{Ker}(\alpha(c)')$ and let $F'(f(z))$ be the unique arrow $h : \mathrm{Ker}(\alpha(c)') \to \mathrm{Ker}(\beta(c)')$ in $\mathcal{B}$ such that

$$\vdash_x \langle \ker(\alpha(c)')x, \ker(\beta(c)')hx\rangle \in |f(c)| \quad \text{in } \mathcal{B}[x].^5$$

We summarize this result as follows.

Proposition 6.4. *$T(\emptyset)(\underline{z})$ as constructed from $\mathcal{L}_1(z)$ is the topos obtained from $T(\emptyset)$ by adjoining an indeterminate arrow $\underline{z} : \underline{1} \to \underline{C}$.*

We shall now look at the special case of this when $\mathcal{B}$ itself is the Freyd cover of $T(\emptyset)(z)$ and $C = \Omega$ or PB.

First take $C = \Omega$, so $\underline{z} : \underline{1} \to \underline{\Omega}$. We need only an arrow $c : \hat{1} \to F(\underline{\Omega}) = \hat{\Omega}$ in $\widehat{\mathcal{A}(\underline{z})}$. Let $\zeta(*) = (\underline{z}, 0)$, then the following square commutes:

$$\begin{array}{ccc} \{*\} & \xrightarrow{\ \zeta\ } & \Gamma(\underline{\Omega}) \dot{\cup} \{\underline{T}\} \\ \big\downarrow & & \big\downarrow \\ \Gamma(\underline{1}) & \xrightarrow[\Gamma(\underline{z})]{} & \Gamma(\underline{\Omega}) \end{array}$$

Therefore, $c \bullet\!=\!\bullet (\zeta, \underline{z})$ is an arrow $\hat{1} \to \hat{\Omega}$ in $\widehat{\mathcal{A}(\underline{z})}$.

By the universal property, there is a unique $F' : \mathcal{A}(\underline{z}) \to \widehat{\mathcal{A}(\underline{z})}$ such that $F'H = F$ and $F'(\underline{z}) \bullet\!=\!\bullet c$. On the other hand, we have Freyd's logical functor $G' : \widehat{\mathcal{A}(\underline{z})} \to$

[5] The construction of $F'(f(z))$ is analogous to that of $F(f)$ in the proof of Theorem 5.3. Note that $\alpha(z)$ is interpreted as an arrow $1 \to PA$ in $\mathcal{B}[z]$ from which we obtain $\alpha(c)$ in $\mathcal{B}$ by the substitution functor sending z onto c; $|f(c)|$ is explained similarly.

$\mathscr{A}(\underset{\sim}{z})$ and $G'F'(\underset{\sim}{z}) \bullet = \bullet\, G'(\zeta, \underset{\sim}{z}) \bullet = \bullet\, \underset{\sim}{z}$. Again, by the universal property, $G'F'$ is the identity functor on $\mathscr{A}(\underset{\sim}{z})$.

Proposition 6.5. *Let z be a variable of type C, where $C = \Omega$ or $C = PB$. Then*

(a) *$T(\emptyset)(\underset{\sim}{z})$ is a retract of its Freyd cover,*

(b) *the terminal object of $T(\emptyset)(\underset{\sim}{z})$ is projective,*

(c) *all arrows from the terminal object to the natural number object in $T(\emptyset)(\underset{\sim}{z})$ are standard numerals.*

Proof. (a) has just been shown in case $C = \Omega$. In case $C = PB$, the problem becomes to define $\hat{z} : \hat{1} \to P\hat{B}$ such that $G'(\hat{z}) \bullet = \bullet\, \underset{\sim}{z}$. In other words, we want to fill in the top row of

$$\begin{array}{ccc}
\{*\} & \longrightarrow & S_{\underline{PB}} \\
\big\downarrow{\scriptstyle \lambda_1} & & \big\downarrow{\scriptstyle \lambda_{\underline{PB}}} \\
\Gamma(\underset{\sim}{1}) & \xrightarrow[\Gamma(\underset{\sim}{z})]{} & \Gamma(\underline{PB})
\end{array}$$

to make the square commutative. Thus, we want to find an element $z' \in S_{\underline{PB}}$ such that $\lambda_{\underline{PB}}(z') \bullet = \bullet\, \underset{\sim}{z}$. Now $S_{\underline{PB}} = \widehat{\mathscr{A}(\underset{\sim}{z})}(\hat{B}, \hat{\Omega})$, so we take $z' \bullet = \bullet\, (\zeta, z')$, where

$$\zeta(s) = (z'\lambda_{\underset{\sim}{B}}(s), 0),$$

so that the following square commutes:

$$\begin{array}{ccc}
S_{\underset{\sim}{B}} & \xrightarrow{\zeta} & \Gamma(\Omega)\dot{\cup}\{\underset{\sim}{T}\} \\
\big\downarrow{\scriptstyle \lambda_B} & & \big\downarrow{\scriptstyle \lambda_\Omega} \\
\Gamma(\underset{\sim}{B}) & \xrightarrow[\Gamma(z')]{} & \Gamma(\underset{\sim}{\Omega})
\end{array}$$

