INDEPENDENCE OF PREMISSES AND THE FREE TOPOS *

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§0. Introduction

In this article we shall be talking about our formulation of intuitionist type theory, two versions of which have been described elsewhere [LS1, LS2] . To make the present discussion more accessible, we shall briefly describe the formation rules of this language L_1. However, for the precise axioms and rules of inference the reader should look at the cited references.

We have a hierarchy of types consisting of three primitive types: 1 (thought of as a one-point set), N (the type of natural numbers) and Ω (the type of truth-values or propositions), and two rules for creating new types from old ones: from A and B one may form A×B (the Cartesian product) and from A one may form PA ($=\Omega^A$, the power-set of A).

Each term of L_1 will belong to some type. In addition to a countable list of variables of each type, there are the following terms, each listed under its type:

1	N	Ω	A×B	PA
$*$	0	$a=a'$	$\langle a,b\rangle$	$\{x\in A \mid \phi(x)\}$
	Sn	$a\in\alpha$		

where the following type assignments are assumed: n has type N, a,a' have type A, b has type B, α has type PA, and $\phi(x)$ has type Ω .

This is the version of L_1 given in [LS2]. In [LS1] equality is defined in terms of the usual logical symbols $\top$, $\wedge$, $\Rightarrow$, and $\forall$. However, these may be defined in terms of equality as follows, where p, q, and $\phi(x)$ are of type Ω :

$$\begin{aligned} \top &\equiv * = * \\ p \wedge q &\equiv \langle p,q\rangle = \langle \top,\top\rangle \\ \forall_{x\in A}\phi(x) &\equiv \{x\in A \mid \phi(x)\} = \{x\in A \mid \top\} \\ p \Rightarrow q &\equiv p \wedge q = p \end{aligned}$$

It is well-known how the remaining symbols may be obtained [P] :

$$\begin{aligned} \bot &\equiv \forall_{t\in\Omega} t \\ p \vee q &\equiv \forall_{t\in\Omega}(((p \Rightarrow t) \wedge (q \Rightarrow t)) \Rightarrow t), \\ \exists_{x\in A}\phi(x) &\equiv \forall_{t\in\Omega}(\forall_{x\in A}(\phi(x) \Rightarrow t) \Rightarrow t) \ , \end{aligned}$$

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$$\neg p \;\equiv\; p \Rightarrow \bot$$

$$\exists!_{x\in A}\phi(x) \;\equiv\; \exists_{x'\in A}(\{x\in A \mid \phi(x)\} = \{x\in A \mid x = x'\})$$

The axioms and rules of inference in terms of equality may be a little unfamiliar, but in terms of $\top$, $\wedge$, $\Rightarrow$ and $\forall$ they are as in the usual intuitionistic predicate calculus, augmented by extensionality, comprehension, Peano's axioms and the obvious axioms governing $*$ and $\langle a,b \rangle$. For details the reader is referred to [LS1] or [LS2].

The term model of L_1 can be made into a topos F, the so-called free topos, actually the topos freely generated by the empty graph. F is an initial object in the category of toposes and logical morphisms. (We shall here assume that a topos contains a natural numbers object.) The objects of F are closed terms of type PA for different types A modulo provable equality. The morphisms are names of provably functional relations modulo provable equality.

Various metatheorems about L_1, e.g. the disjunction rule, the existence rule and the uniformity rule, are equivalent to algebraic properties of the free topos. We originally used realizability methods to prove these metatheorems and deduced properties of F from them. However, Peter Freyd discovered an ingenious direct algebraic method for proving the algebraic theorems [Fr1], from which the metatheorems then follow (see [LS1]).

In the present paper we shall continue this program. In particular, we are interested in closed formulas p for which one has independence of premisses [T], that is, from $\vdash p \Rightarrow \exists_{x\in A}\phi(x)$ one can infer $\vdash \exists_{x\in A}(p \Rightarrow \phi(x))$. It turns out that this property of p is equivalent to a strong form of projectivity of the associated object $\{x\in 1 \mid p\}$ in F. It is also equivalent to strong projectivity of the terminal object 1 in the "free topos on the assumption p", an initial object in the category of all toposes in which p holds.

There are two other properties of closed formulas which imply strong projectivity: "Freydian" [Fr2] and "hereditary". By investigating which closed formulas p have these stronger properties, we obtain two proofs (the first following Freyd) of the fact that any "stable" p (for which $\vdash \neg\neg p \Rightarrow p$) satisfies independence of premisses. As yet our results concerning independence of premisses are not as complete as those obtained by Troelstra for second order arithmetic [T]. The reason for this is that the corresponding properties of open formulas are being left to a future investigation.

§1. The free topos modulo p

Let p be a closed formula in intuitionistic type theory, say the language L_1 in [LS1], then we can define the language L_p just like L_1 but with "$\vdash_X$" replaced by "$p \vdash_X$". Thus the terms of L_p are the same as those of L_1, but deducibility is now "on the assumption p" or "modulo p". Let F_p be the topos constructed from L_p just as the free topos F was constructed from L_1 in [LS1]. Thus the objects of F_p are closed terms of type PA, where A is any type, and equality between objects is provable equality from the assumption p: $\alpha \underset{p}{=} \beta$ if and only if $p \vdash \alpha = \beta$

The morphisms $f: \alpha \to \beta$ of $\mathcal{F}_p$ are triples $(\alpha, |f|, \beta)$, where α is of type PA, β is of type PB and $|f|$ is a closed term of type $P(A\times B)$ such that

$$(1) \quad p \vdash \forall_{x\in A}\forall_{y\in B}(\langle x,y\rangle \in |f| \Rightarrow (x\in\alpha \wedge y\in\beta))$$

$$(2) \quad p \vdash \forall_{x\in A}(x\in\alpha \Rightarrow \exists!_{y\in B}\langle x,y\rangle \in |f|)$$

Again, equality is defined modulo p :

$$f \underset{p}{\cdot\!=\!\cdot} g \quad \text{if and only if} \quad p\vdash |f|=|g| \quad .$$

Note, in particular, that $\mathcal{L}_1 = \mathcal{L}_\top$ and $\mathcal{F} = \mathcal{F}_\top$. An examination of the syntactic construction of the free topos $\mathcal{F}$ (actually, the free topos generated by the empty graph) in [LS1] shows that everything still works in $\mathcal{L}_p$. In particular, we have

PROPOSITION 1.1. $\mathcal{F}_p$ is a topos with canonical subobjects.

At this point we pause to recall that for us a topos is understood to have canonical finite products and power-sets and also a natural numbers object. The free topos is an initial object in the category TOP whose objects are toposes with canonical subobjects [L, Section 9] and whose morphisms are <u>strict logical functors</u>, that is, functors which preserve everything on the nose (including canonical subobjects) .

There is an obvious functor $F_p: \mathcal{F} \to \mathcal{F}_p$ which sends the object α of $\mathcal{F}$ onto the object α of $\mathcal{F}_p$. Actually the former is the equivalence class of the closed term α modulo provable equality, while the latter is the equivalence class of α modulo $\underset{p}{=}$. Similarly F_p sends each morphism $(\alpha, |f|, \beta)$ of $\mathcal{F}$ onto a morphism of $\mathcal{F}_p$ with the same name. It is easily verified that F_p is a strict logical functor. Thus we have:

LEMMA 1.2. The "obvious" functor $F_p:\mathcal{F} \to \mathcal{F}_p$ is the unique strict logical functor from $\mathcal{F}$ to $\mathcal{F}_p$.

