Categorical traces from single-photon linear optics

Peter Hines and Philip Scott

Abstract. We use a single-photon thought experiment, based on a modification of the Sagnac interferometer, to motivate a general construction on linear maps that has a close connection to constructions from algebraic and categorical program semantics. We analyse this general construction in terms of a category of formal power series over linear maps, and exhibit a partial categorical trace, generalising the 'particle-style' trace on Hilbert spaces [**HS10**], that has a physical realisation based on this thought-experiment.

1. Introduction

1.1. Historical background. The Sagnac interferometer is a linear-optics device whose theoretical origins [**Lo93**] predate both quantum mechanics and relativity. An experiment, described in [**Sa13i, Sa13ii**], claimed to 'demonstrate the existence of the luminiferous æther' (*"La preuve de la réalité de l'éther lumineux"*) using an interferometer that split incoming light into two counter-rotating paths around an optical loop. Sagnac's experiment is still a favorite of those who wish to disprove relativity (see *http://www.anti-relativity.com/* for examples). However, as observed by Michelson [**Br02**], the Sagnac effect cannot discriminate between (special-)relativistic and pre-relativistic theories[1].

From essentially 19th century origins, both the Sagnac interferometer and its modern incarnation as the 'Ring Laser' [**St97**] have become immensely important practical tools used in, amongst other devices, highly sensitive gyroscopes and the Sagnac effect is now a key technique in inertial and missile guidance systems (From [**St97**], *"Contrary to the supposed custom in research, the area ... has already proved its commercial and (unfortunately) its military usefulness; it is the scientific potential which has been neglected."*).

1.2. Modifying the Sagnac interferometer. In this paper, we analyse a variant of Sagnac's experiment from a quantum-mechanical perspective. We take a

1991 *Mathematics Subject Classification.* Primary 68Q05; Secondary 81P68.

Key words and phrases. Categorical traces, Formal Power series, Linear Optics.

The first author was supported by E.U. FP6 Grant QICS.

The second author's research is supported by an NSERC Discovery Grant, Canada.

[1] *"We will undertake this [experiment], although my conviction is strong that we shall prove only that the earth rotates on its axis, a conclusion which I think we may be said to be sure of already." – A. Michelson, quoted in* [**Br02**].

single-photon description, and analyse the situation where the splitting of incoming light is not arbitrary (creating a quantum-mechanical 'equal superposition' of paths), but dependent on a certain quantum property (i.e. the polarisation) of the incoming light. The motivation for this is two-fold:

(1) Single-photon thought experiments in linear optics are often used as illustrations of quantum computation and information. By encoding quantum bits (qubits) on either the (superposition of) paths taken by an individual photon, or its polarisation, it is possible to implement a universal set of quantum-computational gates [**CAK05**], and hence all current quantum algorithms[2] and protocols. We give explicit examples in Appendix A.
(2) In the Sagnac interferometer, the replacement of the standard beamsplitter by a polarising beamsplitter (see Section 2.2 for these devices) gives a form of 'conditional looping'. Whether or not a photon enters the optical loop depends on its polarisation; similarly, the exiting of a photon from the optical loop is also dependent on its its polarisation.

Thus, our modified Sagnac Interferometer gives a paradigmatic example of 'quantum conditional iteration'. By analysing our thought-experiment in detail, in the Hilbert space model, we demonstrate that this is very closely related to the form of conditional iteration used in the theory of reversible computation — the 'particle-style' categorical trace (see Section 8).

2. Hilbert space formalism, and the linear optics toolkit

This section presents the standard Hilbert space formalism for quantum information and computation, together with an exposition of how single-photon experiments using linear optics gates may be described within this formalism.

2.1. Basics of quantum computation and information. We use the standard Hilbert space formalism to model our thought-experiments. We emphasise that although we take a very categorical approach, the constructions of this paper live within the traditional Hilbert space formalism, rather than the abstract categorical formalism of [**AbCo05**].

Definition 2.1. Given a vector space V over $\mathbb{C}$, an *inner product* is a Hermitian symmetric form (i.e. a map $\langle \_ | \_ \rangle : V \times V \to \mathbb{C}$ that is linear in the first variable and conjugate-linear in the second) that satisfies $\langle x|x\rangle \geq 0$ and $\langle x|x\rangle = 0$ iff $x = \mathbf{0}$. A *complex Hilbert Space* is then a Banach space (i.e. complete normed vector space) over $\mathbb{C}$ whose norm is defined by an inner product, $\|x\| = (\langle x|x\rangle)^{\frac{1}{2}}$.

By the Reisz representation theorem [**Har83**], for every bounded linear map $L : H \to K$ of Hilbert spaces, there exists a unique bounded linear map $L^* : K \to H$ such that, for all $k \in K$ and $h \in H$,

$$\langle k|L(h)\rangle \; = \; \langle L^*(k)|h\rangle$$

This is called the **Hermitian adjoint** of L, and is often denoted by either $L^\dagger$ (quantum-mechanical notation) or L^H (functional-analysis notation).

[2] It should be emphasised that, in encodings of quantum algorithms, the resources required grow exponentially. Thus such optical circuits are primarily useful either as demonstrations of quantum information, and tests of underlying principles, or in quantum-mechanical communications protocols.

We use Dirac notation for vectors, so $|\psi\rangle \in \mathcal{H}$ is a linear map from $\mathbb{C}$ to $\mathcal{H}$ defined by $|\psi\rangle(z) = z.\psi \in \mathcal{H}$. As is traditional, we abuse notation and refer to $|\psi\rangle$ as a vector of $\mathcal{H}$ (see [**Ab05**] for this concept from a categorical perspective). In quantum computation (especially the *circuit model* [**NC00**]), it is standard to assume that Hilbert spaces are equipped with some fixed orthonormal basis set (the **computational basis**). We use notation derived from a categorical perspective (The l_2 functor of [**Ba92**]), so given some set $X = \{x_i\}_{i\in I}$, the space $l_2(X)$ is the space with a distinguished orthonormal basis $\{|x_i\rangle\}_{i\in I}$. Of particular interest is the **qubit space** $Q = l_2(\{0,1\})$ that plays an analogous role to the set of bits $\{0,1\}$ in classical computation. The assumption that each space is equipped with a fixed orthonormal basis allows us to use matrix representations of linear maps: these are used heavily throughout this paper.

Composite systems are modelled using the tensor product of Hilbert spaces: given spaces $\{\mathcal{H}_k\}_{k=1\ldots n}$ modelling systems $S_1, \ldots S_n$, the composite system is modelled by the space $\otimes_{k=1}^{n}\mathcal{H}_k$. When each space is a copy of the qubit space Q, tensor products of this form are called **quantum registers** of k qubits.

Two key differences between classical and quantum information are the phenomena of superposition and entanglement. We present these mathematically, and refer to any introductory quantum computing text (e.g. [**NC00**]) for physical interpretations.

DEFINITION 2.2. Given a space $\mathcal{H}$ with computational basis $\{|b_0\rangle, \ldots, |b_n\rangle\}$, a state $|\psi\rangle$ is a **superposition** when it is a non-trivial linear combination of computational basis vectors, $|\psi\rangle = \sum_{j=0}^{n} \alpha_j |b_j\rangle$. Note that this concept is *basis-dependent.*

Given another Hilbert space $\mathcal{K}$, with computational basis $\{|c_0\rangle, \ldots |c_m\rangle\}$, the tensor product space $H \otimes K$ has computational basis $\{|b_j c_k\rangle\}_{j=0..n,k=0..m}$. A state $|\phi\rangle = \sum_{j,k} \alpha_{j,k} |b_j c_k\rangle$ is **entangled** when it cannot be written as $|\phi_1\rangle \otimes |\phi_2\rangle$, for any $|\phi_1\rangle \in \mathcal{H}$ and $|\phi_2\rangle \in \mathcal{K}$. Note that, unlike superposition, entanglement is basis-independent.

Physical operations on quantum systems are divided into 2 classes: measurements, and coherent operations. Coherent operations are modelled by *unitary maps*, and measurements are modelled by either *projectors* or *Hermitian operators.* Again, we present these mathematically, and refer to [**NC00**] for physical intuitions.

DEFINITION 2.3. A linear map between Hilbert spaces, $U : \mathcal{H} \to \mathcal{K}$ is **unitary** when it is an inner-product preserving isomorphism[3]. A linear map $P : \mathcal{H} \to \mathcal{K}$ is a **projector** when it is a self-adjoint idempotent. A *measurement* is determined by a self-adjoint operator, or Hermitian matrix. By the spectral decomposition theorem, every (finite) Hermitian matrix has a unique decomposition as the sum of projection operators — in this way a Hermitian matrix describes a set of projections, labelled by eigenvalues, and these are taken to be the experimental outcomes of a measurement.

The class of all Hilbert spaces forms a category, as follows:

[3] A useful characterisation of unitary maps is that, given some matrix representation for $U : \mathcal{H} \to \mathcal{H}$, the *conjugate transpose* of this matrix is the matrix representation for U^{-1}. It is a straightforward exercise to show that this characterisation is basis-independent, and equivalent to the definition given above.

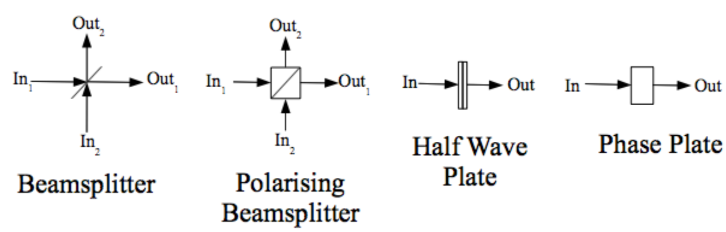
DEFINITION 2.4. The category **Hilb** has all (separable) Hilbert spaces as objects, and all continuous (i.e. bounded) linear maps as arrows. An important subcategory is $\mathbf{Hilb_{FD}}$, which is simply the above category, restricted to finite-dimensional spaces. The category **Hilb** has two distinct symmetric monoidal tensors: the familiar **tensor product** $\otimes$, and the **direct sum** $\oplus$. In [**Hal58**], the direct sum is defined for arbitrary indexed families of Hilbert spaces, as follows: given an indexed family of spaces $\{H_i\}_{i\in I}$, the direct sum $\oplus_{i\in I} H_i$ has elements given by functions $\alpha : I \rightarrow \biguplus_{i\in I} H_i$ such that $\alpha(i) \in H_i$, and $\sum_{i\in I} \|x_i\|^2_{H_i} < \infty$. exists. The inner product of two elements $\alpha, \beta \in \bigoplus_{i\in I} H_i$ is then given by $\langle \alpha | \beta \rangle = \sum_{i\in I} \langle \alpha(i) | \beta(i) \rangle$.

When the indexing set is finite, it is straightforward that the direct sum is indeed a monoidal tensor. Infinitary direct sums can also be given a categorical interpretation, but this is more subtle.

By contrast, the tensor product is defined for finite families only. Although infinitary analogues have also been considered [**JvN38**], these more naturally live within the theory of C^* and von Neumann algebras, and play no part in this paper.

2.2. The optics toolkit. *This section follows very closely the introduction to linear optics given in* [**GK05**]*, and in particular the single-photon case presented in* [**Be05**]. The basic linear optics devices we require are the *Beam Splitter* (**BS**), the *Polarising Beam Splitter* (**PBS**), the *Half Wave Plate* (**HWP**), and the *Phase Plate* (**PP**). These all implement coherent operations, and have standard schematics, as shown in Figure 1.

FIGURE 1. The linear optics toolbox



These all have either 1 or 2 input / output channels[4], and their behaviour may be dependent on the photon polarisation – thus the quantum properties we consider are the *polarisation*, and the *'which channel?'* information.

We adopt the convention that horizontal (resp. vertical) polarisation is denoted $|H\rangle$ (resp. $|V\rangle$), and input (resp. output) channel j is denoted $|in_j\rangle$ (resp. $|out_j\rangle$). Thus, a horizontally polarised photon in input channel 1 corresponds to the state vector $|H\rangle\,|in_1\rangle$, and a vertically polarised photon in an even superposition of both output channels corresponds to the state vector $\frac{1}{\sqrt{2}}\,|V\rangle\,(|out_1\rangle + |out_2\rangle)$.

Their action is then described by unitary maps, as follows:

[4]In fact, all these devices are completely reversible, and the designation of channels as either *input* or *output* depends on the direction of the incident photon. This is important in the analyses of Section 3.

The beamsplitter The beamsplitter is one of the most standard linear optics devices. An input may be in either channel 1 or channel 2, and may be either horizontally or vertically polarized (or, of course, an arbitrary superposition of any of these properties). Given an input in either of the input channels, the output is a superposition of output channels (however, see Remark 2.5). We emphasise that we are only considering the case where there is a single input photon[5].