Then

$$\lambda_{\underline{PB}}(z') = \lambda_{\underline{PB}}((\zeta, z)) = \ulcorner z' \urcorner = z.$$

(b) is shown as in Freyd's argument above.

(c) is shown as in the case $A = N$ of Lemma 6.3.

Corollary 6.6. *If $C = \Omega$ or $C = PB$, every arrow $\underset{\sim}{C} \to \underset{\sim}{N}$ in $T(\emptyset)$ is constant, that is, factors through $\underset{\sim}{1}$.*

Proof. An arrow $f : \underset{\sim}{C} \to \underset{\sim}{N}$ in $T(\emptyset)$ gives rise to an arrow $f(z) : \underset{\sim}{1} \to \underset{\sim}{N}$ in $T(\emptyset)(\underset{\sim}{z})$, where

$$|f(z)| \equiv \{\langle *, x\rangle \mid \langle z, x\rangle \in |f|\}.$$

Now $f(z)$ must be a standard numeral by Proposition 6.5. Thus $f(z) \bullet \underset{\sim}{=} \bullet\, \underset{\sim}{S}^n \underset{\sim}{0}$ for some $n \in \mathbf{N}$. Therefore

$$\vdash_{\{z,x\}} \langle *, x \rangle \in |f(z)| \;\Leftrightarrow\; x = S^n 0,$$

from which it follows that f factors thus:

$$\underset{\sim}{C} \to \underset{\sim}{1} \xrightarrow{\underset{\sim}{S^n 0}} \underset{\sim}{N}.$$

New proof of the Uniformity Property

Let $C = \Omega$ or PB and suppose $\vdash \forall_{z \in C} \exists_{y \in N} \varphi(z, y)$, that is, $\vdash_z \exists_{y \in N} \varphi(z, y)$. Thus, in the topos $T(\emptyset)(\underset{\sim}{z})$ we have an epimorphism $e(z) : \alpha(z) \to \underset{\sim}{1}$, where

$$\alpha(z) \equiv \{y \in N \,|\, \varphi(z, y)\},$$

$$|e(z)| \equiv \{\langle y, * \rangle \in N \times 1 \,|\, \varphi(z, y)\}.$$

We proceed as in the proof of the $\exists$-property, but with a parameter z present. It follows that $\vdash_z \varphi(z, \alpha(z))$ for some term $\alpha(z)$ of type N. However, in $\mathscr{L}_1(z)$ there are no such terms except numerals. Thus $\alpha(z) \equiv \bar{n}$, hence $\vdash \forall_{z \in C} \varphi(z, \bar{n})$, and therefore $\vdash \exists_{y \in N} \forall_{z \in C} \varphi(z, y)$.

Appendix 1

We shall prove the converse of Proposition 3.8 along different lines than those suggested at the end of Section 3. To illustrate the argument, we take $k = 1$.

Let $T(e, m, n)$ be Kleene's T-predicate, as in [13, p. 287] or [22, p. 243]. If $\{f_e | e \in N\}$ is an enumeration of all partical recursive functions in one variable, $T(e, m, n)$ essentially asserts that n is the Gödel number of a proof that $f_e(m)$ is defined. It is known that $T(e, m, n)$ is a recursive predicate, hence representable by a formula $\tau(w, x, y)$ in $\mathscr{L}_1$ in the sense that

if $T(e, m, n)$, then $\vdash \tau(\bar{e}, \bar{m}, \bar{n})$,

if not $T(e, m, n)$, then $\vdash \neg \tau(\bar{e}, \bar{m}, \bar{n})$.

Without loss in generality one may assume that

if $T(e, m, n)$ and $T(e, m, n')$, then $n = n'$,

for example, by taking n as small as possible.

It is known that there is a primitive recursive function U such that $U(n) = f_e(m)$ in case there exist numbers e and m so that $T(e, m, n)$. We shall also assume that a function symbol h has been adjoined to $\mathscr{L}_1$ such that

$\vdash h\bar{n} = \bar{k}$ if and only if $U(n) = k$.