Each type A of $\mathcal{L}_1$ gives rise to an object $A_{\mathcal{A}}$ of each topos $\mathcal{A}$. To avoid heavy notation, we omit the subscripts on objects. Also, in each topos $\mathcal{A}$ we can interpret any closed term a of type A of $\mathcal{L}_1$ as an arrow $a_{\mathcal{A}}: 1 \to A$, in particular, any closed formula p of $\mathcal{L}_1$ as an arrow $p_{\mathcal{A}}: 1 \to \Omega$. In [LS1] the subscripts were suppressed; but here we shall be more pedantic, omitting the subscripts only from such constants as $*$, 0, $\top$, etc.

If $L: \mathcal{A} \to \mathcal{B}$ is a strict logical functor between toposes with canonical subobjects, then clearly $L(a_{\mathcal{A}}) \cdot\!=\!\cdot a_{\mathcal{B}}$. In particular, if $F_{\mathcal{A}} : \mathcal{F} \to \mathcal{A}$ is the unique strict logical functor, we see that $F_{\mathcal{A}}(a_{\mathcal{F}}) \cdot\!=\!\cdot a_{\mathcal{A}}$.

We shall write $\mathcal{A} \models p$ for $p_{\mathcal{A}} \cdot\!=\!\cdot \top$ and say that p <u>holds</u> in $\mathcal{A}$. The theorems of $\mathcal{L}_1$ hold in all toposes. More generally, if $\mathcal{A} \models p$ and $p\vdash q$ then $\mathcal{A} \models q$.

PROPOSITION 1.3. If p and q are closed formulas of $\mathcal{L}_1$, $\mathcal{F}_p \models q$ if and only if $p\vdash q$. In particular, $\mathcal{F}_p \models p$.

<u>Proof</u>: $\mathcal{F}_p \models q$ means that $q_{\mathcal{F}_p} \cdot\!=\!\cdot \top$ in $\mathcal{F}_p$. Now recall from [LS1, Prop. 7.4] that $|q_{\mathcal{F}}| = \{\langle *,q\rangle\}$ in $\mathcal{F}$. Applying the functor F_p to this, and recalling from LEMMA 1.2 that $F_p = F_{\mathcal{F}_p}$, we obtain also that $|q_{\mathcal{F}_p}| = \{\langle *,q\rangle\}$ in $\mathcal{F}_p$. Therefore $q_{\mathcal{F}_p} \cdot\!=\!\cdot \top$ means that $\{\langle *,q\rangle\} = \{\langle *,\top\rangle\}$ in $\mathcal{F}_p$, that is, $p\vdash \{\langle *,q\rangle\} = \{\langle *,\top\rangle\}$, that is, $p\vdash q$.

This well-known result is a kind of completeness theorem. In fact, everything we have said up to now remains valid if we replace the closed formula p by any set of closed formulas. The corollary then looks like the usual completeness theorem in logic.

PROPOSITION 1.5. $\mathcal{F}_p$ is an initial object in the full subcategory of TOP consisting of those toposes $\mathcal{A}$ with canonical subobjects for which $\mathcal{A} \models p$.

<u>Proof:</u> We must show that there is a unique strict logical functor $G\colon \mathcal{F}_p \to \mathcal{A}$ in case $\mathcal{A} \models p$. Consider the diagram

$$\begin{tikzcd} \mathcal{F}_p \arrow[dr, dashed, "G"] & \\ \mathcal{F} \arrow[u] \arrow[r, "F_{\mathcal{A}}"'] & \mathcal{A} \end{tikzcd}$$

where G is defined on objects by $G(\alpha) = F_{\mathcal{A}}(\alpha)$ and on morphisms by $G(f) = F_{\mathcal{A}}(f)$. To see that G is well-defined, we must show that $\alpha \underset{p}{=} \beta$ implies $F_{\mathcal{A}}(\alpha) = F_{\mathcal{A}}(\beta)$, and similarly on morphisms. We shall carry out the argument for objects and leave the morphisms to the reader.

Suppose $\alpha \underset{p}{=} \beta$, that is, $p \vdash \alpha = \beta$. Since $\mathcal{A} \models p$, it follows that $\mathcal{A} \models \alpha = \beta$. Now let $\alpha_{\mathcal{A}} : 1 \to PA$ be the interpretation of α in $\mathcal{A}$, then it follows that $\alpha_{\mathcal{A}} \cdot\!=\!\cdot \beta_{\mathcal{A}}$ (recalling the slogan that internal equality implies external equality [L]). Now the arrow $\alpha_{\mathcal{A}}\colon 1 \to PA$ corresponds to $\alpha_{\mathcal{A}}{}^{\ulcorner} : A \to \Omega$, and so we have $\alpha_{\mathcal{A}}{}^{\ulcorner} \cdot\!=\!\cdot \beta_{\mathcal{A}}{}^{\ulcorner}$. Recall from [LS1, Section 5] that $F_{\mathcal{A}}(\alpha)$ was by definition the canonical subobject of A for which $\alpha^{\ulcorner}$ is the characteristic morphism. It then follows that $F_{\mathcal{A}}(\alpha) = F_{\mathcal{A}}(\beta)$.

It is easily checked that G is a strict logical functor and that it is unique.□

There are two other ways in which the free topos modulo p can be constructed. Let us recall from [LS3] that given an object A of a topos $\mathcal{A}$ with canonical subobjects, it is always possible to adjoin an indeterminate arrow $x\colon 1 \to A$ to $\mathcal{A}$ and obtain a new topos $\mathcal{A}(x)$ with canonical subobjects and with the expected universal property. Moreover, $\mathcal{A}(x)$ is equivalent to the slice category $\mathcal{A}/A$ by a result of Grothendieck, Verdier, Joyal [GV], [LS3]. In particular, given an object α of the free topos $\mathcal{F}$, that is, a closed term of some type PA in the language L_1, we can adjoin an indeterminate $x\colon 1 \to \alpha$ to form $\mathcal{F}(x)$. This comes equipped with a strict logical functor $H\colon \mathcal{F} \to \mathcal{F}(x)$ so that for each topos $\mathcal{A}$ with canonical subobjects and each arrow $a\colon 1 \to F_{\mathcal{A}}(\alpha)$ in $\mathcal{A}$, there is a unique strict logical functor $G\colon \mathcal{F}(x) \to \mathcal{A}$ such that $GH = F_{\mathcal{A}}$ and $G(x) \cdot\!=\!\cdot a$.

$$\begin{tikzcd} \mathcal{F}(x) \arrow[dr, dashed, "G"] & & x\colon 1 \to \alpha \arrow[d, leftrightarrow] \\ \mathcal{F} \arrow[u, "H"] \arrow[r, "F_{\mathcal{A}}"'] & \mathcal{A} & a\colon 1 \to F_{\mathcal{A}}(\alpha) \end{tikzcd}$$

Look now at the special case when $\alpha = \{x \in 1 \mid p\}$. Then $F_{\mathcal{A}}(\alpha) = \mathrm{Ker}(\alpha_{\mathcal{A}}{}^{\ulcorner}) = \mathrm{Ker}(p_{\mathcal{A}})$ according to the definition of $F_{\mathcal{A}}$ [LS1, Section 5], so that we have a pullback:

$$\begin{tikzcd} 1 \arrow[r, "p_A"] & \Omega \\ F_A(\alpha) \arrow[u] \arrow[r] & 1 \arrow[u, "\top"'] \end{tikzcd}$$

Since there is a canonical monomorphism $F_A(\alpha) \to 1$ in A, the arrow $a: 1 \to F_A(\alpha)$ is unique whenever it exists. When does it exist? In view of the pullback if and only if $p_A \doteq \top$, that is, $A \models p$.