The behaviour of the beamsplitter is described by a unitary map, defined by its action on (orthonormal) basis vectors as follows:

$$|H\rangle\,|in_1\rangle \;\mapsto\; \tfrac{1}{\sqrt{2}}\,|H\rangle\,(|out_1\rangle + |out_2\rangle) \qquad |V\rangle\,|in_1\rangle \;\mapsto\; \tfrac{1}{\sqrt{2}}\,|V\rangle\,(|out_1\rangle + |out_2\rangle)$$

$$|H\rangle\,|in_2\rangle \;\mapsto\; \tfrac{1}{\sqrt{2}}\,|H\rangle\,(|out_1\rangle + |out_2\rangle) \qquad |V\rangle\,|in_2\rangle \;\mapsto\; \tfrac{1}{\sqrt{2}}\,|V\rangle\,(|\,|out_1\rangle - |out_2\rangle\rangle)$$

Note that the action of the beamsplitter is independent of the polarisation of the input photon.

The polarising beamsplitter This is closely related to the above example; however, its behaviour is conditional on the polarisation of the input photon. Intuitively, it transmits photons with horizontal polarisation and reflects (through $\pi/2$) photons with vertical polarisation.

Using the same notation as above, the behaviour of the polarising beamsplitter is given by the unitary map defined by:

$$|H\rangle\,|in_1\rangle \;\mapsto\; |H\rangle\,|out_1\rangle \qquad |V\rangle\,|in_1\rangle \;\mapsto\; |V\rangle\,|out_2\rangle$$

$$|H\rangle\,|in_2\rangle \;\mapsto\; |H\rangle\,|out_2\rangle \qquad |V\rangle\,|in_2\rangle \;\mapsto\; |V\rangle\,|out_1\rangle\rangle$$

The phase plate This transmits all photons on the input channel, and rotates the phase by an angle of θ. Given an arbitrary incoming photon $|\psi\rangle$, the action of the phase plate is simply

$$|\psi\rangle \;\mapsto\; e^{i\theta}|\psi\rangle$$

The half-wave plate This again transmits all photons on the input channel, and adds a $\frac{\pi}{4}$ rotation to the polarisation. The action of the half wave plate is given by a unitary map defined by its action on basis vectors as follows:

$$|H\rangle \;\mapsto\; \tfrac{1}{\sqrt{2}}\,(|H\rangle + |V\rangle)$$

$$|V\rangle \;\mapsto\; \tfrac{1}{\sqrt{2}}\,(-\,|H\rangle + |V\rangle)$$

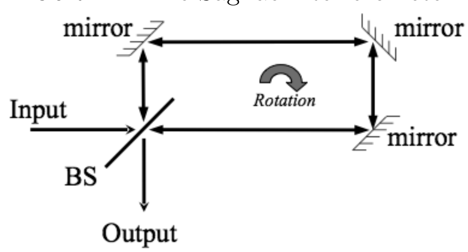
REMARK 2.5. **A note on reflection and phases** Readers familiar with the standard optics toolkit will note that the phases in the outputs of the beamsplitters are non-standard — in linear optics experiments, the reflected path (whether from a

[5]Precisely, we allow for a photon in Channel 1, or Channel 2, or a photon in a superposition of these locations. We do *not* allow for an input photon in each channel – not only would this require a more sophisticated mathematical treatment, but would also take us away from the underlying motivation of single-particle interference. For readers familiar with the usual Fock space description, our treatment also neglects the vacuum states, for simplicity of notation.

mirror, beamsplitter, polarised beamsplitter, or whatever) actually picks up a phase factor of i. We emphasise that all the devices presented can be made to behave exactly as described, simply by using appropriately placed phase plates [**CAK05**]. However, we do not do this explicitly – partly for simplicity of diagrams, and partly to make the connection with the standard circuit model of quantum computation (given in Appendix A) more immediate.

3. The Sagnac Interferometer

FIGURE 2. The Sagnac interferometer



A schematic of the standard Sagnac interferometer is shown in Figure 2. Intuitively, its action is straightforward: incoming light is split into two counter-rotating paths by the beamsplitter indicated. These travel around the optical loop, and recombine at the beamsplitter. This then produces an interference pattern at the output. When the whole apparatus is rotated, the relative length of the respective paths changes, shifting the interference pattern. Thus, absolute rotation is readily detected by changes in the observed interference pattern.

This apparatus has been analysed in detail, by a number of authors, for at least the past 100 years – we do not attempt to add yet another analysis to the literature. A single-photon analysis is conceptually more interesting, but only in that it requires the strongly quantum-mechanical phenomenon of *single-particle interference*. Mathematically this is trivial; we allow for arbitrary superpositions and (complex) phase differences in the 'which path' information for a single photon, as in Section 2. However, the physical interpretation is more remarkable: as stated in [**PSM96**],

> " In his famous introduction [**FLS65**] to the single particle superposition principle, Feynman stated that, '. . . it has in it the heart of quantum mechanics. In fact, it contains the only mystery.' ".

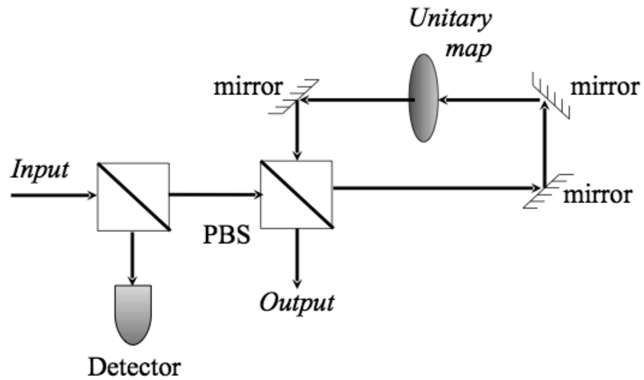
We leave a single-photon analysis as an interesting exercise, and refer to [**PSM96**] for a demonstration of why the many (entangled) particle case is qualitatively different to the single-photon case.

4. A modified Sagnac Interferometer

We now make the following modifications to the Sagnac interferometer, as shown in Figure 3.

(1) We replace the beamsplitter by a polarising beamsplitter.

FIGURE 3. A modified Sagnac interferometer (MSI)



(2) We introduce an arbitrary unitary operation within the optical loop that acts non-trivially on photon polarisation as

$$|H\rangle \mapsto a\,|H\rangle + c\,|V\rangle \quad , \quad |V\rangle \mapsto b\,|H\rangle + d\,|V\rangle$$

for some $a, b, c, d \in \mathbb{C}$.

(3) We post-select for experiments where the detector shown does *not* record a measurement. This ensures that the input to the second PBS (and hence the optical loop) is horizontally polarised.

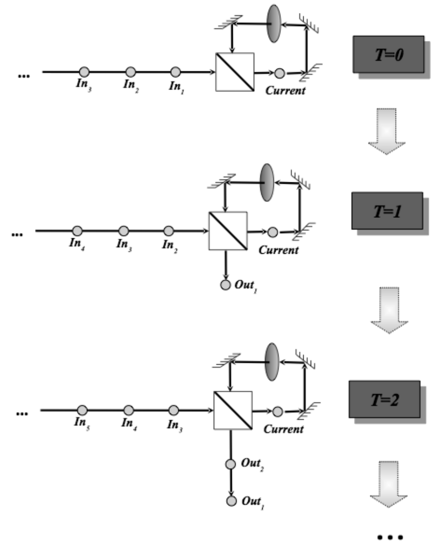
Modifications 1. and 3. ensure that the only possible path traversed around the optical loop is the counter-clockwise path. Thus, this device no longer displays the Sagnac interferometer's extraordinary sensitivity to rotation. However, at any time (possibly excluding the start of the experiment), the photon will be in a non-trivial superposition of locations — both within the feedback loop, and on the output channel. In particular, the 'number of times the photon has traversed the feedback loop' is not a well-defined quantity. Thus, although we no longer have two distinct counter-rotating paths in the optical loop, the phenomenon of single-particle interference still has a large part to play in any formal description.

Because of this temporal aspect, it is not immediate how to give a treatment in terms of input and output spaces. Instead, we describe this apparatus in terms of a unitary operator that is repeatedly applied a space describing the entire state of the system — how to translate this into input / output behaviour, and the correct categorical interpretation forms a substantial part of this paper.

In order to analyse the above thought-experiment, we first use the assumption of discrete space and time — a common assumption used in (for example) the 'toy models' of [**Gr02**] or [**Pen04**]. We make the further assumption that, at the very beginning of the experiment, the output stream is empty — the single photon is not in a superposition of input and output modes. With these assumptions, we may draw the individual time-steps as shown in Figure 4.

Note that when we analyse this experiment using these conventions, we do not have a unitary map from a single space to itself – rather, at each step, the unitary evolution F_j is from space $\mathcal{S}_j$ to space $\mathcal{S}_{j+1}$. This is simply a labelling convention – however, it makes the analysis significantly simpler. From Figure 4,

FIGURE 4. Input / output streams in a modified Sagnac interferometer



and the fact that input and output modes must be horizontally polarised, we may give orthonormal bases for the spaces $\{\mathcal{S}_j\}_{j=0}^{\infty}$, as follows

- the space $\mathcal{S}_0$ has basis

$$\{|current\rangle\,|H\rangle\,, |current\rangle\,|V\rangle\,, |in_1\rangle\,|H\rangle\,, |in_2\rangle\,|H\rangle\,, |in_3\rangle\,|H\rangle\,, |in_4\rangle\,|H\rangle\,, \ldots\}$$

- the space $\mathcal{S}_1$ has basis

$$\{|out_1\rangle\,|H\rangle\,, |current\rangle\,|H\rangle\,|current\rangle\,|V\rangle\,, |in_2\rangle\,|H\rangle\,, |in_3\rangle\,|H\rangle\,, |in_4\rangle\,|H\rangle \ldots\}$$

- the space $\mathcal{S}_2$ has basis

$$\{|out_1\rangle\,|H\rangle\,, |out_2\rangle\,|H\rangle\,, |current\rangle\,|H\rangle\,|current\rangle\,|V\rangle\,, |in_3\rangle\,|H\rangle\,, |in_4\rangle\,|H\rangle \ldots\}$$

- ...

Using the description of the actions of the individual components, we may write down the unitary maps $\{F_i : \mathcal{S}_i \to \mathcal{S}_{i+1}\}_{i\in\mathbb{N}}$ as follows :

- $F_0 = \begin{pmatrix} b & a & 0 & 0 & \dots \\ d & c & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$

- $F_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & b & a & 0 & \dots \\ 0 & d & c & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$

- $F_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & b & a & \dots \\ 0 & 0 & d & c & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$
- $F_3 = \dots$

After T timesteps, the overall state of the system is the state in $\mathcal{S}_T$ given by applying the map $F_{T-1}F_{T-2}\dots F_0$ to the initial state in $\mathcal{S}_0$. Rather than analysing this directly, we consider a generalisation, where a, b, c, d in the above series of unitary maps are not simply complex numbers, but are themselves block matrices.

5. The twisted dagger construction

We now generalise the above analysis to the case where the specified unitary operation is a block matrix, rather than simply a 2×2 complex matrix, and consider the limit as the number of timesteps tends to infinity.

LEMMA 5.1. *Let* $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : X \oplus U \to Y \oplus U$ *be a unitary map, and let the unitary maps*

$$\{F_i : Y^{\oplus i} \oplus U \oplus X^{\oplus\omega} \to Y^{\oplus(i+1)} \oplus U \oplus X^{\oplus\omega}\}_{i=1}^{\infty}$$

be defined by

$$F_0 = \begin{pmatrix} B & A & 0 & \dots \\ D & C & 0 & \dots \\ 0 & 0 & I & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad F_1 = \begin{pmatrix} I & 0 & 0 & 0 & \dots \\ 0 & B & A & 0 & \dots \\ 0 & D & C & 0 & \dots \\ 0 & 0 & 0 & I & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$F_2 = \begin{pmatrix} I & 0 & 0 & 0 & 0 & \dots \\ 0 & I & 0 & 0 & 0 & \dots \\ 0 & 0 & B & A & 0 & \dots \\ 0 & 0 & D & C & 0 & \dots \\ 0 & 0 & 0 & 0 & I & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \dots$$

Then the composition (product of matrices) $F_nF_{n-1}\dots F_1F_0$ *has the block matrix*

$$\begin{pmatrix} B & A & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ BD & BC & A & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ BD^2 & BDC & BC & A & 0 & \dots & 0 & 0 & 0 & \dots \\ BD^3 & BD^2C & BDC & BC & A & \dots & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \\ BD^{n-1} & BD^{n-2}C & BD^{n-3}C & BD^{n-4}C & BD^{n-5}C & \dots & A & 0 & 0 & \dots \\ D^n & D^{n-1}C & D^{n-2}C & D^{n-3}C & D^{n-4}C & \dots & C & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & & 0 & I & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & & 0 & 0 & I & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Proof. We prove this by induction. (Notation: we abbreviate the above infinite matrix, omitting final columns and rows denoted by dots.) As a first step, note that

$$F_1F_0 = \begin{pmatrix} B & A & 0 & 0 \\ BD & BC & A & 0 \\ D^2 & DC & C & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

as required.