Church's Rule [26, p. 258] asserts that if $\vdash \forall_{x \in N} \exists_{y \in N} \varphi(x, y)$, then there is a number e such that $\vdash \forall_{x \in N} \exists_{y \in N} (\tau(\bar{e}, x, y) \wedge \varphi(x, hy))$.

The validity of Church's Rule has been established for second order intuitionist arithmetic [26, p. 322] and for intuitionist type theory by Girard (see his article in the same volume as [7].)

Proposition 7.0. *Church's Rule implies that every strongly representable function is provably recursive.*

Proof. Suppose $f : \mathbf{N} \to \mathbf{N}$ is strongly representable by $\varphi(x, y)$. Then

(1) $\vdash \forall_{x \in N} \exists !_{y \in N} \varphi(x, y)$,

(2) if $f(m) = n$, then $\vdash \varphi(\bar{m}, \bar{n})$.

It follows from (1) that there is a number e such that

$$\vdash \forall_{x \in N} \exists_{y \in N} (\tau(\bar{e}, x, y) \wedge \varphi(x, hy)).$$

Now define the function $g : \mathbf{N} \to \mathbf{N}$ by

$$g(m) = U(\mu_n T(e, m, n)).$$

We claim that (a) g is provably recursive, (b) $g = f$.

This will complete the proof of the proposition.

To prove (a), we replace T by its characteristic function t, so that

$$t(e, m, n) = \begin{cases} 0 & \text{if } T(e, m, n), \\ 1 & \text{otherwise.} \end{cases}$$

We then have

$$g(m) = U(\mu_n (t(e, m, n) = 0)).$$

To prove that g is provably recursive, it suffices to show that $t(e, m, n)$, regarded as a function of its last two arguments, is strongly representable by a formula $\chi_e(x, y, z)$ such that $\vdash \forall_{x \in N} \exists_{y \in N} \chi_e(x, y, 0)$. Let $\chi_e(x, y, z) \equiv (\tau(e, x, y) \wedge z = 0) \vee (\neg\, \tau(e, x, y) \wedge z = 1)$.

Indeed, $t(e, m, n)$ is either 0 or 1. In the first case, $T(e, m, n)$, hence $\vdash \tau(\bar{e}, \bar{m}, \bar{n})$, and so $\vdash \chi_e(\bar{m}, \bar{n}, 0)$. Similarly we see that, in the second case, $\vdash \chi_e(\bar{m}, \bar{n}, \bar{1})$. Moreover,

$$\vdash \forall_{x \in N} \forall_{y \in N} \exists !_{z \in N} \chi_e(x, y, z)$$

holds because

$$\vdash \forall_{x \in N} \forall_{y \in N} (\tau(\bar{e}, x, y) \vee \neg\, \tau(\bar{e}, x, y)).$$

To complete the proof of (a), we must verify that

$$\vdash \forall_{x \in N} \exists_{y \in N} \chi_e(x, y, 0).$$

This holds because

$$\vdash \forall_{x \in N} \exists_{y \in N} \tau(\bar{e}, x, y)$$

in view of Church's Rule.

To prove (b), let $f(m) = n$. Then $\vdash \varphi(\bar{m}, \bar{n})$. Now by Church's Rule,

$$\vdash \forall_{x \in N} \exists_{y \in N} (\tau(\bar{e}, x, y) \wedge \varphi(x, hy)).$$

In view of the Fundamental Theorem, there is a number k such that

$$\vdash \tau(\bar{e}, \bar{m}, \bar{k}) \wedge \varphi(\bar{m}, h\bar{k}).$$

Since $\vdash \exists!_{y \in N} \varphi(\bar{m}, y)$, we have $\vdash h\bar{k} = \bar{n}$, hence $U(k) = n = f(m)$. Now $T(e, m, k)$ and k is the smallest number with this property. (For if also $T(e, m, k')$, then $k = k'$.) Therefore, $f(m) = U(k) = g(m)$.

This completes the proof of (b), hence of Proposition 7.0.

Appendix 2

We shall verify the claim made in the proof of Theorem 5.3 that $m_\alpha \bullet\!=\!\bullet \ker(\alpha')$. To this end we must carefully examine the translation procedure for interpreting the language $\mathscr{L}_1$ of intuitionist type theory in a topos $\mathscr{A}$ in case $\mathscr{A} = T(\emptyset)$ is the free topos. We begin by giving another construction for the predogma $\mathscr{A}[z]$ in this case.

