Tutorial on Geometry of Interaction

Philip Scott (Ottawa)

Reporting on recent work with E. Haghverdi
Traditional (model theory & categorical logic)

\[ [-] : \quad \text{Logic} \quad \rightarrow \quad \text{Model} \]

formulas \quad \mapsto \quad \text{objects}

Proofs \quad \mapsto \quad \text{arrows (functions)}

\[ A \vdash B \quad \Rightarrow \quad [A] \xrightarrow{\pi} [B] \]

More generally:

\[ A_1, \ldots, A_m \vdash B_1, \ldots, B_n \quad \Rightarrow \quad \bigotimes_{i} [A_i] \xrightarrow{\rho} \bigcirc_{j} [B_j] \]

Cut-Elimination: \quad \text{denotations are equal!}

(rewriting) \quad \text{(no dynamics)}

\[ \pi_1 \triangleright \pi_2 \quad \Rightarrow \quad [\pi_1] = [\pi_2] : [A] \rightarrow [B] \]

Girard’s GoI Program (GoI 1-GoI 3 (1989-1995), GoI 4 (2004)) aims to mathematically model the dynamics of cut-elimination via operator algebras. One Goal: dynamical invariants.
Recall Gentzen’s Cut-Rule

\[
\frac{\Gamma \vdash \Delta, A \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{ Cut}
\]

Gentzen proved the following theorem (which applies to many systems of logic):

**Cut-elimination (Gentzen’s Haupsatz, 1934):**
If \( \Pi \) is a proof of \( \Gamma \vdash \Delta \), then there is a proof \( \Pi' \) of \( \Gamma \vdash \Delta \) which does not use the cut-rule.

For usual sequent calculus, Gentzen gave an ND algorithm \( \Pi \leadsto \Pi' \) (the cut-elimination procedure)

\[
\frac{\Gamma \vdash B, B}{\Gamma \vdash B} \quad \frac{B \vdash \Delta}{\Gamma \vdash \Delta} \text{ Cut}
\]

reduces to (w.r.t. appropriate measure)

\[
\frac{\Gamma \vdash B, B \quad B \vdash \Delta}{\Gamma \vdash B, \Delta} \quad \frac{\Gamma \vdash \Delta, \Delta}{\Gamma \vdash \Delta} \text{ Cut}
\]

\[
\frac{\Gamma \vdash \Delta, \Delta}{\Gamma \vdash \Delta} \text{ Cut}
\]
In Girard's papers:

Proofs $\Rightarrow$ matrix operators on a $C^*$-algebra $B(\mathcal{H})$

Idea of Girard's work (Details Later!) for MELL:

\[ \pi : \vdash [\Delta], \Gamma \quad \text{where } \Delta \text{ is a list of Cut formulas (e.g. } (A, A^\perp, B, B^\perp, C, C^\perp, \ldots) \text{) } \quad |\Delta| = 2^m \quad |\Gamma| = n \]

**Dynamic View:** A Proof = I/O box (with feedback) in a graphical network.

\[ (*) \]

\[ \begin{array}{c}
\Rightarrow \\
\Delta \\
\Rightarrow \\
\Gamma \rightarrow \Gamma
\end{array} \quad \begin{array}{c}
\Rightarrow \\
\Delta \\
\Rightarrow \\
\Gamma \rightarrow \Gamma
\end{array} \quad \begin{array}{c}
\Rightarrow \\
\Delta \\
\Rightarrow \\
\Gamma \rightarrow \Gamma
\end{array} \]

\[ \begin{array}{c}
\Rightarrow \\
\Delta \\
\Rightarrow \\
\Gamma \rightarrow \Gamma
\end{array} \quad \begin{array}{c}
\Rightarrow \\
\Delta \\
\Rightarrow \\
\Gamma \rightarrow \Gamma
\end{array} \quad \begin{array}{c}
\Rightarrow \\
\Delta \\
\Rightarrow \\
\Gamma \rightarrow \Gamma
\end{array} \]

\[ \begin{array}{c}
\Rightarrow \\
\Delta \\
\Rightarrow \\
\Gamma \rightarrow \Gamma
\end{array} \quad \begin{array}{c}
\Rightarrow \\
\Delta \\
\Rightarrow \\
\Gamma \rightarrow \Gamma
\end{array} \quad \begin{array}{c}
\Rightarrow \\
\Delta \\
\Rightarrow \\
\Gamma \rightarrow \Gamma
\end{array} \]