Now assume that for some $k \geq 1$,

$$F_kF_{k-1}\ldots F_0 = \begin{pmatrix} B & A & 0 & 0 & \ldots & 0 & 0 \\ BD & BC & A & 0 & \ldots & 0 & 0 \\ BD^2 & BDC & BC & A & \ldots & 0 & 0 \\ BD^3 & BD^2C & BDC & BC & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \\ BD^{k-1} & BD^{k-2}C & BD^{k-3}C & BD^{k-4}C & \ldots & A & 0 \\ D^k & D^{k-1}C & D^{k-2}C & D^{k-3}C & \ldots & C & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & I \end{pmatrix}$$

Then direct calculation gives that $F_{k+1}F_kF_{k-1}\ldots F_0 =$

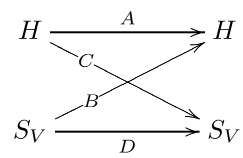
$$\begin{pmatrix} B & A & 0 & 0 & 0 & \ldots & 0 & 0 \\ BD & BC & A & 0 & 0 & \ldots & 0 & 0 \\ BD^2 & BDC & BC & A & 0 & \ldots & 0 & 0 \\ BD^3 & BD^2C & BDC & BC & A & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \\ BD^k & BD^{k-1}C & BD^{k-2}C & BD^{k-3}C & BD^{k-4}C & \ldots & A & 0 \\ D^{k+1} & D^kC & D^{k-1}C & D^{k-2}C & D^{k-3}C & \ldots & C & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & I \end{pmatrix}$$

Our result thus follows by induction. □

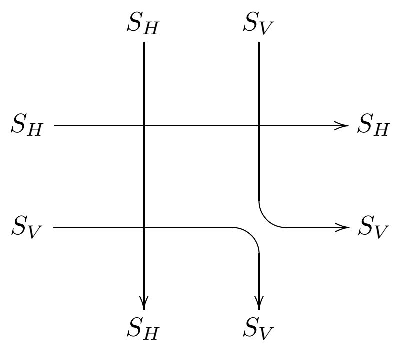
Remark 5.2. The above matrix calculations are based on a slight generalisation of the thought experiment of Section 4 — this more general case can be thought of as describing a single-particle interferometry experiment where the particle in question carries a number of quantum properties in addition to the polarisation. Let us assume that this particle may, as before, be horizontally or vertically polarised (H or V), and has further independent quantum properties $(k_1, k_2, \ldots, k_n)$. Let the 'polarisation space' P have orthonormal basis $\{|H\rangle, |S_V\rangle\}$ and let the 'additional properties space' K have orthonormal basis $\{|k_1\rangle, |k_2\rangle, \ldots, |k_n\rangle\}$. The state vector for a single particle in such an experiment is then a member of $S = P \otimes K$. We may take a direct sum decomposition of S as

$$S = S_H \oplus S_V \quad \text{where} \quad \begin{cases} S_H & \text{has basis} \quad |H\rangle|k_1\rangle \ , \ |H\rangle|k_2\rangle \ , \ \ldots \ , |H\rangle|k_n\rangle \\ S_V & \text{has basis} \quad |V\rangle|k_1\rangle \ , \ |V\rangle|k_2\rangle \ , \ \ldots \ , |V\rangle|k_n\rangle \end{cases}$$

and using this direct sum decomposition, give a block matrix representation for some unitary $U : S \to S$, as $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : S_H \oplus S_V \to S_H \oplus S_V$. Such a matrix may be represented schematically as



and using similar conventions, the polarising beam splitter is represented as



(Note that in the above two schematic diagrams, lines are separated by an implicit direct sum, rather than tensor product. Thus they should be interpreted as categorical string diagrams for the direct sum structure — an interpretation within the quantum circuit paradigm is not appropriate).

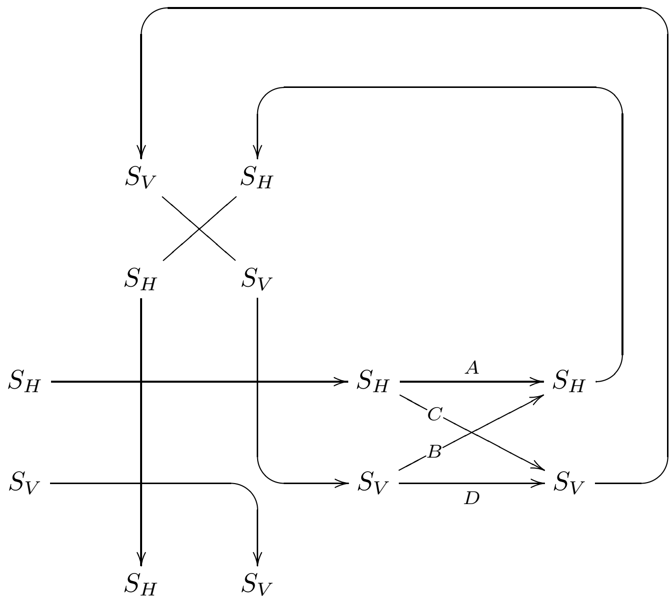
We may now compose these two diagrams, with the output fed back into the input as in the thought experiments of Section 4, to produce the schematic diagram shown in figure 5. As well as structure very similar to the usual diagram for a particle-style categorical trace, note the presence of the symmetry map for the direct sum.

Definition 5.3. Let $L : X \oplus U \to Y \oplus U$ be a finite-dimensional linear map, with block matrix $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. We define the **twisted dagger** of L, w.r.t. this decomposition, to be the matrix

$$\dagger^U(L) \;=\; \begin{pmatrix} B & A & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ BD & BC & A & 0 & 0 & 0 & 0 & 0 & \dots \\ BD^2 & BDC & BC & A & 0 & 0 & 0 & 0 & \dots \\ BD^3 & BD^2C & BDC & BC & A & 0 & 0 & 0 & \dots \\ BD^4 & BD^3C & BD^2C & BDC & BC & A & 0 & 0 & \dots \\ BD^5 & BD^4C & BD^3C & BD^2C & BDC & BC & A & 0 & \dots \\ BD^6 & BD^5C & BD^4C & BD^3C & BD^2C & BDC & BC & A & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Remark 5.4. The terminology 'twisted dagger' comes from the similarity of the above construction (especially the finite approximations of Lemma 5.1) with the Elgot dagger — up to an additional twist. Note that we do *not* claim the twisted

FIGURE 5. A schematic diagram for a generalised Sagnac interferometer



dagger is, in every case, the matrix representation of a continuous (i.e. bounded) linear map. In general, it is simply a formal matrix.

We now investigate the existence and properties of this matrix, with particular emphasis on when it describes either a continuous, or a unitary, linear map. We first require a trivial result:

LEMMA 5.5. *Given* $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{Hilb_{FD}}(X \oplus U, Y \oplus U)$, *and arbitrary* $\phi = \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \vdots \end{pmatrix} \in U \oplus X^{\oplus \omega}$, *let us define the formal matrix* $\zeta = \begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \zeta_2 \\ \vdots \end{pmatrix}$ *to be given by the formal matrix product* $\zeta = \dagger^U(L)(\phi)$, *so*

$$\zeta_n = \begin{cases} BD^n(\phi_0) + \sum_{i=1}^n BD^{n-i}C(\phi_i) + A(\phi_{n+1}) & (n > 0) \\ B(\phi_0) + A(\phi_1) & (n = 0) \end{cases}$$

Then ζ_n *exists, for all* $n \in \mathbb{N}$.

PROOF. Observe that ζ_n is given by a finite sum of continuous linear maps applied to a finite vector of elements. □

Note that we do not claim that $\sum_{n=0}^{\infty} \|\zeta_n\|^2$ exists in general, or (equivalently) that ζ is an element of $Y^{\oplus \omega}$ — a sufficient condition for this is given in Theorem 5.6 below.

THEOREM 5.6. *Let* $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : X \oplus U \to Y \oplus U$ *be a linear map between finite-dimensional Hilbert spaces. Then:*

(1) *A sufficient condition for* $\dagger^U(L)$ *to be the matrix representation of a bounded linear map between Hilbert spaces is that the component* D *is a strict contraction (i.e.* $Sup_{\|\psi\|=1}\|D(\psi)\| < 1$*).*
(2) *When* L *is a unitary map, a sufficient condition for* $\dagger^U(L)$ *to be unitary is that the component* D *is a strict contraction.*

PROOF. Let the family of unitary maps $\{F_i\}_{i\in\mathbb{N}}$ be as defined in Lemma 5.1, and define

$$\{G_n : U \oplus X^{\oplus\omega} \to Y^{\oplus(n+1)} \oplus U \oplus X^{\oplus\omega}\}_{n=0}^{\infty}$$

by $G_n = F_nF_{n-1}F_{n-2}\ldots F_0$. (we refer to Lemma 5.1 for explicit fomulæ for G_n). It is immediate that F_i is a well-defined linear map for all $i \geq 0$, and is unitary exactly when L is unitary. Similarly, the maps $\{G_k\}_{k\in\mathbb{N}}$ are bounded linear maps, and unitary exactly when L is unitary.

We now use these preliminaries to prove (1) and (2) above:

(1) Consider arbitrary $\phi \in U \oplus X^{\oplus\omega}$. We now study the sequence

$$\phi = \phi^{(0)} \xrightarrow{F_0} \phi^{(1)} \xrightarrow{F_1} \phi^{(2)} \xrightarrow{F_2} \phi^{(3)} \xrightarrow{F_3} \ldots$$

so $\phi^{(n)} = G_n(\phi)$. We write $\phi^{(n)}$ explicitly as

$$\phi^{(n)} = \begin{pmatrix} \phi_0^{(n)} \\ \phi_1^{(n)} \\ \phi_2^{(n)} \\ \phi_3^{(n)} \\ \vdots \end{pmatrix} \quad \text{where} \quad \begin{cases} \phi_n^{(n+i)} \in Y \\ \phi_n^{(n)} \in U \\ \phi_{n+i}^{(n)} \in X \end{cases} \quad \text{for all } n \in \mathbb{N}\ ,\ i > 0$$

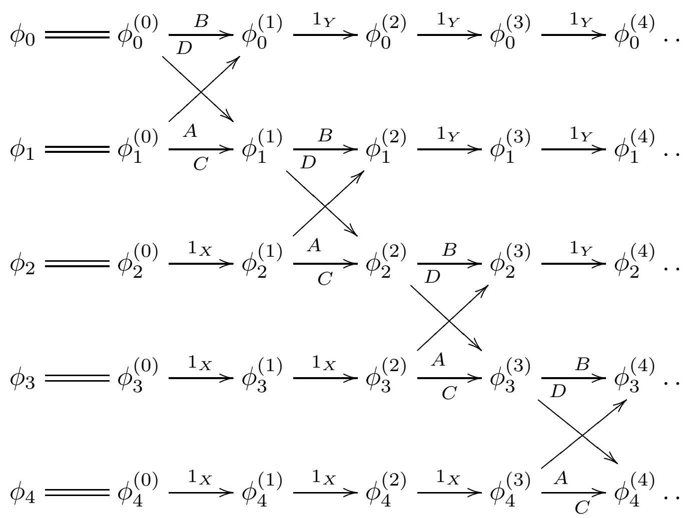
In particular, we make the identification $\phi_i^{(0)} = \phi_i$, for all $i \in \mathbb{N}$.

From the explicit description of $\{F_i\}_{i\in\mathbb{N}}$, we may use standard diagrammatic notation for matrix composition, and draw the calculation of the components of $\phi^{(n)}$ as shown in Figure 6.

From either this diagram, or by direct calculation, we may inductively calculate these components for all $p, q > 0$, as follows:

$$\phi_q^{(p)} = \begin{cases} \phi_q^{(0)} & q > p \\ B\left(\phi_{p-1}^{(p-1)}\right) + A\left(\phi_p^{(0)}\right) & q = p-1 \\ D\left(\phi_{p-1}^{(p-1)}\right) + C\left(\phi_p^{(0)}\right) & p = q \\ \phi_q^{(q+1)} & p > q+1 \end{cases}$$

FIGURE 6. Calculating components of $\phi^{(n)}$



By comparing these elements with the formal matrix $\zeta = \begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \zeta_2 \\ \vdots \end{pmatrix}$ from Lemma 5.5, it is immediate that that $\zeta_i = \phi_i^{(j)}$ for all $i < N$ and $j > i$.

By direct calculation, and the Cauchy-Bunyakovski-Schwarz inequality,

$$\|\phi_k^{(k)}\| \leq \|D^k\|.\|\phi_0^{(0)}\| + \sum_{n=0}^{k-1} \|D^n\|.\|C\|.\|\phi_{k-n}^{(0)}\|$$

However, by assumption $D : U \to U$ is a strict contraction map, so $\|D\| < 1$. Also, $\phi \in X^{\oplus\omega}$ and so $\sum_{i=0}^{\infty} \|\phi_i^{(0)}\|^2 < \infty$. Therefore, we deduce that $\sum_{k=0}^{\infty} \|\phi_k^{(k)}\|^2 < \infty$, and hence the 'diagonal element'

$$\Delta_\phi = \begin{pmatrix} \phi_0^{(0)} \\ \phi_1^{(1)} \\ \phi_2^{(2)} \\ \vdots \end{pmatrix}$$

is a member of $U^{\oplus\omega}$. Finally, observe that

$$\zeta = \begin{pmatrix} B & 0 & 0 & \cdots \\ 0 & B & 0 & \cdots \\ 0 & 0 & B & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \phi_0^{(0)} \\ \phi_1^{(1)} \\ \phi_2^{(2)} \\ \vdots \end{pmatrix} + \begin{pmatrix} A & 0 & 0 & \cdots \\ 0 & A & 0 & \cdots \\ 0 & 0 & A & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \phi_0^{(0)} \\ \phi_1^{(0)} \\ \phi_2^{(0)} \\ \vdots \end{pmatrix}$$

and hence $\zeta \in Y^{\oplus\omega}$, as required.