Thus, if $F(x)$ is obtained from F by adjoining an indeterminate arrow $x: 1 \to \{x \in 1 \mid p\}$, $F(x)$ is also an initial object in the category of all toposes A with canonical subobjects such that $A \models p$. Since initial objects are unique up to isomorphism, we have thus established the following:

PROPOSITION 1.6. F_p is isomorphic to $F(x)$, with $x: 1 \to \{x \in 1 | p\}$, and equivalent to $F/\{x \in 1 | p\}$.

§2. Description modulo p

The $\exists!$-rule for L_p at type A asserts: if $p \vdash \exists!_{x \in A} \phi(x)$, then $p \vdash \phi(a)$ for some closed term a of type A. If this is so for all types A, we say the $\exists!$-rule holds for L_p. In the language of Kleene [K], this means that, in the absence of parameters, descriptions are eliminable in L_p not just by contextual but by explicit definitions.

Whether the $\exists!$-rule holds for L_p depends on the formula p. For example, when $p \equiv \top$ then $L_\top = L_1$ is ordinary intuitionist type theory, for which the $\exists!$-rule was proved in [LS1, Lemma 6.3]. As we shall see later, the $\exists!$-rule holds for a large class of formulas, the strongly projective ones. On the other hand, if p is the Boolean axiom $\beta \equiv \forall_{t \in \Omega}(t \vee \neg t)$, then the $\exists!$-property fails.

To see this, let γ be Godel's classically undecidable sentence and put

$$\phi(x) \equiv (x=0 \Rightarrow \gamma) \wedge (x \neq 0 \Rightarrow \neg\gamma)$$

In classical type theory L_β, clearly $\beta \vdash \exists_{x \in N} \phi(x)$. Let

$$\psi(x) \equiv \phi(x) \wedge \forall_{y \in N}(\phi(y) \Rightarrow x \leq y)$$

then clearly $\beta \vdash \exists!_{x \in N} \psi(x)$. However, there is no closed term t of type N in L_β such that $\beta \vdash \psi(t)$. For the only closed terms of type N are the standard numerals 0, S0, S(S0), ... If a standard numeral t were to satisfy ψ, hence ϕ, then we would be able to decide Godel's sentence, depending on whether t was 0 or not.

We may say that ψ defines a nonstandard numeral, the unique x for which $\psi(x)$, which happens to have no name in the language L_β. Note that ψ gives rise to an arrow $f: 1 \to N$ in F_β with $|f| \equiv \{\langle *, x\rangle | \psi(x)\}$, which is not of the form 0, S0, S(S0), S(S(S0)), . . . etc.

The above discussion shows that the $\exists!$-rule may fail for L_p because nonstandard numerals are definable. In fact, this is the only thing that can go wrong.

PROPOSITION 2.1. The $\exists!$-rule holds for L_p if and only if it holds at type N, that is, all the arrows $1 \to N$ in F_p are induced by standard numerals.

Proof: The proof for L_1 [LS1, Lemma 6.3] by induction on types remains valid at types 1, Ω, $B\times C$ and PB, and so the only difficult case is type N, which we assume.□

If the $\exists!$-property does not hold for L_p, we may restore it by introducing a description operator. As we have seen, it suffices to introduce a minimization operator μ. Given any formula $\phi(x)$ with x of type N such that

$$(*) \qquad p \vdash \exists_{x\in N}(\phi(x) \wedge \forall_{y\in N}(\phi(y) \Rightarrow x\leq y)) ,$$

we adjoin the term $\mu\phi \equiv \mu_{x\in N}\phi(x)$ together with the additional axiom:

$$p \vdash \phi(\mu\phi) \wedge \forall_{y\in N}(\phi(y) \Rightarrow \mu\phi\leq y)$$

The resulting language will be called L_p^μ. Note that the $\exists!$-rule holds in L_p if and only if $L_p^\mu = L_p$.

One may be tempted to assume in place of (*) above merely that

$$(**) \qquad p \vdash \exists_{x\in N}\phi(x)$$

and hope that (*) is a consequence of (**). The implication (**) $\Rightarrow$ (*) is the so-called "least number principle" on the assumption p. It holds for some p, e. g., if p is the Boolean axiom, but not for others, e. g., when $p \equiv \top$.

§3. Projective formulas

We shall call the closed formula p of L_1 projective if the associated subobject of 1, namely $\{x\in 1|p\}$, is a projective object in the topos F in the usual sense, that is, if all epimorphisms $\alpha \to \{x\in 1|p\}$ split. For example, we know from [LS1] that $\top$ is projective. As we shall see, there are plenty of other projective formulas.

In the following, let α and β be closed terms of types PA and PB respectively.

LEMMA 3.1. An arrow $\alpha \to \beta$ in the free topos F is an epimorphism if and only if it is "provably surjective," that is, $\vdash \forall_{y\in B}(y\in\beta \Rightarrow \exists_{x\in A}(\langle x,y\rangle\in|e|))$.

Proof: ($\Leftarrow$) See [LS1, Lemma 6.2]

($\Rightarrow$) Consider the diagram $\alpha \xrightarrow{e} \beta \overset{f}{\underset{g}{\rightrightarrows}} \Omega$

where f and g are defined as follows. Put

$$\mathrm{Im}(e) \equiv \{y\in B|\exists_{x\in A}(\langle x,y\rangle\in|e|)\}$$

and let $\mathrm{im}(e): \mathrm{Im}(e) \to \beta$ be the canonical monomorphism with

$$|\mathrm{im}(e)| \equiv \{\langle y,y\rangle\in B\times B|y\in\mathrm{Im}(e)\}$$

Put $f \cdot\!=\!\cdot\ \mathrm{char}(\mathrm{im}(e))$ and $g \cdot\!=\!\cdot\ \top O_\beta$. Then, by [LS1, Section 4],

$$|f| \equiv \{\langle y,t\rangle\in B\times\Omega|t=\exists_{z\in B}(\langle y,z\rangle\in|\mathrm{im}(e)|)\}$$

Straightforward calculations now show that

$$\vdash |f| = \{\langle y,t\rangle\in B\times\Omega|t=\exists_{x\in A}(\langle x,y\rangle\in|e|)\}$$

$$\vdash |g| = \{\langle y,t\rangle \in B\times\Omega \mid t = \top \wedge y\in\beta\}$$

whence it is easy to compute that $\vdash |fe| = |ge|$, hence $fe \cdot\!=\!\cdot\, ge$. Since e is an epimorphism, $f \cdot\!=\!\cdot\, g$, and so $\vdash |f| = |g|$. From this it follows that

$$\vdash \forall_{y\in B}(y\in\beta \Rightarrow \exists_{x\in A}(\langle x,y\rangle \in |e|))$$

as was to be proved.

LEMMA 3.2. If A is an object of the topos $\mathcal{A}$, A is projective if and only if the terminal object is projective in $\mathcal{A}/A$.

Proof: The result follows immediately from the following two observations:

(a) $e: B \to A$ is an epimorphism in $\mathcal{A}$ if and only if the commutative square

$$\begin{array}{ccc} B & \xrightarrow{e} & A \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle 1_A} \\ A & \xrightarrow[1_A]{} & A \end{array}$$

is an epimorphism in $\mathcal{A}/A$;

(b) the former splits if and only if the latter does.