We construct the category $T(\emptyset)[\underline{z}]$ whose objects are the same as those of $T(\emptyset)$ and whose arrows $f(z) : \alpha \to \beta$ are triplets $(\alpha, |f(z)|, \beta)$, where $|f(z)|$ is a term of type $P(A \times B)$ containing no free variables other than the "parameter" z of type C, subject to the conditions (1) and (2) of Definition 4.3 with $|f|$ replaced by $|f(z)|$.

$T(\emptyset)[\underline{z}]$ is easily seen to be a predogma, its predogma structure being inherited from $T(\emptyset)$. Moreover, the inclusion functor $H : T(\emptyset) \to T(\emptyset)[\underline{z}]$ preserves the predogma structure. The predogma $T(\emptyset)[\underline{z}]$ contains the special arrow $\underline{z} : \underline{1} \to \underline{C}$ with $|\underline{z}| \equiv \{\langle *, z \rangle\}$. We shall see that $\underline{z}$ plays the rôle of an indeterminate, so that $T(\emptyset)[\underline{z}]$ has the same universal property as $T(\emptyset)[z]$, hence is isomorphic to it. We first establish what has been called "functional completeness" [16].

Proposition 7.1. *For every arrow $f(z) : \alpha \to \beta$ in $T(\emptyset)[\underline{z}]$ there is a unique arrow $g : \underline{C} \times \alpha \to \beta$ in $T(\emptyset)$ such that $g\langle \underline{z} *_\alpha, 1_\alpha \rangle \bullet\!=\!\bullet f(z)$ in $T(\emptyset)[\underline{z}]$.*

Proof. Take

$$|g| \equiv \{\langle\langle z, x \rangle, y \rangle \in (C \times A) \times B \mid \langle x, y \rangle \in |f(z)|\}$$

and check that the required equations holds. As for uniqueness, from $h\langle \underline{z} *_\alpha, 1_\alpha \rangle \bullet\!=\!\bullet f(z)$ it easily follows that $h \bullet\!=\!\bullet g$.

Corollary 7.2. $T(\emptyset)[\underline{z}] \cong T(\emptyset)[z]$.

The corollary is an immediate consequence of the proposition in view of the following "recognition lemma".

Lemma 7.3. *Suppose $\mathscr{A}'$ is a predogma extending the predogma $\mathscr{A}$, having the same objects as $\mathscr{A}$ and containing an arrow $\zeta : 1 \to C$ so that for every arrow $f : A \to B$ in $\mathscr{A}'$ there is a unique arrow $g : C \times A \to B$ in $\mathscr{A}$ such that $g\langle \zeta *_A, 1_A \rangle \bullet\!=\!\bullet f$. Then, as* *extensions of* $\mathscr{A}$, $\mathscr{A}' \equiv \mathscr{A}[z]$, ζ *corresponding to the indeterminate* $z : 1 \to C$ *under the isomorphism.*

Proof. It suffices to show that $\mathscr{A}'$ has the expected universal property. Suppose $F : \mathscr{A} \to \mathscr{B}$ is any functor into a predogma $\mathscr{B}$ which preserves the predogma structure exactly and $c : 1 \to F(C)$ is a given arrow in $\mathscr{B}$. We claim that there is a unique functor $F' : \mathscr{A}' \to \mathscr{B}$ preserving the predogma structure such that $F'H = F$ and $F'(\zeta) \bullet = \bullet\, c$. (Recall that H is the inclusion functor.)

Define F' on objects and arrows as follows:

$$F'(A) = F(A),\ F'(g\langle \zeta *_A, 1_A \rangle) \bullet = \bullet\, F(g)\langle c *_{F(A)}, 1_{F(A)} \rangle .$$

One easily verifies that F' has the required properties and is unique. We omit the routine calculations.

We remark that this lemma remains valid if the word "predogma" is replaced by "cartesian category", "cartesian closed category" or "dogma".

We now return to the interpretation of $\mathscr{L}_1$ in the free topos.

Proposition 7.4. *Let* $t = t(x_1, \dots, x_n)$ *be a term of type* A *in the language* $\mathscr{L}_1$. *If* t *is interpreted as an arrow* $t : \underset{\sim}{1} \to \underset{\sim}{A}$ *in* $T(\emptyset)[\underset{\sim}{x}_1, \dots, \underset{\sim}{x}_n]$, *then* $\vdash_{\{x_1, \dots, x_n\}} |t| = \{\langle *, t \rangle\}$.

Proof. We proceed by induction on the construction of t.

(1) If t is a variable x of type A, then its interpretation in $\mathscr{A}[x]$ is the indeterminate $x : 1 \to A$. In particular, its interpretation in $T(\emptyset)[\underset{\sim}{x}]$ is the arrow $\underset{\sim}{x} : \underset{\sim}{1} \to \underset{\sim}{A}$, where $|\underset{\sim}{x}| \equiv \{\langle *, x \rangle\}$.