To show that the condition $\|D\| < 1$ is not a necessary condition, consider the simplest possible counterexample – the identity matrix $\begin{pmatrix} 1_X & 0 \\ 0 & 1_U \end{pmatrix}$. It is immediate that

$$\dagger^U(L) = \begin{pmatrix} 0 & 1_X & 0 & 0 & 0 & \dots \\ 0 & 0 & 1_X & 0 & 0 & \dots \\ 0 & 0 & 0 & 1_X & 0 & \dots \\ 0 & 0 & 0 & 0 & 1_X & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

this is clearly not unitary, but is a partial isometry – the *shift map.*

(2) *In this part of the proof, we use the characterisation of unitary maps as,* "Partial isometries, with full initial and final subspaces". *This is immediate from the definition of partial isometries* [**Hal58**].

We know from 1/ above that $\dagger^U(L)$ exists, for all unitary $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ satisfying $\|D\| < 1$. We now need to show that:

(a) $\dagger^U(L)$ is a partial isometry,
(b) The initial and final subspaces of $\dagger^U(L)$ are the whole of $U \oplus X^{\oplus\omega}$ and $Y^{\oplus\omega}$ respectively.

These results may be seen as follows :

(a) We first define $Term_N : Y^{\oplus(N+1)} \oplus U \oplus X^{\oplus\omega} \to Y^{\oplus\omega}$ for all $N \in \mathbb{N}$, by

$$Term_N \begin{pmatrix} y_0 \\ \vdots \\ y_{N-1} \\ u \\ x_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} y_0 \\ \vdots \\ y_N \\ 0_Y \\ 0_Y \\ \vdots \end{pmatrix}$$

Clearly, $Term_N$ is a linear map, and is a partial isometry, with initial subspace $Y^{\oplus(N+1)} \subseteq Y^{\oplus(N+1)} \oplus U \oplus X^{\oplus\omega}$. Hence, as $G_N : U \oplus X^{\oplus\omega} \to Y^{\oplus(N+1)} \oplus U \oplus X^{\oplus\omega}$ is unitary, the composite $Term_N G_N : U \oplus X^{\oplus\omega} \to Y^{\oplus\omega}$ is a partial isometry.
Now consider arbitrary fixed $\phi \in U \oplus X^{\oplus\omega}$. From part 1/ above, for all $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that

$$\|Term_M(G_M(\phi)) - \dagger^U(L)(\phi)\| < \epsilon$$

By completeness, $lim_{N\to\infty} Term_N(G_N(\phi)) = \dagger^U(L)(\phi)$, and so in the space $\mathbf{Hilb}(U \oplus X^{\oplus\omega}, Y^{\oplus\omega})$, the series of partial isometries $\{Term_N G_N\}_{N=0}^{\infty}$ converges to $\dagger^U(L)$. By [**AnCo04, AnCo05**], the set of partial isometries between spaces H_1, H_2 forms a smooth closed submanifold of the space $\mathbf{Hilb}(H_1, H_2)$ Therefore, we deduce that the limit $\dagger^U(L)$ is a partial isometry.

(b) *We prove that the inital subspace is full by contradiction.*
Assume there exists some $u \in U$ such that $B(u) = 0$. Then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ D(u) \end{pmatrix}$$

However, $\left\| \begin{pmatrix} 0 \\ u \end{pmatrix} \right\| = \|u\|$, and by the assumption that D is a strong contraction, $\left\| \begin{pmatrix} 0 \\ D(u) \end{pmatrix} \right\| = \|D(u)\| < \|u\|$. This is a contradiction of the unitarity of L, so we deduce that $B(u) \neq 0$, for all $u \in U$.

Now let $\chi \in U \oplus X^{\oplus\omega}$ be in the complement of the initial subspace of $\dagger^U(L)$, so $\dagger^U(L)(\chi) = 0$. As $lim_{n\to\infty} Term_n(G_n(\phi)) = \dagger^U(L)(\phi)$, we deduce that $\{\chi^{(n)} = Term_n(G_n(\chi))\}_{n=0}^{\infty}$ is a series of elements of $Y^{\oplus\omega}$ that converges to 0. Writing these explicitly as

$$\chi^{(n)} = \begin{pmatrix} \chi_0^{(n)} \\ \chi_1^{(n)} \\ \chi_2^{(n)} \\ \vdots \end{pmatrix}$$

We observe from part 1. that $\chi_{n+k}^{(n)} = \chi_{n+2}^{(n)}$ for all $k \geq 2$. Hence $\chi = 0$ implies that $\chi_{n+2}^{(n)} = 0$, for all $n \in \mathbb{N}$. However, by close inspection of Figure 6, this is only possible when $B(u) = 0$, for some $u \in U$, contradicting the preliminary result above.

We now demonstrate that the final subspace is full
Consider arbitrary $\zeta \in Y^{\oplus\omega}$, written as

$$\zeta = \begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \zeta_2 \\ \vdots \end{pmatrix}$$

As $\zeta \in Y^{\oplus\omega}$, for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that $\sum_{i=N}^{\infty} \|\zeta_i\|^2 < \epsilon$. Using the adjoint of the partial isometry $Term_M$ above, it is immediate that

$$Term_M^*(\zeta) = \begin{pmatrix} \zeta_0 \\ \vdots \\ \zeta_M \\ 0_U \\ 0_X \vdots \end{pmatrix}$$

and, for all $M > N$, $\|\zeta\|^2 - \|Term_M^*(\zeta)\|^2 < \epsilon$. We now define $\lambda^{(M)} \in U \oplus X^{\oplus\omega}$ by $\lambda^{(M)} = G_M^{-1}(Term_M^*(\zeta))$, where the unitary map G_M^{-1} is given by $F_M^{-1}F_{M-1}^{-1}\dots F_0^{-1}$, for F_i as defined in part 1. Since G_M^{-1} is unitary, $\|\lambda^{(M)}\| = \|Term_M^*(\zeta)\|$. By taking sufficiently large $M > N \in \mathbb{N}$, it follows that $\dagger^U(L)(\lambda^{(M)}) \to \zeta$ as $M \to \infty$, and as ζ was chosen arbitrarily, the terminal subspace of $\dagger^U(L)$ is exactly $Y^{\oplus\omega}$. This then completes our proof of unitarity.

□

6. Setting initial conditions, and compositionality

The intention of this paper is to use the twisted dagger to motivate general categorical constructions — thus, we need an appropriate setting in which such operations give rise to arrows in a category.

As a motivating example, we consider the special case where the apparatus of Figure 4 satisfies the further initial condition, that that the 'internal state' of the optical loop is empty: at the start of the experiment, the probability that an observation of the internal state detects a photon is 0. This will allow us to treat the apparatus of Figure 4 (via the associated 'twisted dagger') as defining a map from the **input space** $l_2(\{in_j\}_{j=0}^\infty)$ to the **output space** $l_2(\{out_j\}_{j=0}^\infty)$. We then generalise this to arbitrary twisted dagger operations.

6.1. Initial conditions, and inclusion maps. In the apparatus of Figure 4, we wish to impose the initial condition that the 'internal state' of the optical loop is empty. i.e. at the start of the experiment, the probability that an observation on the internal state detects a photon is 0.

Given a state representing an 'input stream' (i.e. a single photon in a superposition of input modes),

$$|in\rangle = \alpha_0 |in_0\rangle + \alpha_1 |in_1\rangle + \alpha_2 |in_2\rangle + \dots \ \in \ l_2(\{in_j\}_{j=0}^\infty)$$

the state in the larger space S_0, describing both the input stream and (empty) 'internal states' is $\iota(|in\rangle) \ = \ \alpha_0 |in_0\rangle + \alpha_1 |in_1\rangle + \alpha_2 |in_2\rangle + \dots \ \in \ S_0$, where

$$\iota : l_2(\{in_j\}_{j=0}^\infty) \ \to \ S_0 = l_2(\{in_j\}_{j=0}^\infty) \oplus l_2(\{(Current, H), (Current, V)\})$$

is simply the canonical inclusion map associated with the direct sum. This trivially satisfies the measurement condition of Section 6, since

$$\langle Current, H| \cdot \iota \cdot |in\rangle = 0 = \langle Current, V| \cdot \iota \cdot |in\rangle$$

Thus, the input-output map associated with the apparatus of Figure 4, along with this initial condition, is the composite

$$\dagger^U \begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \iota : l_2(\{in_j\}_{j=0}^\infty) \to l_2(\{out_k\}_{k=0}^\infty)$$

We now generalise this intuition to the general case, for arbitrary finite-dimensional Hilbert spaces, as follows:

Definition 6.1. Let $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{Hilb_{FD}}(X \oplus U, Y \oplus U)$ be a linear map where $\dagger^U(L)$ is defined (i.e. is an arrow of $\mathbf{Hilb}(U \oplus X^{\oplus\omega}, Y^{\oplus\omega})$). We define the **input-output behaviour** of $\dagger^U(L)$ to be the map

$$\Delta^U(L) \ = \ \dagger^U(L) \cdot \iota \in \mathbf{Hilb}(X^{\oplus\omega}, Y^{\oplus\omega})$$

where $\iota : X^{\oplus\omega} \to U \oplus X^{\oplus\omega}$ is the canonical inclusion associated with the direct sum.

The following facts about such input-output behaviours are straightforward, but will be essential in the following sections:

LEMMA 6.2. *Let* $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{Hilb_{FD}}(X \oplus U, Y \oplus U)$ *be a linear map such that* $\dagger^U(L) \in \mathbf{Hilb}(U \oplus X^{\oplus\omega}, Y^{\oplus\omega})$*, and let* $\Delta^U(L) \in \mathbf{Hilb}(X^{\oplus\omega}, Y^{\oplus\omega})$ *be as defined above. Then*

(1) $\Delta^U(L)$ *is a continuous (i.e. bounded) linear map.*
(2) $\Delta^U(L)$ *has a matrix representation of the form* $\Delta^U(L) = [M_{ij}]$*, where*

$$M_{ij} = \begin{cases} 0_{XY} & j - i < 0 \\ p_{j-i} & j - i \geq 0 \end{cases}$$

for some family of linear maps $\{p_k \in \mathbf{Hilb_{FD}}(X, Y)\}_{k=0}^{\infty}$.

PROOF.

(1) This is immediate: $\Delta^U(L) = \dagger^U(L) \cdot \iota$ is the composite of two arrows in the same category, and hence is also an arrow in this category.
(2) From the explicit matrix for the twisted dagger given in Definition 5.3, the composite $\Delta^U(L) \cdot \iota : X^{\oplus\omega} \to Y^{\oplus\omega}$ has the following matrix:

$$\Delta^U(L) \;=\; \begin{pmatrix} A & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ BC & A & 0 & 0 & 0 & 0 & 0 & \dots \\ BDC & BC & A & 0 & 0 & 0 & 0 & \dots \\ BD^2C & BDC & BC & A & 0 & 0 & 0 & \dots \\ BD^3C & BD^2C & BDC & BC & A & 0 & 0 & \dots \\ BD^4C & BD^3C & BD^2C & BDC & BC & A & 0 & \dots \\ BD^5C & BD^4C & BD^3C & BD^2C & BDC & BC & A & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Thus our result follows from this explicit matrix description, by taking $p_k = \begin{cases} A & k = 0 \\ BD^{k-1}C & k > 0 \end{cases}$

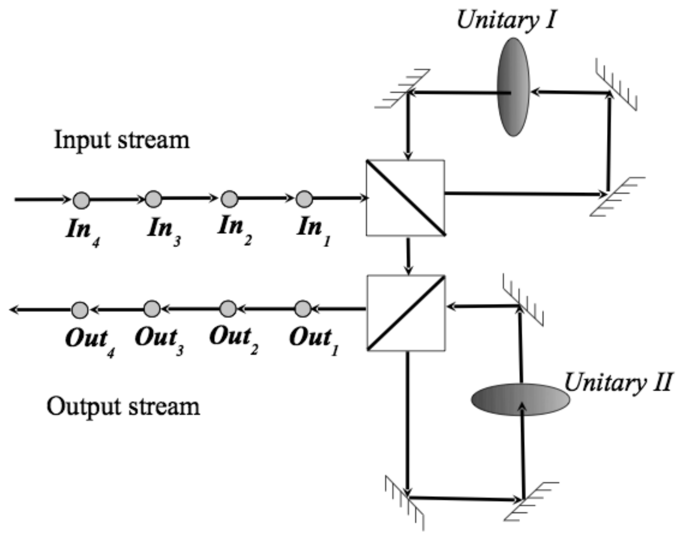
□

6.2. Input-output behaviours, and compositionality. We now consider how to compose the input-output behaviours, as defined above. We again motivate this by the single-photon linear optics case, where our interpretation must correspond to treating the input stream of one such experiment as the output stream of another, as shown in Figure 7. (Note that, in this figure, we omit the detectors and post-conditioning, for clarity. Rather, we simply assume that the input photon is guaranteed to be horizontally polarised. Of course, from the experimental set-up, the output of the first MSI - and hence the input of the second MSI - is also horizontally polarised).