THEOREM 3.3. The following statements are equivalent:

(i) the terminal object is projective in $\mathcal{F}_p$;

(ii) p is projective, that is, $\{x\in 1 \mid p\}$ is projective in $\mathcal{F}$;

(iii) the $\exists$-rule holds in $\mathcal{L}_p^\mu$, that is, for any type A in $\mathcal{L}_p^\mu$, if $p \vdash \exists_{x\in A}\phi(x)$ then $p \vdash \phi(a)$ for some closed term of type A.

Note that, in case the $\exists!$-rule holds for $\mathcal{L}_p$, $\mathcal{L}_p^\mu$ in the above theorem may be replaced by $\mathcal{L}_p$.

Proof: By Lemma 3.2, $\{x\in 1 \mid p\}$ is projective in $\mathcal{F}$ if and only if the terminal object is projective in the slice topos $\mathcal{F}/\{x\in 1 \mid p\}$. Moreover, by Proposition 1.6, the latter topos is equivalent to $\mathcal{F}_p$. Thus (i) $\Leftrightarrow$ (ii). We shall now prove that (ii) $\Leftrightarrow$ (iii).

Suppose (ii) and assume that $p \vdash \exists_{x\in A}\phi(x)$. Put $\alpha \equiv \{x\in A \mid \phi(x)\}$ and $\beta \equiv \{y\in 1 \mid p\}$, and define $e: \alpha \to \beta$ by

$$|e| \equiv \{\langle x,y\rangle \in A\times 1 \mid \phi(x) \wedge y = *\}$$

By Lemma 3.1 and hypothesis, e is an epimorphism. Since β is projective, e splits, that is, there is an arrow $m: \beta \to \alpha$ such that $em \cdot\!=\!\cdot\, 1_\beta$. Since m is a morphism in $\mathcal{F}$, we have

(1) $\vdash \forall_{x\in A}(\langle *,x\rangle \in |m| \Rightarrow (p \wedge x\in\alpha))$,

(2) $\vdash p \Rightarrow \exists!_{x\in A}(\langle *,x\rangle \in |m|)$

If we put $\psi(x) \equiv \langle *,x\rangle \in |m|$, these two conditions may be rewritten as follows:

(1)′ $\vdash \forall_{x\in A}(\psi(x) \Rightarrow \phi(x))$,

(2)′ $p \vdash \exists!_{x\in A}\psi(x)$.

In view of the $\exists!$-rule for $\mathcal{L}_p^\mu$, it follows from (2)′ that there is a closed term a of type A in L_p^{μ} such that $p \vdash \psi(a)$. Therefore it follows from (1)′ that $p \vdash \phi(a)$ and so (ii) ⇒ (iii) .

Conversely, suppose (iii) and assume that $e: \alpha \to \beta \equiv \{y \in 1 | p\}$ is an epimorphism in F . To deduce (ii), we want to show that e splits. Now, by Lemma 3.1,

$$\vdash \forall_{y \in 1}(y \in \beta \Rightarrow \exists_{x \in A}(\langle x,y\rangle \in |e|) ,$$

that is,

$$p \vdash \exists_{x \in A}(\langle x,*\rangle \in |e|)$$

By (iii), there is a closed term a in L_p such that

$$p \vdash \langle a,*\rangle \in |e|$$

Since $e: \alpha \to \beta$ is a morphism in F , it follows that $p \vdash a \in \alpha$. Now define $m: \beta \to \alpha$ by putting $|m| \equiv \{\langle *,a\rangle | p\}$ and check that $em \cdot\!=\!\cdot 1_{\beta}$. Thus (iii) ⇒ (ii), and our proof is complete.□

Theorem 3.3 explains, among other things, why we ought to be interested in projective formulas. Intuitionists are fond of introducing assorted new axioms into mathematics, and we may ask: which closed formulas may conceivably be adopted as axioms? Since the ∃-rule is a matter of dogma for intuitionists, a necessary condition for a closed formula to be an axiom candidate is surely that it be projective.

COROLLARY 3.4. If p is a closed formula of L_1 , the following statements are equivalent:

(o) the terminal object is projective in F_p and all arrows $1 \to N$ in F_p are (induced by) standard numerals;

(i) p is projective and the ∃!-rule holds for L_p ;

(ii) the ∃-rule holds for L_p ;

(iii) p satisfies independence of premisses, that is, for any type A, from $\vdash p \Rightarrow \exists_{x \in A}\phi(x)$ one may infer $\vdash \exists_{x \in A}(p \Rightarrow \phi(x))$.

Proof: That (o) ⇔ (i) follows immediately from Theorem 3.3 and Proposition 2.1. Similarly (i) ⇔ (ii) follows from Theorem 3.3. We shall now prove that (ii) ⇒ (iii).

Suppose (ii) and $\vdash p \Rightarrow \exists_{x \in A}\phi(x)$. Then $p \vdash \exists_{x \in A}\phi(x)$, hence $p \vdash \phi(a)$ for some closed term a of type A in L_1 by (ii). Therefore $\vdash p \Rightarrow \phi(a)$, hence $\vdash \exists_{x \in A}(p \Rightarrow \phi(x))$. Thus (ii) ⇒ (iii).

Suppose (iii) and $p \vdash \exists_{x \in A}\phi(x)$. Then $\vdash p \Rightarrow \exists_{x \in A}\phi(x)$, hence $\vdash \exists_{x \in A}(p \Rightarrow \phi(x))$ by (iii). Applying the ∃-rule for L_1 [LS1, Theorem 1.3] , we obtain $\vdash p \Rightarrow \phi(a)$ for some closed term a of type A. Therefore $p \vdash \phi(a)$, and so (iii) ⇒ (ii).

If the equivalent conditions of Corollary 3.4 are satisfied, we shall call the closed formula p and also the associated object $\{x \in 1 | p\}$ of F strongly projective.

PROPOSITION 3.5. If p is strongly projective then p is indecomposable, that is, from $p \vdash q \vee r$ one may infer that either $p \vdash q$ or $p \vdash r$.

Proof: If $p \vdash q \vee r$ then surely we have

$$p \vdash \exists_{x \in N}((x=0 \wedge q) \vee (x \neq 0 \wedge r)) \quad .$$

If p is strongly projective, there is then a standard numeral n so that

$$p \vdash (n=0 \wedge q) \vee (n \neq 0 \wedge r) \quad ;$$

But we can check whether $n=0$ or $n \neq 0$. In the first case $p \vdash q$, in the second $p \vdash r$.

§4. The Freyd cover of a topos

Peter Freyd [F1] discovered an algebraic method of proving the strong projectivity of objects in the free topos. As we saw from Theorem 3.3, this leads to new proofs of syntactic properties of intuitionistic type theory and has already been exploited in [LS1] . In this section we give a brief review of the Freyd cover of a topos, which is crucial to his method, and make some detailed calculations that will prove useful in establishing strong projectivity of closed formulas later.