**Models** Concrete $\otimes$-categories $\mathcal{C}$ with distinguished "reflexive" object $U \in \mathcal{C}$, with additional structure.
Proofs are modeled as follows:

$$\pi : \vdash [\Delta], \Gamma \rightarrow ([\pi_1], \sigma)$$

where $[\pi_1] : U^{n+2m} \rightarrow U^{n+2m}$, and $U^{2m} \xrightarrow{\sigma} U^{2m}$ represents the cuts $\Delta$, where $|\Delta| = 2m$ and $|\Gamma| = n$. (Here $U^k = U \otimes \cdots \otimes U$ $k$-times).

If $\Delta = \emptyset$, $\pi$ is cut-free and $\sigma = 0$ will be a zero map ($\mathcal{C}$ is a semi-additive $\otimes$-category).

Write $[\pi_1]$ as a block matrix:

$$[\pi_1] = \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix}$$

A version of feedback/trace

(The Execution Formula)

$$Ex([\pi_1], \sigma) = \pi_{11} + \sum_{n \geq 0} \pi_{12}(\sigma\pi_{22})^n(\sigma\pi_{21})$$

Here we can think of $Ex([\pi_1], \sigma) : U^n \rightarrow U^n$. 
In Girard's Hilbert-space models, the Execution Formula has a special form (described later).

**Theorem (Girard):** (MELL and System \( \mathcal{F} \))

- \( \text{Ex}(\llbracket \pi \rrbracket, \sigma) \) is a finite sum.
- \( \text{Ex}(\llbracket \pi \rrbracket, \sigma) \) is an invariant of cut-elimination. (Under certain restrictions on types for MELL)
- If \( \pi' \) is a cut-free normal form of \( \pi \), then
  \[
  \text{Ex}(\llbracket \pi \rrbracket, \sigma) = \text{Ex}(\llbracket \pi' \rrbracket, 0) = \| \pi' \|
  \]
- Above suggests new idea: GoI computing (à la Curry-Howard). (Recently studied in Complexity Theory, by Harry Mairson).

Girard also introduced fundamental operator algebra encodings which we need to categorize for:

- Types and Orthogonalities (cf. Hyland-Schalk)
- Algorithms
- Data
Later Works on GoI:

**Girard-Style:**
- Danos (1990)
- Danos-Regnier (1992–96)
- Malacaria-Regnier (1991)

**GoI-style normalization & Complexity:**
- Abadi, Gonthier, Levy (1992): Optimal Reduction (Lamping)
- Girard-Scedrov-Scott (1992): Bounded LL
- H. Mairson (2002–): GoI & Complexity Classes

**Categorical Frameworks:**
- Abramsky (1997): Siena Lectures
- AHS (2002): GoI & LCA’s
- Lenisa-Honsell: \( \lambda \)-Calc. & "wave-style" GoI
- H-S (2004): 2 papers on UDC-based GoI
Basic Algebraic Framework

- GoI Situations (Abramsky '97, AHS'02)
  - Traced Monoidal Category $C$
  - Endofunctor $T : C \to C$ with monoidal retracts: $TT \triangleleft T$, $Id \triangleleft T$, $T \otimes T \triangleleft T$, $K_f \triangleleft T$
  - Reflexive object $U \in C$ with retractions $U \otimes U \triangleleft U$, $I \triangleleft U$, $TU \triangleleft U$

GoI Situations isolate basic algebraic structure of GoI. We obtain Linear Combinatory Algebras on $C(U, U)$ “modelling” full LL.

Variants of GoI

- $\otimes = +$ (Sum or “particle”-style)
- $\otimes = \times$ (Product or “wave”-style)

Theorem: $\ell_2[\text{PInj}]$ is exactly Girard’s GoI 1.
Unique Decomposition Categories (UDC’s)

- Symmetric $\otimes$-Category $\mathcal{C}$

- Axioms saying: homsets have infinitary partially-additive-monoid operation $\sum_{i \in I} f_i$ for countable families (compatible with composition). In particular zero morphisms $0_{XY} \in \mathcal{C}(X,Y)$.