Trivially, the output stream of the upper MSI becomes the input stream of the lower MSI. Thus, if the input-output behaviour of the upper MSI is $P = [P_{ij}]$, and the input-output behavior of the lower MSI is $Q = [Q_{ij}]$, then the input-output behaviour of the entire apparatus is simply given by the matrix product QP.

6.3. A relevant subcategory of Hilb. We now demonstrate that bounded linear maps with matrices satisfying the special form given in part (2) of Lemma 6.2 are the arrows of a subcategory of **Hilb**.

FIGURE 7. Composing MSIs



PROPOSITION 6.3. *Arrows in* **Hilb** *with matrix representations of the form*

$$M_{ij} = \begin{cases} 0_{XY} & j-i<0 \\ p_{j-i} & j-i \geq 0 \end{cases} \quad \textit{for some } \{p_k \in \mathbf{Hilb_{FD}}(X,Y)\}_{k=0}^{\infty}$$

are closed under composition, and include identity maps, and hence define a subcategory of **Hilb**.

PROOF. Consider arrows $L \in \mathbf{Hilb}(X^{\oplus\omega}, Y^{\oplus\omega})$ and $M \in \mathbf{Hilb}(Y^{\oplus\omega}, Z^{\oplus\omega})$ where

$$M_{ij} = \begin{cases} 0_{XY} & j-i<0 \\ q_{j-i} & j-i \geq 0 \end{cases} \quad \text{for some } \{q_k \in \mathbf{Hilb_{FD}}(X,Y)\}_{k=0}^{\infty}$$

$$L_{ij} = \begin{cases} 0_{XY} & j-i<0 \\ p_{j-i} & j-i \geq 0 \end{cases} \quad \text{for some } \{p_k \in \mathbf{Hilb_{FD}}(X,Y)\}_{k=0}^{\infty}$$

Then from the standard formula for matrix multiplication,

$$[ML]_{i,k} = \sum_{j=0}^{\infty} [M]_{i,j}[L]_{j,k}$$

their composite $ML \in \mathbf{Hilb}(X^{\oplus\omega}, Z^{\oplus\omega})$ has matrix representation

$$[ML]_{i,k} = \begin{cases} 0_{XY} & k-i<0 \\ r_{k-i} & k-i \geq 0 \end{cases} \quad \text{where } r_c = \sum_{c=b+a} q_b p_a$$

and hence is of the required form. Finally, observe that the identity matrix is trivially of this form. Thus, bounded linear maps of this form specify a subcategory of **Hilb**. □

We observe the similarity between the composition in Proposition 6.3 above, and the *Cauchy product* of power series. This motivates the general categorical setting below.

7. A categorical setting for twisted daggers

We now introduce a categorical setting for twisted daggers (or rather, their input-output behaviours), based on the above observation that composition may be expressed as a Cauchy product (i.e. the usual composition of single-variable power series).

7.1. A category of formal power series.

DEFINITION 7.1. We define the category $\mathbf{Hilb_{FD}[z]}$ of **formal power series** over **Hilb** as follows:

- Objects of $\mathbf{Hilb_{FD}[z]}$ are finite-dimensional complex Hilbert spaces.
- The hom-set $\mathbf{Hilb_{FD}[z]}(H,K)$ is the set of all formal power series in z over $\mathbf{Hilb}(H,K)$. Thus $p \in \mathbf{Hilb_{FD}[z]}(H,K)$ may be written as

  $$p = \sum_{n=0}^{\infty} p_n.z^n \quad \text{where} \;\; p_n \in \mathbf{Hilb}(H,K) \;\; \forall n \in \mathbb{N}$$

  We will equivalently and interchangeably refer to the formal power series

  $$p = p_0 + p_1.z + p_2.z^2 + \ldots \in \mathbf{Hilb_{FD}[z](H,K)}$$

  and the function $p : \mathbb{N} \to \mathbf{Hilb_{FD}}(H,K)$ where $p(k) \in \mathbf{Hilb_{FD}}(H,K)$ is the coefficient of z^k in $p_0 + p_1.z + p_2.z^2 + \ldots$.
- Composition is the usual *Cauchy product* [**Ti83**]. Given $p \in \mathbf{Hilb_{FD}[z]}(H,K)$ and $q \in \mathbf{Hilb_{FD}[z]}(K,L)$, then their composite $r = qp \in \mathbf{Hilb_{FD}[z]}(H,L)$ is the formal power series

  $$r = r_0 + r_1.z + r_2.z^2 + r_3.z^3 + \ldots$$

  where $r_c = \sum_{c=b+a} q_b p_a \in \mathbf{Hilb}(H,L)$, for all $c \in \mathbb{N}$.

We emphasize that these power series are *formal*, rather than convergent. Indeed, we may identify them with an infinite sequence $(p_n)_{n\in\mathbb{N}}$ of bounded linear operators. We do not assume that the sum $p_0 + p_1\zeta + p_2\zeta^2 + \ldots$ converges for any non-zero $\zeta \in \mathbb{C}$ or even that $\sum_{k=0}^{\infty} \|p_k\|_\nu$ converges, for any particular operator norm $\| \; \|_\nu$ — although it may be observed that imposing such requirements defines various subcategories of $\mathbf{Hilb_{FD}[z]}$.

PROPOSITION 7.2. *$\mathbf{Hilb_{FD}[z]}$, as defined above, is a category.*

PROOF. First note that, for all $p \in \mathbf{Hilb_{FD}[z]}(H,K)$ and $q \in \mathbf{Hilb_{FD}[z]}(K,L)$, their composite

$$r = r_0 + r_1.z + r_2.z^2 + r_3.z^3 + \ldots \quad \text{where } r_c = \sum_{c=b+a} q_b p_a \in \mathbf{Hilb}(H,L)$$

is defined, since for all $k \in \mathbb{N}$

$$r_k = q_k p_0 + q_{k-1} p_1 + \ldots + q_1 p_{k-1} + q_0 p_k$$

is a finite sum of linear maps. It remains to show that composition is associative, and has identities at each object. However, associativity of composition of formal power series is long-established ([**Ti83**]), and the identity map I_H at an object H is simply the formal power series $1_H + 0_H.z + 0_H.z^2 + 0_H.z^3 + \ldots$. Thus, $\mathbf{Hilb_{FD}[z]}$ is a category. □

LEMMA 7.3. *There exists a canonical inclusion $\iota : \mathbf{Hilb_{FD}} \to \mathbf{Hilb_{FD}[z]}$.*

PROOF. The canonical inclusion is simply given by

- **Objects** $\iota(A) = A \in Ob(\mathbf{Hilb_{FD}}[\mathbf{z}])$, for all $A \in Ob(\mathbf{Hilb_{FD}})$.
- **Arrows** Given $f \in \mathbf{Hilb_{FD}}(X,Y)$, then $\iota(f) \in \mathbf{Hilb_{FD}}[\mathbf{z}](X,Y)$ is defined by

$$\iota(f)(n) = \begin{cases} f \in Hilb_{FD}(X,Y) & n = 0 \\ 0_{X,Y} & \text{otherwise.} \end{cases}$$

It is immediate from the definition of composition in $\mathbf{Hilb_{FD}}[\mathbf{z}]$ that this is an injective functor. □

We are now able to identify a suitable category in which the input-output behaviour of twisted daggers (as in Section 6.3) lives, and by extension a category suitable for reasoning about the thought-experiment of Section 4.

DEFINITION 7.4. Let us denote the category of Proposition 6.3 by **THilb**, so

- $H^{\oplus\omega} \in Ob(\mathbf{THilb})$, for all $H \in Ob(\mathbf{Hilb_{FD}})$.
- $M \in \mathbf{Hilb}(H^{\oplus\omega}, K^{\oplus\omega})$ is an arrow of **THilb** iff M has a block matrix representation of the form

$$M_{ij} = \begin{cases} 0_{XY} & j-i<0 \\ p_{j-i} & j-i \geq 0 \end{cases} \quad \text{for some } \{p_k \in \mathbf{Hilb_{FD}}(H,K)\}_{k=0}^{\infty}$$

THEOREM 7.5. *There exists an embedding of* **THilb** *into* $\mathbf{Hilb_{FD}}[\mathbf{z}]$.

PROOF. This is immediate by mapping matrices of the form

$$M_{ij} = \begin{cases} 0_{HK} & j-i<0 \\ p_{j-i} & j-i \geq 0 \end{cases} \quad \text{for some } \{p_k \in \mathbf{Hilb_{FD}}(H,K)\}_{k=0}^{\infty}$$

to power series defined by

$$M = p_0 + p_1.z + p_2.z^2 + \ldots \in \mathbf{Hilb_{FD}}[\mathbf{z}](H,K)$$

□

REMARK 7.6. *The intuition behind categories of power series* Abstractly, an arrow in a category $f \in \mathcal{C}(X,Y)$ may be thought of as a process that transforms data of type X into data of type Y. Categories of formal power series extend this intuition by considering the time associated with such transformations, and associating with each input-output pairing a discrete number of time-steps. Thus, the arrow $f(n) \in \mathcal{C}(X,Y)$ describes the input-output mappings of f that take exactly n timesteps. Given a suitable notion of summability of arrows (as in $\mathbf{Hilb_{FD}}$), the Cauchy product of power series then has a natural interpretation: the input-output mappings of gf that take c timesteps arise as the sum of all processes that take b timesteps in g, and a timesteps in f, where $c = b + a$.

7.2. Additional structure on the category of power series. Our category of formal power series has a natural notion of summation on its hom-sets, derived from the familiar summation of arrows of $\mathbf{Hilb_{FD}}$, as we now demonstrate:

DEFINITION 7.7. Let $\{f_j \in \mathbf{Hilb_{FD}}[\mathbf{z}](H,K)\}_{j\in J}$ be a countably indexed family of arrows. We say that this family is **summable** when $\sum_{j\in J}(f_j(n))$ exists (in the sense of absolute convergence of countable families of linear maps), for all $n \in \mathbb{N}$.

When the family $\{f_j\}_{j\in J}$ above is summable, its **formal sum**

$$F = \sum_{j\in J} f_j \ \in \ \mathbf{Hilb_{FD}}[\mathbf{z}](H,K)$$

is the function $F : \mathbb{N} \to \mathbf{Hilb_{FD}}(H,K)$ given by

$$F(n) = \sum_{j\in J} f_j(n) \in \mathbf{Hilb}_{FD}(H,K)$$

Note that, as this notion of summation is based on absolute convergence at each power of z, we have distributivity of composition over arbitrary sums. Also, it is straightforward from its definition in terms of absolute convergence that this notion of summation satisfies a one-sided version of the *partition-associativity* axiom of [**Ha00, HS04**], which itself arose from the theory of partially-additive semantics in [**MA86**].

The weak partition-associativity property: Let $\{f_i \in \mathbf{Hilb_{FD}}[\mathbf{z}](H,K)\}_{i\in I}$ be a countably indexed summable family, and let $\{I_j\}_{j\in J}$ be a countable partition[6] of I. Then $\{f_i\}_{i\in I_j}$ is summable for every $j \in J$, as is $\{\sum_{i\in I_j} f_i\}_{j\in J}$, and

$$\sum_{i\in I} f_i = \sum_{j\in J}\left(\sum_{i\in I_j} f_i\right)$$

Informally, this may be phrased as: 'sub-families of summable families are themselves summable, and replacing any sub-sum by its sum neither affects summability nor changes the result'. Finally, note that, by contrast with the axioms of [**Ha00, HS04**], the implication in the above property is strictly one-way.

Given this notion of summation, we may give matrix representations of such power series, as follows:

DEFINITION 7.8. Consider some formal power series $p \in \mathbf{Hilb_{FD}}[\mathbf{z}](A,B)$. Further assume that A and B are given as (finite) direct sum decompositions, so

$$A = \bigoplus_{i=1}^{m} A_i \quad \text{and} \quad B = \bigoplus_{j=1}^{n} B_j$$

Since the direct sum is a biproduct on $\mathbf{Hilb_{FD}}$, $p(t)$ may be written as an $(m\times n)$ matrix, where $[p(t)]_{x,y} : A_y \to B_x$, for all $t \in \mathbb{N}$. Using this matrix decomposition for each $p(t) \in \mathbf{Hilb_{FD}}(A,B)$, we define the **matrix** of $p \in \mathbf{Hilb_{FD}}[\mathbf{z}](A,B)$ to be the $n\times m$ matrix of arrows of $\mathbf{Hilb_{FD}}[\mathbf{z}]$ defined by

$$[p]_{x,y}(t) \overset{def.}{=} [p(t)]_{x,y} \ \in \ \mathbf{Hilb_{FD}}(A_y, B_x)$$

Composition of such matrices of power series is defined in the natural way, with composition defined by the Cauchy product of power series (i.e. composition in $\mathbf{Hilb_{FD}}[\mathbf{z}]$), and summation is as in Definition 7.7 above.