DEFINITION 4.1. The Freyd cover of a topos $\mathcal{A}$ is the comma category $\hat{\mathcal{A}} = (\text{Sets}, \Gamma_{\mathcal{A}})$, where $\Gamma_{\mathcal{A}} = \mathcal{A}(1,-) : \mathcal{A} \to \text{Sets}$. Its objects are triples (X,ξ,A), where X is a set, A an object of $\mathcal{A}$ and $\xi: X \to \Gamma_{\mathcal{A}}(A)$ a mapping. Its arrows $(X,\xi,A) \to (Y,\eta,B)$ are pairs of arrows $(\phi: X \to Y,\ f: A \to B)$ such that the following square commutes:

$$\begin{array}{ccc} X & \xrightarrow{\ \phi\ } & Y \\ \Big\downarrow{\scriptstyle \xi} & & \Big\downarrow{\scriptstyle \eta} \\ \Gamma_{\mathcal{A}}(A) & \xrightarrow{\ \Gamma_{\mathcal{A}}(f)\ } & \Gamma_{\mathcal{A}}(B) \end{array}$$

If $\mathcal{A}$ is a topos (with natural numbers object) having canonical subobjects, then so is $\hat{\mathcal{A}}$. We shall describe some of the structure of $\hat{\mathcal{A}}$.

We recall that every type A in $\mathcal{L}_1$ gives rise to an object $A = A_{\mathcal{A}}$ in each topos $\mathcal{A}$. We wish to calculate the corresponding object $\hat{A} = A_{\hat{\mathcal{A}}}$ in $\hat{\mathcal{A}}$. As explained in [LS1, Section 6] , we have $\hat{A} = (S_A, \lambda_A, A)$, where S_A is defined by induction on A:

$$S_1 = \{*\}\ ,\quad S_N = \mathbb{N}\ ,\quad S_{A\times B} = S_A \times S_B\ ,\quad S_\Omega = \Gamma_{\mathcal{A}}(\Omega) \cup \{\top\}\ ,\quad S_{PA} = \mathcal{A}(A,\Omega)$$

The mappings $\lambda_A: S_A \to \Gamma_{\mathcal{A}}(A)$ are the obvious ones.

More generally, products and power objects in $\hat{\mathcal{A}}$ are defined as follows:

$$(X,\xi,A) \times (Y,\eta,B) = (X \times Y, \zeta, A \times B)\ ,$$

where ζ is the compound mapping $X \times Y \xrightarrow{\xi \times \eta} \Gamma_{\mathcal{A}}(A) \times \Gamma_{\mathcal{A}}(B) \xrightarrow{\sim} \Gamma_{\mathcal{A}}(A \times B)$, and

$$P(X,\xi,A) = (\hat{\mathcal{A}}((X,\xi,A),\hat{\Omega}), \theta, PA)$$

where for any arrow $(\lambda,h)\colon (X,\xi,A) \to \hat{\Omega}$ we have $\theta(\lambda,h) = \ulcorner h \urcorner\colon 1 \to PA$, corresponding to $h\colon A \to \Omega$.

Freyd covers are of interest because of the following observation made by Freyd [Fr1] , which is easily checked:

LEMMA 4.2. If $\mathcal{A}$ is any topos, the terminal object 1 in $\hat{\mathcal{A}}$ is strongly projective, in the sense that 1 is projective in $\hat{\mathcal{A}}$ and all arrows $1 \to N$ in $\hat{\mathcal{A}}$ are induced by standard numerals.

The remainder of this section will be devoted to the question: how are closed terms of L_1 interpreted in $\hat{\mathcal{A}}$, given their interpretation in $\mathcal{A}$?

First observe that, for any topos $\mathcal{A}$ with canonical subobjects, there is a logical functor $G_{\mathcal{A}}\colon \hat{\mathcal{A}} \to \mathcal{A}$ given by $G_{\mathcal{A}}(X,\xi,A) = A$, $G_{\mathcal{A}}(\phi,f) \cdot=\cdot f$. Note that $G_{\mathcal{A}}F_{\hat{\mathcal{A}}} = F_{\mathcal{A}}$ where $F_{\mathcal{A}}\colon \mathcal{F} \to \mathcal{A}$ is the unique strict logical functor from the initial object $\mathcal{F}$ of TOP.

Recall that every closed term a of L_1 has an interpretation $a_{\mathcal{A}}$ in each topos $\mathcal{A}$. If $L\colon \mathcal{A} \to \mathcal{B}$ is a strict logical functor, $L(a_{\mathcal{A}}) \cdot=\cdot a_{\mathcal{B}}$. In particular, $F_{\mathcal{A}}(a_{\mathcal{F}}) = a_{\mathcal{A}}$, hence $G_{\mathcal{A}}(a_{\hat{\mathcal{A}}}) \cdot=\cdot G_{\mathcal{A}}F_{\hat{\mathcal{A}}}(a_{\mathcal{F}}) \cdot=\cdot F_{\mathcal{A}}(a_{\mathcal{F}}) \cdot=\cdot a_{\mathcal{A}}$. It follows that $a_{\hat{\mathcal{A}}}$ is given by a commutative square:

$$\begin{tikzcd}
\{*\} \arrow[r, "S_a"] \arrow[d, "\lambda_1"'] & S_A \arrow[d, "\lambda_A"] \\
\Gamma_{\mathcal{A}}(1) \arrow[r, "\Gamma_{\mathcal{A}}(a_{\mathcal{A}})"] & \Gamma_{\mathcal{A}}(A)
\end{tikzcd}$$

Writing a^+ for $S_a(*)$, this commutative square is expressed by the equation

$$(\dagger) \qquad \lambda_A(a^+) \cdot=\cdot a_{\mathcal{A}}$$

Clearly, S_a and $a_{\hat{\mathcal{A}}}$ may be recaptured from a^+ . (Actually, the symbols S, λ and $^+$ should all carry a subscript or superscript $\mathcal{A}$, but this would make the notation too heavy.)

We shall calculate a^+ in some cases, but a complete determination would have to proceed by induction on the length of a and involve discussion also of open terms, which we shall not carry out in this paper. The following cases are immediate consequences of $(\dagger)$:

$$*^+ \cdot=\cdot * \ , \quad (S^n 0)^+ \cdot=\cdot S^n 0 \ , \quad (\langle a,b\rangle)^+ \cdot=\cdot (a^+,b^+)$$; we also note the equation

$$\top^+ \cdot=\cdot (\top,1)$$

which has been utilized already in [LS1 , Section 6]. Some less obvious cases are contained in Proposition 4.3 and Corollary 4.4 below.

First note that, if p is any closed formula, p^+ is an element of $S_\Omega = \Gamma(\Omega) \uplus \{\top\}$ hence must have the form $(p_{\mathcal{A}},0)$ or $(\top,1)$. In the latter case the equation $\lambda_\Omega(p^+) \cdot=\cdot p_{\mathcal{A}}$ tells us that $p_{\mathcal{A}} \cdot=\cdot \top$, that is $\mathcal{A} \models p$. Also note that $p^+ \cdot=\cdot (\top,1)$ if and only if $\hat{\mathcal{A}} \models p$.

PROPOSITION 4.3. If a and b are closed terms of type A, then $\hat{\mathcal{A}} \models a=b$, that is, $(a=b)^+ \cdot\!=\!\cdot (\top,1)$ if and only if $a^+ \cdot\!=\!\cdot b^+$.