(2) If t is $*$ of type 1, its interpretation in Section 5 was the identity arrow $1_{\underset{\sim}{1}}$. According to Definition 4.3,

$$\vdash |1_{\underset{\sim}{1}}| = \{\langle x, x \rangle \in 1 \times 1 \mid x \in \underset{\sim}{1}\} = \{\langle *, * \rangle\}.$$

(3) If t is of type N other than a variable, it will be 0 or Sn for some n of type N. In the first case, it is interpreted as $\underset{\sim}{0}$, and $|\underset{\sim}{0}| \equiv \{\langle *, 0 \rangle\}$ by Definition 4.3. In the second case, it is interpreted as $\underset{\sim}{S}n$, where

$$|\underset{\sim}{S}n| \equiv \{\langle x, z \rangle \in 1 \times N \mid \exists_{y \in N}(\langle x, y \rangle \in |n| \wedge \langle y, z \rangle \in |\underset{\sim}{S}|)\}.$$

By inductional assumption and Definition 4.3,

$$\begin{aligned}\vdash |\underset{\sim}{S}n| &= \{\langle x, z \rangle \in 1 \times N \mid x = * \wedge \langle n, z \rangle \in |\underset{\sim}{S}|\} \\ &= \{\langle *, Sn \rangle\}.\end{aligned}$$

(4) If t is of type $A \times B$ other than a variable, the result follows easily by inductional assumption.

(5) If t is of type Ω other than a variable, there are two cases: t may be $a = a'$ or $a \in \alpha$. In the first case we have

$$|a = a'| \equiv |\delta_{\underset{\sim}{A}}\langle a, a' \rangle|.$$

Using the fact that

$$|\delta_A| \equiv |\text{char } \langle 1_A, 1_A \rangle|$$
$$\equiv \{\langle\langle x,x'\rangle, t\rangle \in (A \times A) \times \Omega \mid t = (x = x')\},$$

we easily calculate $\vdash |a = a'| = \{\langle *, a = a'\rangle\}$. In the second case we have

$$|a \in \alpha| \equiv |\in_A \langle a, \alpha\rangle|.$$

Using the definition of $|\in_\alpha|$ in Definition 4.3 with $\alpha = A$, one proceeds similarly.

(6) If t is of type PA other than a variable, say $\alpha \equiv \{\langle x \in A \mid \varphi(x)\}$, we have, by inductional assumption,

$$\vdash_x |\varphi(x)| = \{\langle *, \varphi(x)\rangle\} = \{\langle *, x \in \alpha\rangle\}.$$

On the other hand,

$$\vdash_x \varphi(x) = |\in_A \langle \alpha, x\rangle|$$
$$= \{\langle *, t\rangle \in 1 \times \Omega \mid \exists_{u \in PA}(\langle *, u\rangle \in |\alpha| \wedge t = (x \in u))\}$$

after some calculation. Therefore

$$\vdash_{\{x,t\}} t = (x \in \alpha) \;\Leftrightarrow\; \exists_{u \in PA}(\langle *, u\rangle \in |\alpha| \wedge t = (x \in u)).$$

From this we easily infer that

$$\vdash_u \langle *, u\rangle \in |\alpha| \;\Leftrightarrow\; u = \alpha,$$

whence $\vdash |\alpha| = \{\langle *, \alpha\rangle\}$.

Corollary 7.5. *In the free topos $T(\emptyset)$, $m_\alpha \bullet\!=\!\bullet \ker(\alpha')$.*

Proof. We recall that

$$\alpha' \bullet\!=\!\bullet \in_A \langle \alpha *_A, 1_A\rangle.$$

(See the definition preceding Lemma 5.2.) From this one easily calculates

$$\vdash |\alpha'| = \{\langle x, t\rangle \in A \times \Omega \mid \exists_{u \in PA}(t = (x \in u) \wedge \langle *, u\rangle \in |\alpha|)\}.$$

Using the fact that $\vdash |\alpha| = \{\langle *, \alpha\rangle\}$, by Proposition 7.4, one then deduces that

$$\vdash |\alpha'| = \{\langle x, t\rangle \in A \times \Omega \mid t = (x \in \alpha)\}.$$

Hence, by Definition 4.3,

$$\vdash |\ker \alpha'| = \{\langle x, x\rangle \in A \times A \mid \langle x, T\rangle \in |\alpha'|\}$$
$$= \{\langle x, x\rangle \in A \times A \mid x \in \alpha\} = |m_\alpha|.$$

This completes the proof.

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