As an illustrative example, this is simply the familiar interchangeability of formal power series whose coefficients are matrices of linear maps with matrices

[6]Following [**MA86**], we also allow countably many I_j to be empty.

whose coefficients are formal power series of linear maps, e.g. the equivalence of the power series of matrices

$$\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.z \; + \; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.z^2$$

with the matrix of power series

$$\begin{pmatrix} i + \frac{1}{\sqrt{2}} z^2 & z + \frac{1}{\sqrt{2}} z^2 \\ z + \frac{1}{\sqrt{2}} z^2 & 1 - \frac{1}{\sqrt{2}} z^2 \end{pmatrix}$$

Proposition 7.9. *The interpretation of formal power series as matrices above is compatible with composition: the matrix of the composite is the product of the matrices. Precisely, given*

$$q \in \mathbf{Hilb_{FD}}[\mathbf{z}] \left(\bigoplus_{j=1}^{m} B_j, \bigoplus_{k=1}^{n} C_k \right) \quad \textit{and} \quad p \in \mathbf{Hilb_{FD}}[\mathbf{z}] \left(\bigoplus_{i=1}^{l} A_i, \bigoplus_{j=1}^{m} B_j \right)$$

then $[q][p] = [qp]$.

Proof. This is a simple-index-chasing argument, that follows from comparing the definition of matrix multiplication of Definition 7.8 with the definition of composition in $\mathbf{Hilb_{FD}}[\mathbf{z}]$.

□

7.3. A monoidal tensor on $\mathbf{Hilb_{FD}}[\mathbf{z}]$. We use the above notion of matrices in $\mathbf{Hilb_{FD}}[\mathbf{z}]$ to provide a monoidal tensor:

Definition 7.10. We define $\oplus : \mathbf{Hilb_{FD}}[\mathbf{z}] \times \mathbf{Hilb_{FD}}[\mathbf{z}] \rightarrow \mathbf{Hilb_{FD}}[\mathbf{z}]$ as follows:

- **Objects** Given $A, B \in Ob(\mathbf{Hilb_{FD}}[\mathbf{z}])$, then $A \oplus B \in Ob(\mathbf{Hilb_{FD}}[\mathbf{z}])$ is simply the direct sum of A and B, as in Definition 2.4.
- **Arrows** Given $p \in \mathbf{Hilb_{FD}}[\mathbf{z}](H, J)$ and $q \in \mathbf{Hilb_{FD}}[\mathbf{z}](L, M)$, then

$$(p \oplus q) : \mathbb{N} \rightarrow \mathbf{Hilb}(H \oplus L, J \oplus M)$$

  is simply defined by

$$(p \oplus q)(k) = p(k) \oplus q(k) \in \mathbf{Hilb}(H \oplus L, J \oplus M)$$

Theorem 7.11. *The map* $\oplus : \mathbf{Hilb_{FD}}[\mathbf{z}] \times \mathbf{Hilb_{FD}}[\mathbf{z}] \rightarrow \mathbf{Hilb_{FD}}[\mathbf{z}]$ *defined above is a symmetric monoidal tensor.*

Proof. *For clarity, this proof uses matrix notation for both linear maps and formal power series. This is justified by Proposition 7.8 above.*

- **compositionality** Consider arrows

$$p = \sum_{i=0}^{\infty} p_i z^i \in \mathbf{Hilb_{FD}}[\mathbf{z}](H, J) \;\text{ and }\; q = \sum_{j=0}^{\infty} q_j z^j \in \mathbf{Hilb_{FD}}[\mathbf{z}](L, M)$$

  and similarly

$$r = \sum_{i=0}^{\infty} r_i z^i \in \mathbf{Hilb_{FD}}[\mathbf{z}](J, K) \;\text{ and }\; s = \sum_{j=0}^{\infty} s_j z^j \in \mathbf{Hilb_{FD}}[\mathbf{z}](M, N)$$

Then

$$(p \oplus q) = \sum_{i=0}^{\infty} \begin{pmatrix} p_i & 0 \\ 0 & q_i \end{pmatrix} z_i \quad \text{and} \quad (r \oplus s) = \sum_{j=0}^{\infty} \begin{pmatrix} r_j & 0 \\ 0 & s_j \end{pmatrix} z_j$$

By definition of composition in $\mathbf{Hilb_{FD}}[\mathbf{z}]$,

$$((r \oplus s)(p \oplus q))(c) = \sum_{c=b+a} \begin{pmatrix} r(b) & 0 \\ 0 & s(b) \end{pmatrix} \begin{pmatrix} p(a) & 0 \\ 0 & q(a) \end{pmatrix}$$

$$= \sum_{c=b+a} \begin{pmatrix} r(b)p(a) & 0 \\ 0 & s(b)q(a) \end{pmatrix} = \begin{pmatrix} \sum_{c=b+a} r(b)p(a) & 0 \\ 0 & \sum_{c=b+a} s(b)q(a) \end{pmatrix}$$

However, this is simply $(rp \oplus sq)(c)$, as required.

- **Identities** The identity $1_A \in \mathbf{Hilb_{FD}}[\mathbf{z}](A, A)$ is given by

$$1_A(n) = \begin{cases} 1_A \in \mathbf{Hilb}(A, A) & n = 0 \\ 0_{A,A} & \text{otherwise.} \end{cases}$$

It is immediate from the definition that $1_A \oplus 1_B \in \mathbf{Hilb_{FD}}[\mathbf{z}](A \oplus B, A \oplus B)$ satisfies

$$(1_A \oplus 1_B)(n) = \begin{cases} 1_{A\oplus B} \in \mathbf{Hilb}(A \oplus B, A \oplus B) & n = 0 \\ 0_{A\oplus B, A\oplus B} & \text{otherwise.} \end{cases}$$

- **Associativity** Let us denote the canonical associativity isomorphisms for $(\mathbf{Hilb_{FD}}, \oplus)$ by $t_{A,B,C}$. The corresponding associativity isomorphisms for $\mathbf{Hilb_{FD}}[\mathbf{z}]$ are given by $\iota(t_{A,B,C})$, where $\iota : \mathbf{Hilb_{FD}} \to \mathbf{Hilb_{FD}}[\mathbf{z}]$ is given in Lemma 7.3. It is straightforward to verify that $\iota : \mathbf{Hilb_{FD}} \to \mathbf{Hilb_{FD}}[\mathbf{z}]$ is a monoidal functor, and MacLane's pentagon condition then follows by functoriality.
- **Symmetry** Similarly to the associativity isomorphisms, the symmetry isomorphisms for $\mathbf{Hilb_{FD}}[\mathbf{z}]$ are given by $\iota(s_{A,B})$, where $s_{A,B}$ is the family of symmetry isomorphisms for $\mathbf{Hilb_{FD}}$. The commutativity hexagon again follows by functoriality.
- **Units objects and arrows** Finally, it is immediate that the unit object for $(\mathbf{Hilb_{FD}}, \oplus)$ is also the unit object for $(\mathbf{Hilb_{FD}}[\mathbf{z}], \oplus)$.

□

8. Partial traces in symmetric monoidal categories

The notion of categorical trace was introduced by Joyal, Street and Verity in an influential paper [**JSV96**]. The motivation for their work arose in algebraic topology and knot theory, although the authors were aware of applications in Computer Science, where they include such notions as feedback, fixedpoints, etc. The starting point for our investigation of the categorical structures associated with the Twisted Dagger operation of Section 5 was the observation in Remark 5.2 that the columns of this formal matrix bear a close similarity with the summands of the particle-style categorical trace; for some of the history of these ideas, see [**Ab96, AHS02, HS10**]. There are also deep connections of traced monoidal categories with the proof theory of linear logic, and Girard's Geometry of Interaction program, as described in [**Hi98, AHS02, HS04, HS10a**], which motivated the introduction of the axioms of partial traces below.

In categories associated with linear maps on Hilbert spaces it is more natural to consider *partial traces* defined in terms of infinite sums (or, in the case of [**HS10**], the invertibility of certain operators). We follow the definitions of [**HS10**], where a partial particle-style trace on Hilbert spaces with direct sums is exhibited. We also contrast this with an alternative definition of partial trace on Hilbert spaces with tensor products given in [**ABP98**].

Recall, following Joyal, Street, and Verity [**JSV96**], a (parametric) trace in a symmetric monoidal category $(C, \otimes, I, s)$ is a family of maps

$$Tr^U_{X,Y} : C(X \otimes U, Y \otimes U) \to C(X, Y),$$

satisfying various naturality equations. A *partial* (parametric) trace requires instead that each $Tr^U_{X,Y}$ be a partial map (with domain denoted $\mathbb{T}^U_{X,Y}$) satisfying various closure conditions.

The following definitions are taken from [**HS10, HS10a**].

DEFINITION 8.1. Let $(C, \otimes, I, s)$ be a symmetric monoidal category. A *(parametric) trace class* in C is a choice of a family of subsets, for each object U of C, of the form

$$\mathbb{T}^U_{X,Y} \subseteq C(X \otimes U, Y \otimes U) \text{ for all objects } X, Y \text{ of } C$$

together with a family of functions, called a *(parametric) partial trace*, of the form

$$Tr^U_{X,Y} : \mathbb{T}^U_{X,Y} \to C(X, Y)$$

subject to the following axioms. Here the parameters are X and Y and a morphism $f \in \mathbb{T}^U_{X,Y}$, by abuse of terminology, is said to be *trace class*.

(1) **Naturality** in X and Y: For any $f \in \mathbb{T}^U_{X,Y}$ and $g : X' \to X$ and $h : Y \to Y'$,
$$(h \otimes 1_U) f (g \otimes 1_U) \in \mathbb{T}^U_{X',Y'},$$
and $$Tr^U_{X',Y'}((h \otimes 1_U) f (g \otimes 1_U)) = h\, Tr^U_{X,Y}(f)\, g.$$

(2) **Dinaturality** in U: For any $f : X \otimes U \to Y \otimes U'$, $g : U' \to U$,
$$(1_Y \otimes g) f \in \mathbb{T}^U_{X,Y} \text{ iff } f(1_X \otimes g) \in \mathbb{T}^{U'}_{X,Y},$$
and $$Tr^U_{X,Y}((1_Y \otimes g) f) = Tr^{U'}_{X,Y}(f(1_X \otimes g)).$$

(3) **Vanishing I**: $\mathbb{T}^I_{X,Y} = C(X \otimes I, Y \otimes I)$, and for $f \in \mathbb{T}^I_{X,Y}$
$$Tr^I_{X,Y}(f) = \rho_Y f \rho_X^{-1}.$$
Here $\rho_A : A \otimes I \to A$ is the right unit isomorphism of the monoidal category.

(4) **Vanishing II**: For any $g : X \otimes U \otimes V \to Y \otimes U \otimes V$, if $g \in \mathbb{T}^V_{X \otimes U, Y \otimes U}$, then
$$g \in \mathbb{T}^{U \otimes V}_{X,Y} \text{ iff } Tr^V_{X \otimes U, Y \otimes U}(g) \in \mathbb{T}^U_{X,Y},$$
and $$Tr^{U \otimes V}_{X,Y}(g) = Tr^U_{X,Y}(Tr^V_{X \otimes U, Y \otimes U}(g)).$$

(5) **Superposing**: For any $f \in \mathbb{T}^U_{X,Y}$ and $g : W \to Z$,
$$g \otimes f \in \mathbb{T}^U_{W \otimes X, Z \otimes Y},$$
and $$Tr^U_{W \otimes X, Z \otimes Y}(g \otimes f) = g \otimes Tr^U_{X,Y}(f).$$

(6) **Yanking**: $s_{UU} \in \mathbb{T}^U_{U,U}$, and $Tr^U_{U,U}(s_{U,U}) = 1_U$.

A symmetric monoidal category $(C, \otimes, I, s)$ with such a trace class is called a *partially traced category*, or a *category with a trace class*. If we let X and Y be I (the unit of the tensor), we get a family of operations $Tr^U_{I,I} : \mathbb{T}^U_{I,I} \to C(I, I)$ defining what we call a *non-parametric* (or *scalar-valued*) trace.

In the case when the domain of Tr is the entire hom-set $C(X \otimes U, Y \otimes U)$, the axioms above reduce to the original notion of traced monoidal category in [**JSV96**]. It is also important to notice the extremely subtle "conditional" nature of Vanishing II in the partial case.

In [**HS10**], there are a number of examples of such partial traces, of both "particle" as well as "wave" style (using the terminology of [**Ab96, AHS02**]). One relevant example here,with many variations is the following:

DEFINITION 8.2. consider the symmetric monoidal category $(\mathbf{Hilb_{FD}}, \oplus)$ of finite dimensional complex Hilbert spaces.

We shall say an $f : X \oplus U \to Y \oplus U$ is *trace class* iff $(I - f_{22})$ is invertible, where I is the identity matrix, and I and f_{22} have size $dim(U)$. In that case, we can define

$$Tr^U_{X,Y}(f) = f_{11} + f_{12}(I - f_{22})^{-1} f_{21}$$

9. Weak partial traces

We have observed the subtle 'conditional' nature of Vanishing II in the above axioms for a partial trace. However, it is not uncommon to find examples that satisfy all the axioms except the *existence* conditions of Vanishing II. We first present an axiomatisation of this situation, and then a naturally occurring example closely related to the twisted dagger construction.