Proof: Recall that $(a=b)_A \cdot\!=\!\cdot \delta_A\langle a_A, b_A\rangle$, where $\delta_A : A\times A \to \Omega$ is the characteristic morphism of the diagonal $\langle 1_A, 1_A\rangle : A \longrightarrow A\times A$. We want to calculate this with A replaced by $\hat{A}$. Using the description of the characteristic morphism in $\hat{A}$ given in [LS1, Section 6] , we find that δ_A in $\hat{A}$ is given by the commutative square:

$$\begin{tikzcd} S_A \times S_A \arrow[r, "\sigma"] \arrow[d, "\lambda_{A\times A}"'] & S_\Omega \arrow[d, "\lambda_\Omega"] \\ \Gamma_A(A\times A) \arrow[r, "\Gamma_A(\delta_A)"] & \Gamma_A(\Omega) \end{tikzcd}$$

where, for $x,y \in S_A$,

$$\sigma(x,y) \cdot\!=\!\cdot \begin{cases} (\top,1) & \text{if } x \cdot\!=\!\cdot y \\ (\delta_A\langle\lambda_A(x),\lambda_A(y)\rangle, 0) & \text{otherwise} \end{cases}$$

Now we calculate $(a=b)^+ \cdot\!=\!\cdot S_{a=b}(*) \cdot\!=\!\cdot \sigma\langle S_a, S_b\rangle(*) \cdot\!=\!\cdot (a^+, b^+) \cdot\!=\!\cdot (\top,1)$ if and only if $a^+ \cdot\!=\!\cdot b^+$.

COROLLARY 4.4. If p and q are closed formulas, then

(i) $\hat{A} \models p \wedge q$ if and only if $\hat{A} \models p$ and $\hat{A} \models q$, that is, $(p \wedge q)^+ \cdot\!=\!\cdot (\top,1)$ if and only if $p^+ \cdot\!=\!\cdot q^+ \cdot\!=\!\cdot (\top,1)$,

(ii) $\hat{A} \models p \Rightarrow q$ if and only if $A \models p \Rightarrow q$ and $\hat{A} \models p$ implies $\hat{A} \models q$, that is, $A \models p \Rightarrow q$ and $p^+ \cdot\!=\!\cdot (\top,1)$ implies $q^+ \cdot\!=\!\cdot (\top,1)$.

Proof: (i) is clear. (ii) Since $p \Rightarrow q$ may be defined as $(p \wedge q) = p$, we obtain from Proposition 4.3: $(p \Rightarrow q)^+ \cdot\!=\!\cdot (\top,1)$ if and only if $(p \wedge q)^+ \cdot\!=\!\cdot p$. We may assume the necessary condition $A \models p \Rightarrow q$, that is, $(p \Rightarrow q)_A \cdot\!=\!\cdot p_A$, whenever required. We now distinguish two cases:

In case $p^+ \cdot\!\neq\!\cdot (\top,1)$, also $(p \wedge q)^+ \cdot\!\neq\!\cdot (\top,1)$, by (i), hence $p^+ \cdot\!=\!\cdot (p_A, 0) \cdot\!=\!\cdot ((p \wedge q)_A, 0) \cdot\!=\!\cdot (p \wedge q)^+$.

In case $p^+ \cdot\!=\!\cdot (\top,1)$, $(p \wedge q)^+ \cdot\!=\!\cdot p^+$ if and only if $(p \wedge q)^+ \cdot\!=\!\cdot (\top,1)$, that is, $q^+ \cdot\!=\!\cdot (\top,1)$, by (i). The proof is now complete.

It would seem to be of interest to calculate a^+ and p^+ in all remaining cases. If $A = F$, the assignment $p \longmapsto p^+$ bears a remarkable resemblance to the mapping $L \to L^+$ appearing in Friedmann's adaptation of Kleene's realizability (see [LS1, Section 2]), while λ_A corresponds to the mapping $L^+ \xrightarrow{-} L$ used there.

5. Iterated Freyd covers

In this section we shall consider what happens if the construction of the Freyd cover is repeated. For typographical reasons we write $A^\wedge$ for the Freyd cover of the topos A . We define

$$A^{\wedge 0} = A \qquad A^{\wedge n+1} = (A^{\wedge n})^\wedge$$

We shall obtain an explicit characterization of the topos $A^{\wedge n}$.

In what follows, $\underline{n}$ is the n-element chain regarded as a category. Put $\Gamma_A = A(1,-): A \to \text{Sets}$ and let $[\text{Sets}^{\underline{n}},\Gamma_A]$ be the category defined as follows. Its objects are triples (X^n,η,A), where $X^n = X_n \to X_{n-1} \to \cdots \to X_1$ is an object of $\text{Sets}^{\underline{n}}$, A is an object of A and $\eta: X_1 \to \Gamma_A(A)$ is a mapping. This object may be depicted by the diagram: $X_n \to X_{n-1} \to \cdots \to X_1 \to \Gamma_A(A)$. A morphism from this to the object depicted by $Y_n \to Y_{n-1} \to \cdots \to Y_1 \to \Gamma_A(B)$ is given by an $(n+1)$-tuple of mappings $(\phi_n,\phi_{n-1},\cdots,\phi_1,f)$, where $\phi_i: X_i \to Y_i$ and $f: A \to B$ are such that the obvious diagram commutes.

PROPOSITION 5.1. For each natural number $n \geq 1$, $A^{\wedge n} \simeq [\text{Sets}^{\underline{n}},\Gamma_A]$.

Proof: We argue by induction on n. For $n=1$ this is the definition of the Freyd cover. Assume the result for n, then we want to show that

$$[\text{Sets}^n,\Gamma_A]^\wedge \simeq [\text{Sets}^{\underline{n+1}},\Gamma_A]$$

Now an object of the LHS is a triple (X_{n+1},ξ,A'), where $A' = (X^n,\eta,A)$ as above and $\xi: X_{n+1} \to \Gamma(A')$. We have omitted the subscript on Γ, which should be A. Thus $\Gamma(A')$ is the set of all $(n+1)$-tuples $(\phi_n,\phi_{n-1},\ldots,\phi_1,f)$ such that the following diagram commutes:

$$(\dagger)\qquad \begin{tikzcd}
\{*\} \arrow[r] \arrow[d, "\phi_n"'] & \{*\} \arrow[r] \arrow[d, "\phi_{n-1}"'] & \cdots \arrow[r] & \{*\} \arrow[r] \arrow[d, "\phi_1"'] & \Gamma_A(1) \arrow[d, "\Gamma_A(f)"] \\
X_n \arrow[r] & X_{n-1} \arrow[r] & \cdots \arrow[r] & X_1 \arrow[r] & \Gamma_A(A)
\end{tikzcd}$$

Clearly, $\phi_{n-1}, \ldots, \phi_1$, and $f: 1 \to A$ may be computed in terms of ϕ_n and the given mappings in the bottom row, hence $\Gamma(A')$ consists essentially just of pairs (A',ϕ_n), with $\phi_n: \{*\} \to X_n$. Now $\phi_n(*)$ is just any element of X_n, and so ξ is determined by a mapping $X_{n+1} \to X_n$. Therefore (X_{n+1},ξ,A') may be depicted by the diagram: $X_{n+1} \to X_n \to \cdots \to X_1 \to \Gamma_A(A)$ which also depicts an object of the RHS of $(\dagger)$. We thus have a bijection between the objects of the LHS and the objects of the RHS of $(\dagger)$, and one may easily obtain a functorial bijection between the corresponding Hom-sets. □

The isomorphism of Proposition 5.1 is one between categories, perhaps with terminal objects. It becomes an isomorphism between toposes if we define the topos structure of $[\text{Sets}^{\underline{n}},\Gamma_A]$ appropriately.

We wish to apply Proposition 5.1 to the case where A is a degenerate topos.

PROPOSITION 5.2. If A is any topos, the following statements are equivalent:

(i) $A \models \bot$,

(ii) $0 \simeq 1$ in A,

(iii) there is an arrow $1 \to 0$ in A,

(iv) for any pair of objects A,B of A there is an arrow $A \to B$,

(v) for any pair of objects A,B of A there is at most one arrow $A \to B$,

(vi) all objects of A are isomorphic .