DEFINITION 9.1. Let $(C, \otimes, I, s)$ be a symmetric monoidal category. We define a *weak parametric trace class* to be a parametrised family of subsets

$$\mathbb{W}^U_{X,Y} \subseteq C(X \otimes U, Y \otimes U) \text{ for all } U, X, Y \in Ob(\mathcal{C})$$

together with a family of functions, called a *weak (parametric) partial trace*, of the form

$$wTr^U_{X,Y} : \mathbb{W}^U_{X,Y} \to C(X, Y)$$

. These are required to satisfy axioms (1)-(3) and (5)-(6) of Definition 8.1 above, and the following weaker version of Vanishing II:

(3') *Weak vanishing II* Let $g : X \otimes U \otimes V \to Y \otimes U \otimes V$ be an arrow in the intersection of $\mathbb{W}^V_{X\otimes U, Y\otimes U}$ and $\mathbb{W}^{U\otimes V}_{X,Y}$ satisfying

$$Tr^V_{X\otimes U, Y\otimes U}(g) \in \mathbb{W}^U_{X,Y}$$

then

$$wTr^{U\otimes V}_{X,Y}(g) = wTr^U_{X,Y}(wTr^V_{X\otimes U, Y\otimes U}(g))$$

Note that any partial trace is trivially a weak partial trace. Also, when the domain of $wTR^U_{X,Y}$ is the entire homset $\mathcal{C}(X \otimes U, Y \otimes U)$, for all X, Y, U, then wTr is exactly a categorical trace in the sense of [**JSV96**].

We may find weak partial trace classes as subsets of partial trace classes, as the following example in the category of finite-dimensional Hilbert spaces demonstrates:

DEFINITION 9.2. An arrow

$$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{Hilb_{FD}}(X \oplus U, Y \oplus U)$$

is *(strictly) lower-right contractive (LRC)* when $|d| < 1$ (using the operator norm). Let us use the notation:

$$F \in \ LRC^U_{X,Y} \subseteq \mathbf{Hilb_{FD}}(X \oplus U, Y \oplus U)$$

REMARK 9.3. Note that the LRC condition is exactly that required by Theorem 5.6, to ensure that the twisted dagger of a unitary map is itself unitary. For a physical interpretation, recall the generalised interferometry experiment of Remark 5.2. in this setting, the LRC condition may be interpreted as stating that, when prepared with a particle in state $|\psi\rangle$ that is with probability 1 within the feedback loop of Figure 5, the probability $\|d^T\,|\psi\rangle\,\|^2$ of observing the particle within this loop at some later time T tends to zero, as T increases.

We now demonstrate that the LRC arrows define a weak partial trace class.

PROPOSITION 9.4. *In $\mathbf{Hilb_{FD}}$, the parametric family of arrows*

$$LRC^U_{X,Y} \subseteq \mathbf{Hilb_{FD}}(X \oplus U, Y \oplus U)$$

defined above, together with the parametric function $wTr^U_{X,Y} : LRC(X \oplus U, Y \oplus U) \to \mathbf{Hilb_{FD}}(X, Y)$ given by

$$wTr^U_{X,Y} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ = \ a + \sum_{j=0}^{\infty} bd^j c$$

specifies a weak partial trace.

PROOF. First recall that, in any Banach algebra B, for any element $x \in B$ satisfying $lim_{n\rightarrow\infty}x^n \ = \ 0$, the element $(1 - x)$ is invertible, with inverse given by $(1 - x)^{-1} \ = \sum_{j=0}^{\infty} x^j$. Thus, in *finite-dimensional* Hilbert spaces, for any strictly contractive map $|d| < 1$, the element $I - d$ is invertible, and $(I - d)^{-1} = \sum_{j=0}^{\infty} d^j$.

From this, we deduce that $LRC^U_{X,Y} \subset \mathbb{T}^U_{X,Y}$, for all $X, Y, U \in Ob(\mathbf{Hilb_{FD}})$ (this inclusion is strict, since $(I_U - \alpha I_U)$ is invertible, for arbitrary $\alpha > 1 \in \mathbb{C}$), and for all $f \in LRC^U_{X,Y}$,

$$Tr^U_{X,Y}(f) = wTr^U_{X,Y}(f) = a + b(I_U - d)^{-1}c = a + b\left(\sum_{j=0}^{\infty} d^j\right) c \ = \ a + \sum_{j=0}^{\infty} bd^j c$$

It is then almost immediate that axioms (1)-(3) and (5)-(6) for a partial categorical trace are satisfied, and the fact that wTR and Tr coincide when both are defined is enough to establish the equality required for the weak Vanishing II axiom of Definition 9.1.

To see that LRC, wTr does *not* define a partial trace in the sense of Definition 8.1, consider the 3×3 complex matrix

$$M = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & \frac{1}{2} \end{pmatrix} : X \oplus U \oplus V \to Y \oplus U \oplus V$$

where $X \cong Y \cong U \cong V \cong \mathbb{C}$.

Then $M \in LRC^V_{X\oplus U,Y\oplus U}$, and

$$wTr^V_{X\oplus U,Y\oplus U}(M) = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \in LRC^U_{X,Y}$$

Thus the hypothesis for the ($\Leftarrow$) implication of Vanishing II is satisfied. However,

$$\begin{pmatrix} -2 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} : U \oplus V \to U \oplus V$$

is clearly not strictly contractive, and thus M is *not* a member of $LRC^{U\oplus V}_{X,Y}$. (The above counterexample was motivated by a similar calculation taken from [**MSS11**]).

□

9.1. A weak partial trace ($\mathbf{Hilb_{FD}}[\mathbf{z}], \oplus$). We now use the result of Proposition 9.4 to give a weak partial trace on $\mathbf{Hilb_{FD}}[\mathbf{z}]$ satisfying the axioms of Definition 9.1 above.

DEFINITION 9.5. For each $X, Y, U \in Ob(\mathbf{Hilb_{FD}}[\mathbf{z}])$, we define

$$\mathbb{W}^U_{X,Y} \subseteq \mathbf{Hilb_{FD}}[\mathbf{z}](X \oplus U, Y \oplus U)$$

as follows: An arrow $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{Hilb_{FD}}[\mathbf{z}](X \oplus U, Y \oplus U)$ is in $\mathbb{W}^U_{X,Y}$ when the following condition is satisfied:

- For all norm-1 vectors $|\psi\rangle \in U$,

$$\sum_{j=0}^{\infty} \| d^j |\psi\rangle \| < 1$$

Note that the above condition implies both that d_j is strictly contractive, for all $j \in \mathbb{N}$, and that $\sum_{j=0}^{\infty} d^j$ exists.

Given $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{W}^U_{X,Y}$, we define

$$wTr^U_{X,Y}(f) = a + \sum_{j=0}^{\infty} bd^j c$$

(The existence of this sum is a straightforward corollary of Proposition 9.4 above).

REMARK 9.6. Note that the condition of Definition 9.5 above implies, but it is not implied by, the condition we might assume from a physical motivation – that $\sum_{j=0}^{\infty} \| d^j |\psi\rangle \|^2 \leq 1$ for all norm-1 vectors $|\psi\rangle$. Thus, in using this as the definition of our weak trace class, we are not simply requiring that arrows be amenable to a physical interpretation.

THEOREM 9.7. *Definition 9.5 above specifies a weak partial trace on* $\mathbf{Hilb_{FD}}[\mathbf{z}]$*, as defined in Definition 9.1.*

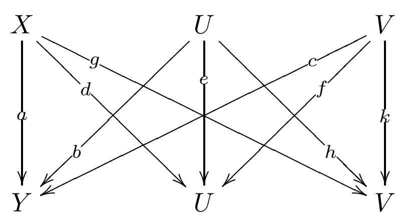
PROOF. By direct calculation, it is relatively straightforward to verify axioms (1)-(3) and (5)-(6) of Definition 8.1, with reference to Propositions 7.9 and 9.4. As before, the only non-trivial point is the weak Vanishing II axiom of Definition 9.1.

Consider some $P \in \mathbf{Hilb}[\mathbf{z}](X \oplus U \oplus V, Y \oplus U \oplus V)$, given explicitly in matrix form as

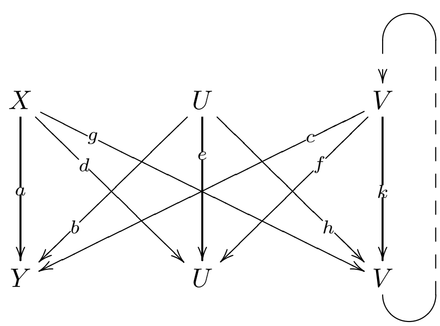
$$P = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} : X \oplus U \oplus V \to Y \oplus U \oplus V$$

and assume further that P is a member of both $\mathbb{W}^V_{X\oplus U, Y\oplus U}$ and $\mathbb{W}^{U\oplus V}_{X,Y}$. Thus, both $\sum_{j=0}^{\infty} k^j$ and $\sum_{j=0}^{\infty} \begin{pmatrix} e & f \\ h & k \end{pmatrix}$ exist.

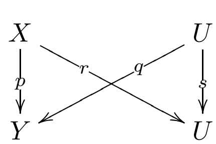
We may give an explicit formula for $wTr^V(P)$; however, it is more instructive to give P itself as the following digraph:



and observe that the matrix of $wTr(P)$ may be found by summing over all paths in the following diagram.



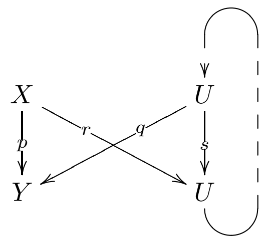
giving $wTr^V_{X\oplus U,Y\oplus U}(P) = \begin{pmatrix} a + \sum_{j=0}^{\infty} ck^j g & b + \sum_{j=0}^{\infty} ck^j h \\ p + \sum_{j=0}^{\infty} fk^j g & e + \sum_{j=0}^{\infty} fk^j h \end{pmatrix}$. Using similar graphical notation, we draw this as



where $\begin{cases} p = a + \sum_{j=0}^{\infty} ck^j g & q = b + \sum_{j=0}^{\infty} hk^j c \\ r = d + \sum_{j=0}^{\infty} gk^j f & s = c + \sum_{j=0}^{\infty} hk^j f \end{cases}$

We will now make the assumption (for the *weak* version of vanishing II) that $wTr^V_{X\oplus U,Y\oplus U}(P) \in \mathbb{W}^U_{X,Y}$, and give the following diagrammatic representation for

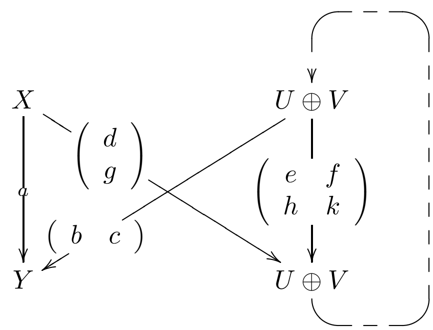
$wTr^U_{X,Y}\left(wTr^V_{X\oplus U,Y\oplus U}(P)\right)$



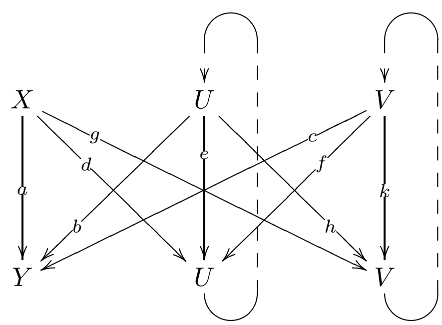
where p,q,r,s are as above.

It is straightforward, although unenlightening, to derive an explicit formula for $wTr^U_{X,Y}\left(wTr^V_{X\oplus U,Y\oplus U}(P)\right)$ from this diagram.

We now use the same formalism to calculate $Tr^{U\oplus V}_{X,Y}(P)$. Let us draw $P : X\oplus(U\oplus V)\to Y\oplus(U\oplus V)$ (together with the appropriate feedback loop) as



Let first us replace all matrix-labelled arrows by the appropriate digraphs, giving



We may then sum over all paths from X to Y, giving an explicit formula for $Tr^{U\oplus V}_{X,Y}(P)$. It may be verified – either by diagrammatic manipulations, or converting these into explicit calculations, that under the *existence* conditions imposed by the weak version of Vanishing II, that $Tr^U_{X,Y}(Tr^V_{X\oplus U,Y\oplus U}(P)) = Tr^{U\oplus V}_{X,Y}(P)$. $\square$

Note that this weak trace of formal power series has a useful interpretation as the weak trace of Proposition 9.4, where components are 'split up' according to the number of iterative cycles they require.

9.2. Relating the trace and the thought-experiment. We are now able to relate the weak categorical trace on $\mathbf{Hilb_{FD}}[\mathbf{z}]$ to the thought-experiment of Section 4 and its generalisation given in Section 5. Consider a unitary map

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : X \oplus U \to Y \oplus U$$

where d is strictly contractive. Using the interpretation of categories of formal power series given in Remark 7.6, we assume that a single application of this unitary map takes a single timestep. Thus, we consider the formal power series

$$p = 0 + L.z + 0.z^2 + 0.z^3 + \dots$$

We observe that $p \in \mathbf{Hilb_{FD}}[\mathbf{z}](X \oplus U, Y \oplus U) \in \mathbb{W}^U_{X,Y}$, and by definition

$$wTr^U_{X,Y}(p) = 0.z^0 + a.z^1 + bc.z^2 + bdc.z^3 + bd^2c.z^4 + \dots$$

Now compare this to the formal matrices of Lemma 6.2, and the embedding of Theorem 7.5. We observe that the input-output behaviour of the general form of our thought-experiment is given by the weak trace on $\mathbf{Hilb_{FD}}[\mathbf{z}]$. However, the trace has an additional leading zero (the component of z^0). This fits in well with the intuition of formal power series as 'timing' iteration. A non-zero coefficient of z^0 interprets as a computation (or physical process) that takes no time at all. Due to the labelling conventions of Section 4, this is not immediately apparent from a straightforward analysis, but drops naturally out of the categorical description.

10. Conclusions and applications

It is perhaps unsurprising that the theory of conditional iteration, whether in the quantum or the classical setting, should be related to the theory of categorical traces. What is more unexpected is that not only is this apparent from minor modifications to a 100 year old experiment, but that this gives a previously unobserved decomposition of the usual particle-style categorical trace. Moreover, we still find it slightly mysterious that there should be an apparent connection with the Elgot dagger.

In terms of applications, quantum circuits that implement the operations of Lemma 5.1, and thus provide finitary approximations to the twisted dagger in the quantum circuit model, are used heavily in [**Hi09**]. Given the original motivation for the twisted dagger in terms of a simple linear optics experiment, and the encoding of the standard circuit model into this optical framework presented in Appendix A, it is not unreasonable to suppose that the (rather complicated) circuits of [**Hi09**] have a simple, almost trivial, realisation in terms of similar experiments.

From a more mathematical point of view, a great deal of theory remains to be developed. The construction of a category of formal power series from $\mathbf{Hilb_{FD}}$ is clearly a special case of a general categorical construction. This is developed further, in a much more general framework, in [**Hi10**]. However, although the analogous construction in [**Hi10**] is shown to be functorial, its interaction with monoidal tensors is not considered. Many other questions remain open. The general theory of partially traced categories considered above (from [**HS10**]) is developed in detail in the thesis of Octavio Malherbe [**Mal10**] with an eye towards models of quantum programming languages. Such partial traces are completely characterized in the paper [**MSS11**].

Finally, this paper has been phrased very concretely in terms of physical experiments. Similarly, the applications proposed above are very concrete constructions involving quantum circuits. Despite this, we should not forget the motivation from the Geometry of Interaction program in linear logic [**HS10, HS10a**], which attempts to model the invariants for the dynamics of normalization (rewriting) of formal proofs.

References

[Ab96] S. Abramsky 1996, Retracing some paths in Process algebra, In *CONCUR 96, Lecture Notes in Computer Science*, pp.1-17, Springer-Verlag.

[ABP98] S. Abramsky, R. Blute, P. Panangaden 1998, Nuclear and Trace Ideals in Tensor ∗-Categories, *Journal of Pure and Applied Algebra*, Special edition in honour of M. Barr.

[AHS02] S. Abramsky, E. Haghverdi, P. Scott 2002 Geometry of interaction and linear combinatory algebras, *Math. Struct. Comput. Sci. 12 (5)* 625-665

[Ab05] S. Abramsky 2005, Abstract Scalars, Loops, and Free Traced and Strongly Compact Closed Categories, in *Proceedings of CALCO 2005, Springer LNCS 3629*, 1–31

[AbCo05] S. Abramsky, B. Coecke 2005, A categorical semantics of quantum protocols. In *Proc. 19th Annual IEEE Symp. on Logic in Computer Science (LICS 2004)*, IEEE Computer Soc. Press, 415-425.

[AnCo04] E. Andruchow, G. Corach 2004, Differential geometry of partial isometries and partial unitaries, *Illinois J. Math. 48(1)*, 97-120

[AnCo05] E. Andruchow, G. Corach 2005, Metrics in the set of partial isometries with finite rank, *Rend. Mat. Acc Lincei s 9, v.16* (31-44)

[Ba92] M. Barr 1992, Algebraically Compact Functors *Journal of Pure and Applied Algebra 82* Elsevier (1992) 211-231

[Be05] A. Beige 2005 Quantum Optics, CONQUEST research network Lectures, available as *http://www.quniverse.sk/conquest/lectures.php*

[Br02] K. Brown, Reflections on Relativity, *Mathpages.com online texts* (2002). Available as: *http://www.mathpages.com/rr/refrel2.htm*

[CAK05] N. Cerf, C. Adami, P. Kwiat 1998, Optical Simulation of quantum logic, *Physical Review A 57(3)*

[FLS65] R. Feynman, R. Leighton, M. Sands 1965, *The Feynman lectures on Physics, Vol. 3*, Addison-Wesley, Reading, Mass.

[GK05] C. Gerry, P. Knight 2005 *Introductory quantum optics*, Cambridge University Press.

[Gr02] R. Griffiths 2002 *Consistent Quantum Theory*, Cambridge University Press

[Ha00] E. Haghverdi 2000, A categorical approach to linear logic, geometry of proofs and full completeness, *PhD Thesis, Univ. Ottawa*

[HS04] Haghverdi, E. and Scott, P.J. A categorical model for the Geometry of Interaction, *Theoretical Computer Science* Volume 350, Issues 2-3 , Feb 2006, pp. 252-274.

[HS10] E. Haghverdi, P. Scott 2010 Towards a Typed Geometry of Interaction, *Mathematical Structures in Comp. Science*, Vol. 20, Camb. U. Press, pp. 473-521. (preliminary version in: CSL2005 (Computer Science Logic), Luke Ong, Ed. SLNCS 3634, pp. 216-231.)

[HS10a] Haghverdi, E. and Scott, P.J. 2010. Geometry of Interaction and the Dynamics of Proof Reduction: a tutorial, in: *New Structures for Physics*, B. Coecke (ed), Lecture Notes in Physics, Vol. 813, Springer-Verlag, pp. 339–397.

[Hal58] P.R. Halmos 1958, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, 2^{nd} Ed. Chelsea Publishing, New York

[Har83] D. G. Hartig 1983, The Reisz Representation Theorem Revisited, *American Mathematical Monthly (90)4* 277-280

[Hi98] P. Hines 1998, The algebra of self-similarity and its applications, *PhD Thesis , University of Wales Bangor*

[Hi09] P. Hines 2010, Quantum circuit oracles for abstract machine computations *Theoretical Computer Science 411* (2010) 1501-1520

[Hi10] P. Hines, A categorical analogue of monoid semirings *(under revision)* *Available as http://www.peterhines.net/downloads/papers/Cauchy.pdf*

[Je97] A. Jeffrey (1997), Premonoidal categories and a graphical view of programs. *Electr. Notes Theor. Comput. Sci. 10*

[JSV96] A. Joyal, R. Street, D. Verity 1996 Traced Monoidal categories, *Math. Proc. Camb. Phil. Soc.* 425-446

[McL98] S. MacLane 1998, *Categories for the working mathematician*, Graduate texts in mathematics 5, 2^{nd} ed. Springer-Verlag, New York

[Mal10] O. Malherbe 2010, *Phd thesis*, Dept. of Mathematics, U. Ottawa. *http://www.fing.edu.uy/∼malherbe/octavio/personal.html*

[MSS11] O. Malherbe, P. Scott, P. Selinger 2011, Partially Traced Categories, to appear, *J. Pure and Applied Algebra*.

[MA86] E. Manes, M. Arbib 1986, *Algebraic Approaches to Program Sematics* Springer-Verlag

[NC00] M. Nielsen, I. Chuang 2000 *Quantum Computation and Quantum Information*, Cambridge University Press

[Pen04] R. Penrose 2004 *The Road To Reality: A complete guide to the laws of the universe* Random House Ltd., London.

[PSM96] T. B. Pittman, D. V. Strekalov, A. Migdall, M. H. Rubin, A. V. Sergienko, Y. H. Shih 1996 Can Two-Photon Interference be Considered the Interference of Two Photons?, *Physical Review Letters 77 (10)*, 1917-1920

[Sa13i] G. Sagnac, *L'éther lumineux dèmontré par éffet du vent relatif d'éther dans un interféromètre en rotation uniforme*, Comptes Rendus 157 (1913) 708-710

[Sa13ii] G. Sagnac *Sur la preuve de la réalité de l'éther lumineux paer l'experience de l'interférographe tournant* Comptes Rendus 157 (1913), S. 1410-1413

[St97] G.E. Stedman *Ring-laser tests of fundamental physics and geophysics* Rep. Prog. Phys. 60 (1997) 615688

[Ti83] E. C.Titchmarsh (1983) *The theory of Functions,* 2^{nd} *Edition*, Oxford University Press

[Lo93] O. Lodge *Abberation Problems* Philosophical Transactions of the Royal Society (London) 184A 727 (1893)

[JvN38] J. von Neumann *On infinite direct products Compos. Math. 6: 177* (1938)

Appendix A: linear optics devices as quantum logic gates

This appendix is expository, and details how the optics toolkit presented in Section 2 may be used to implement a universal set of quantum logic gates[7]*. For a fuller treatment, and applications (including linear optics circuits for teleportation), we refer to* [**CAK05**]).

Recall that — for single-photon experiments — the basic optics devices in Section 2 are modelled by unitary operations on a 4-dimensional Hilbert space. By considering this 4-dimensional space as the tensor product of two 2-dimensional spaces, we may give a treatment in terms of qubits, and the standard quantum computational logic gates.

The quantum information is encoded on:

(1) The choice of channel.
(2) The photon polarisation.

Thus, we consider the first qubit to be encoded on the choice of channel, with channel 1. (resp. channel 2.) corresponding to $|0\rangle$ (resp. $|1\rangle$). Similarly, the second qubit is encoded on the photon polarisation, with horizontal (resp. vertical) polarisation corresponding to $|0\rangle$ (resp. $|1\rangle$).

This then gives a straightforward encoding of 2-qubit states, e.g.

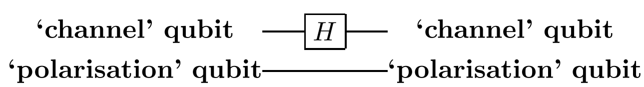
- The pure state $|0\rangle\,|1\rangle$ corresponds to a vertically polarised photon in channel 1.

[7]We emphasise, as noted in Section 1.2, this encoding is *not* efficiently scalable.

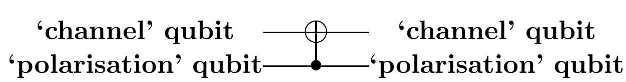
- The superposition $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle$ corresponds to a horizontally polarised photon in a superposition of channels 1. and 2.
- The entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ corresponds to a photon that is in a superposition of horizontal polarisation in channel 1. and vertical polarisation in channel 2. This is the highly important *Bell state*, used in both quantum teleportation and cryptography [**NC00**].

This encoding also gives a neat realisation of the standard quantum logic gates in terms of linear optics devices, as in the following examples:

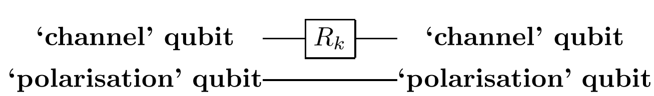
- **The Hadamard gate**
  The action of the beamsplitter is simply to apply a Hadamard gate $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ to the first qubit (encoded on the choice of channel) and leave the second one (encoded on the polarisation) alone. In the standard quantum circuit formalism, this is drawn as:



- **The controlled-not gate**
  The polarised beamsplitter applies a NOT gate $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to the channel qubit when the polarisation qubit is $|1\rangle$, and leaves the channel qubit unchanged otherwise. This is the controlled-not, or CNOT gate, drawn in the standard circuit formalism as

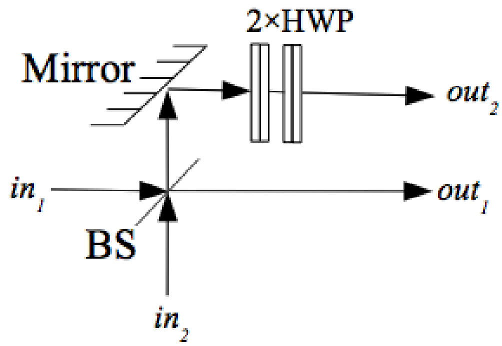


- **The phase shift gate**
  By placing a phase plate in the second channel only of such an experiment, a phase shift gate $R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix}$ may be implemented. Note also that this has no effect on the polarisation. In the standard circuit formalism, this is drawn as:



Using this encoding, it is easy to verify that the Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ may be produced by introducing a horizontally polarised photon into channel 1. of the apparatus shown in Figure 8.

FIGURE 8. Linear-optics apparatus to produce the Bell state



Once the Bell state has been produced, it is a short step to an implementation of quantum teleportation. The theoretical details of how to implement teleportation using linear optics are given in [**CAK05**].

COMPUTER SCIENCE, UNIVERSITY OF YORK
Current address: Department of Computer Science, University of York, York, U.K.
E-mail address: Peter.Hines@cs.york.ac.uk

DEPARTMENT OF MATHEMATICS & STATISTICS, UNIVERSITY OF OTTAWA, CANADA
E-mail address: phil@site.uottawa.ca