Proof: Most of the following implications are obvious: (v) $\Longleftrightarrow$ (i) $\Longleftrightarrow$ (ii) (iii) $\Longleftrightarrow$ (iv) , (iv) $\wedge$ (v) $\Longleftrightarrow$ (vi). We shall extablish only the less obvious ones. (i) $\Rightarrow$ (ii). In any topos the initial object 0 is given by the equalizer diagram $0 \to 1 \overset{\top}{\underset{\perp}{\rightrightarrows}} \Omega$. If $A \models \perp$, then $\top \cdot\!=\!\cdot \perp$ in A , hence $0 \simeq 1$. (ii) $\Rightarrow$ (i). If $0 \simeq 1$ the characteristic morphisms of $0 \to 1$ and $1 \to 1$ must coincide, hence $\top \cdot\!=\!\cdot \perp$ in A that is, $A \models \perp$. (iii) $\Rightarrow$ (ii). Composing the arrows $0 \to 1$ and $1 \to 0$ both ways, we get the identity arrow each time. (i) $\Rightarrow$ (v). Since $A \models \perp$, $A \models \forall_{x \in A} fx = gx$. Therefore $f \cdot\!=\!\cdot g$ in A , in view of the slogan that internal equality implies external equality.

COROLLARY 5.3. Up to isomorphism there is only one topos satisfying the equivalent conditions of Proposition 5.2, namely $F_\perp$

We shall call $F_\perp$ the <u>degenerate</u> topos. Note that it is a terminal object in TOP.

PROPOSITION 5.4. $(F_\perp)^{\wedge n} \simeq \mathrm{Sets}^{\underline{n}}$ for all $n \geq 0$.

Proof: This is a consequence of Proposition 5.1 if we think of both sides as categories. If we think of them as toposes, one should also verify that the isomorphism preserves the topos structure. We skip the details.

§6. Conditions implying projectivity

In this section we shall study a number of conditions which entail strong projectivity, yet are easier to verify.

PROPOSITION 6.1. Given a closed formula p of L_1 the following statements are equivalent:

(i) F_p is a retract of $\hat{F}_p$, that is, $(F_p)^\wedge$ in TOP;

(ii) there is a strict logical functor $F\colon F_p \to \hat{F}_p$

(iii) $\hat{F}_p \models p$

Proof: (i) asserts that there are strict logical functors $F\colon F_p \to \hat{F}_p$ and $G\colon \hat{F}_p \to F_p$ such that GF is the identity functor on F_p . Clearly (i) $\Rightarrow$ (ii).

Suppose (ii). Since $F_p \models p$, we have $p_{F_p} \cdot\!=\!\cdot \top$. Applying F to this, we get $p_{\hat{F}_p} \cdot\!=\!\cdot \top$, that is, $\hat{F}_p \models p$. Thus (ii) $\Rightarrow$ (iii).

Suppose (iii) By Proposition 1.2, there is a unique strict logical functor $F\colon F_p \to \hat{F}_p$. Hence $GF\colon F_p \to F_p$ is the unique logical functor, which must be the identity. Thus (iii) $\Rightarrow$ (i).□

When p satisfies the equivalent conditions of Proposition 6.1, we shall call p, or the associated object $\{x \in 1 | p\}$ <u>Freydian</u>. This concept (though not the name) is due to Peter Freyd [Fr2] .

We record some obvious facts.

(1) Provable $\Rightarrow$ Freydian. For theorems hold in every topos, in particular in

Freyd covers.

(2) Freydian $\Rightarrow$ strongly projective. For suppose F_p is a retract of $\hat{F}_p$, then it inherits from $\hat{F}_p$ the two important properties: 1 is projective and all arrows $1 \to N$ are induced by standard numerals (see Lemma 4.2).

The following result was first announced by Peter Freyd [Fr2].

THEOREM 6.2. If p and q are closed formulas of L_1, then

(i) $\neg p$ is Freydian if and only if it is not refutable;

(ii) $p \Rightarrow q$ is Freydian if $p \Rightarrow q$ does not entail p.

Proof: (i) Suppose $\neg p$ is Freydian, but $\vdash \neg\neg p$. Since $\neg\neg p$ is provable, it holds in every topos, in particular in $\hat{F}_{\neg p}$. Therefore $\hat{F}_{\neg p}$ is degenerate. But no Freyd cover is degenerate, as is easily seen, for example, by observing that there is no arrow from $\{*\} \to \Gamma_A(1)$ to $\phi \to \Gamma_A(0)$. This proves the implication one way. The converse follows from (ii) by taking $q \equiv \bot$.

(ii) The formula $p \Rightarrow q$ is Freydian if and only if $\hat{F}_{p\Rightarrow q} \models p \Rightarrow q$. In view of Corollary 4.4, since $F_{p\Rightarrow q} \models p \Rightarrow q$ by Proposition 1.3, this happens if and only if either p does not hold in $\hat{F}_{p\Rightarrow q}$ or q does. A sufficient condition is therefore that p does not hold in $F_{p\Rightarrow q}$, in view of the strict logical functor $\hat{F}_{p\Rightarrow q} \to F_{p\Rightarrow q}$ that is, that $p \Rightarrow q$ does not entail p, in view of Proposition 1.3.□

Unfortunately, the set of Freydian formulas does not have nice closure properties; for example, it is not closed under conjunction. For let γ be Godel's undecidable sentence, then $\neg\gamma$ and $\neg\neg\gamma$ are both Freydian by Theorem 6.2, yet their conjunction is equivalent to $\bot$, which is not Freydian, as it fails to hold in $\hat{F}_\bot$, that is, in Sets.

We therefore investigate a related notion. A closed formula p of L_1, or its associated object $\{x \in 1 \mid p\}$ in F, will be called hereditary if, for all nondegenerate toposes A, whenever $A \models p$ then $\hat{A} \models p$. Clearly, $A \models p$ implies $\hat{A} \models p$ for all toposes if and only if p is hereditary and it holds in $\hat{F}_\bot \simeq$ Sets.

We now continue the list of implications begun earlier in this section.

(3) Hereditary $\Rightarrow$ Freydian or refutable. For let P be hereditary and take $A = F_p$. Then $\hat{F}_p \models p$ unless F_p is degenerate, that is, $F_p \models \bot$, that is, $\vdash p \Rightarrow \bot$.

(4) Hereditary and true in Sets $\Rightarrow$ true in $\text{Sets}^{\hat{n}}$ for all $n \geq 0$. For let p be hereditary and true in A = Sets. Then $\text{Sets}^\wedge \models p, \ldots$ etc. Use Proposition 5.4.

(5) Provable or refutable $\Rightarrow$ hereditary. Clear.

(6) Hereditary $\Rightarrow$ strongly projective. For let p be hereditary. If p is refutable, it is equivalent to $\bot$, hence strongly projective. If p is not refutable, use (3) and (2).

The class of hereditary closed formulas has some nice closure properties, reminiscent of the Harrop formulas [Du].

THEOREM 6.3.

(i) $\bot$ is hereditary

(ii) If p and q are hereditary, so is $p \wedge q$.

(iii) If q is hereditary, then so is $p \Rightarrow q$ for any closed formula p whatsoever.

<u>Proof</u>: (i) is obvious, as $\perp$ cannot hold in a nondegenerate topos.

(ii) Suppose p and q are hereditary, A is nondegenerate and $A \models p \wedge q$. Then $A \models p$ and $A \models q$, hence $\hat{A} \models p$ and $\hat{A} \models q$, and so $\hat{A} \models p \wedge q$.

(iii) Suppose q is hereditary, A nondegenerate and $A \models p \Rightarrow q$. We claim that $\hat{A} \models p \Rightarrow q$. In view of Corollary 4.4, we must show that $\hat{A} \models p$ implies $\hat{A} \models q$. So suppose $\hat{A} \models p$, then $A \models p$, in view of the strict logical functor $\hat{A} \longrightarrow A$. Since $A \models p \Rightarrow q$, also $A \models q$. Therefore $\hat{A} \models q$, as q is hereditary.

§7. <u>Some counterexamples</u>

We summarize the implications established in Section 6 and Proposition 3.5 in the following picture:

$$\begin{tikzcd}
 & \text{Freydian} \arrow[r, Rightarrow] & \text{strongly projective} \arrow[dr, Rightarrow] & \\
\text{provable} \arrow[r, Rightarrow] & \text{hereditary and true in Sets} \arrow[u, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{hereditary} \arrow[u, Rightarrow] & \text{projective and indecomposable} \\
 & \text{true in } \mathrm{Sets}^{\underline{n}}\text{, for all } n\geq 0 & &
\end{tikzcd}$$

We shall now exhibit some examples to show, among other things, that most of these implications cannot be reversed.

EXAMPLE 7.1. Strongly projective $\not\Rightarrow$ Freydian.

Clearly, $\perp$ is strongly projective. It also holds in $F_\perp$; so, if it were Freydian, it would also hold in $\hat{F}_\perp \simeq \mathrm{Sets}$, in view of Proposition 5.4.

EXAMPLE 7.2. Freydian $\not\Rightarrow$ true in Sets.

Take any closed formula of the form $\neg p$ which is false in Sets yet not refutable, e.g., $\neg\gamma$, where γ is Godel's classically undecidable sentence. Then $\neg p$ is Freydian, by Theorem 6.2.

EXAMPLE 7.3. Hereditary $\not\Rightarrow$ Freydian.

For instance, $\perp$ is hereditary, by Theorem 6.3, but not Freydian, by Theorem 6.2.

EXAMPLE 7.4. Hereditary and true in Sets $\not\Rightarrow$ provable.

$\neg\neg\gamma \equiv \neg\gamma \Rightarrow \perp$ is hereditary, by Theorem 6.3, yet it is not provable. Else γ would be provable classically, contradicting Godel.

EXAMPLE 7.5. True in $\mathrm{Sets}^{\underline{n}}$ for all $n \geq 0 \not\Rightarrow$ indecomposable.

Let $\delta \equiv (\gamma \Rightarrow \neg\gamma) \vee (\neg\gamma \Rightarrow \gamma)$. Then δ holds in $\mathrm{Sets}^{\underline{n}}$ for all $n\geq 0$, however it is not indecomposable. For otherwise either $\delta \vdash \gamma \Rightarrow \neg\gamma$ or $\delta \vdash \neg\gamma \Rightarrow \gamma$. Now δ is a tautology, so $\beta \vdash \delta$, where $\beta \equiv \forall_{t\in\Omega}(t \vee \neg t)$ is the Boolean axiom. Therefore, $\beta \vdash \gamma \Rightarrow \neg\gamma$ or $\beta \vdash \neg\gamma \Rightarrow \gamma$, that is, $\beta \vdash \neg\gamma$ or $\beta \vdash \gamma$, contradicting Godel.

EXAMPLE 7.6. Freydian and true in Sets $\not\Rightarrow$ true in $\mathrm{Sets}^{\underline{n}}$, for all $n\geq 0$, thus Freydian $\not\Rightarrow$ hereditary and so strongly projective $\not\Rightarrow$ hereditary. A direct calculation ~~shows that $\neg\beta$ is true in $\mathrm{Sets}^{\underline{2}} \simeq \mathrm{Sets}^{\bullet\to\bullet}$, hence $\neg\beta$ is an example of a formula~~

No!

See Lambek-Scott "New Proofs of Int. Principles", ZFL (1983) Footnote, p. 50

~~as required;~~ however the following nonconstructive argument is quite amusing. ~~Either $\neg\neg\beta$ is true in Sets^2 or not. If not, then $\neg\neg\beta$ has the required properties as mentioned above. On the other hand, should $\neg\neg\beta$ actually be true in Sets^2~~ , consider the formula $\neg\neg\beta \Rightarrow \beta$. This formula holds in Sets but not in Sets^2, else the latter would be Boolean using the hypothesis above. Furthermore $\neg\neg\beta \Rightarrow \beta$ is Freydian, for by Theorem 6.2 it suffices to show that $\neg\neg\beta \Rightarrow \beta$ does not entail $\neg\neg\beta$. If it did, since $\neg\beta \vdash \neg\neg\beta \Rightarrow \beta$, it would follow that $\neg\beta \vdash \neg\neg\beta$, that is, $\vdash \neg\neg\beta$, which is known to be false.

At the time of writing the following questions are open: to find examples of formulas showing that true in Sets^n for all n $\not\Rightarrow$ projective and that projective or indecomposable $\not\Rightarrow$ strongly projective. However we haven't really given much thought to these questions and the answers may be quite easy. It is known, for example, that N (not a subobject of 1) is projective but not strongly projective (for F/N has non-standard numerals).

It is clear that not all objects of F can be projective, else the full axiom of choice would hold in F . But then F would be Boolean, by Diaconescu's Theorem [Di], and therefore all toposes would be Boolean, contrary to fact. It is a little more difficult to point out a specific nonprojective object. It <u>can</u> be shown that the object of Dedekind reals is not projective, although the proof uses techniques not discussed here. More interesting in the present context is a nonprojective subobject of 1 .

EXAMPLE 7.7. The Boolean axiom β is not projective.
It is clear that β is not strongly projective, since, by Godel's Theorem, it is not indecomposable. Therefore the $\exists$-rule does not hold in L_β. We assert more: the $\exists$-rule does not hold in L_β^μ , that is, even if a description operator is thrown in. Indeed, if the $\exists$-rule did hold in L_β^μ , one would hardly be justified in criticizing classical mathematics for being nonconstructive !

To say that β is not projective is to assert for some open formula $\phi(x)$ that $\beta \vdash \exists_{x\in A}\phi(x)$ but not $\beta \vdash \phi(a)$ for any closed term a of type A (the type of x) in L_β^μ. In particular, take $\phi(x) \equiv \exists_{x\in A}\psi(x) \Rightarrow \psi(x)$, then clearly $\beta \vdash \exists_{x\in A}\phi(x)$, so it suffices to find $\psi(x)$ such that $\beta, \exists_{x\in A}\psi(x) \vdash \psi(a)$ for no closed term a of type A.

To this end let B be a high enough type so that the set of Dedekind real numbers may be constructed as an element of PB. Now take $A \equiv P(B\times B)$ and let $\psi(x) \equiv$ x is a well-ordering of the reals. We claim that there is no closed term a of type A in L_β^μ i.e. in classical type theory with description, such that $\beta, \exists_{x\in A}\psi(x) \vdash \psi(a)$. For we can interpret L_β^μ in Zermelo-Fraenkel set theory with axiom of choice so that $\exists_{x\in A}\psi(x)$ becomes provable, yet $\psi(a)$ is not provable for any closed term a, that is, there is no definable well-ordering of the reals, as was proved by Feferman [Fe] (See also [FB, pages 68-69]).

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