- Finite tensors are quasi biproducts: there are quasi-injections $\iota_j : X_j \rightarrow \bigotimes_i X_i$ and quasi-projections $\rho_j : \bigotimes_i X_i \rightarrow X_j$ satisfying:

  1. $\rho_k \iota_j = \begin{cases} 1_{X_j} & \text{if } j = k \\ 0_{X_j X_k} & \text{else} \end{cases}$

  2. $\sum_{i \in I} \iota_i \rho_i = 1_{\bigotimes_i X_i}$

Arrows as Matrices: Given $f : \bigotimes_j X_j \rightarrow \bigotimes_j Y_i$ in a UDC with $|I| = m$ and $|J| = n$, there exists a unique family $\{f_{ij}\}_{i \in I, j \in J} : X_j \rightarrow Y_i$ with $f = \sum_{i \in I, j \in J} \iota_i f_{ij} \rho_j$, namely, $f_{ij} = \rho_i f_{ij}$. We write $f$ as a matrix $f = [f_{ij}]$.

Composition in UDC’s = matrix multiplication.
Symmetric monoidal categories \((C, I, \otimes, \alpha)\), equipped with a family of functions called a trace \(Tr^U_{X,Y} : C(X \otimes U, Y \otimes U) \to C(X, Y)\) subject to some axioms. In our models, think of \(Tr^U_{X,Y}(f)\) given by “feedback”.

\[ (*) \]

\[
\begin{array}{cccc}
X & \rightarrow & \text{Tr}(f) & \rightarrow & Y \\
U & \downarrow & & \downarrow & \rightarrow U
\end{array}
\]

**Natural in** \(X\), \(Tr^U_{X,Y}(f)g = Tr^U_{X',Y}(f(g \otimes 1_U))\) where \(f : X \otimes U \to Y \otimes U\), \(g : X' \to X\),

**Natural in** \(Y\), \(gTr^U_{X,Y}(f) = Tr^U_{X,Y}((g \otimes 1_U)f)\) where \(f : X \otimes U \to Y \otimes U\), \(g : Y \to Y'\),

**Dinatural in** \(U\), \(Tr^U_{X,Y}(1_Y \otimes g)f = Tr^U_{X,Y}(f(1_X \otimes g))\) where \(f : X \otimes U \to Y \otimes U'\), \(g : U' \to U\).

**Vanishing (1.2)**.
\(Tr^I_{X,Y}(f) = f\) and \(Tr^U_{X,Y}(g) = Tr^U_{X,Y}(Tr^Y_{X \otimes U, Y \otimes U}(g))\) (cf. Bekić)
Superposing,

\[ \text{Tr}^U_{X,Y}(f) \otimes g = \text{Tr}^U_{X \otimes W,Y \otimes Z}((1_Y \otimes \sigma_{U,Z})(f \otimes g)(1_X \otimes \sigma_{W,U})) \]

for \( f : X \otimes U \to Y \otimes U \) and \( g : W \to Z \),

Yanking \( \text{Tr}^U_{U,U}(\sigma_{U,U}) = 1_U \).

There is a general geometric calculus for reasoning about such TMC's...

Some Examples of TMC's

- \( \text{Rel}_\times, \, \text{Vec}_{cd} \), more generally any compact category (where \( \otimes \cong \phi \)) has a canonical trace

- \( \omega\text{-CPO}_\bot \) where trace given by \( Y \) combinator

- Unique Decomposition Categories

  (Iterative Traces:) \( \text{Rel}_\bot, \, \text{SRel}, \, \text{Pfn}, \, \text{PInj} \),

  and in general all partially additive categories (Manes-Arbib), etc.

- (Selinger): Quantum TMC's
Figure 4: Dinaturality in $U$

Figure 5: Naturality in $X$

Figure 6: Naturality in $Y$

Figure 7: Vanishing

Figure 8: Vanishing II, $U \odot V$ denotes the simultaneous feedback on the lines $U$ and $V$

Figure 9: Yanking
Traced UDC's: Let $g$ be a UDC. If for every $f: X \otimes U \to Y \otimes U$ the sum

$$f_{11} + \sum_{n=0}^{\infty} f_{12} f_{22}^{-1} f_{21}$$

exists

then

1. $g$ is traced
2. $\text{Tr}_{X,Y}^U(f) = f_{11} + \sum_{n=0}^{\infty} f_{12} f_{22}^{-1} f_{21}$

E.g. $X \otimes U \to Y \otimes U = [g \circ_0 h]$ \n
$\text{Tr}_{X,Y}^U(f) = \text{Tr} [g \circ_0 h]$ \n
$= g + \sum_n o h^n o = g + o = g$
E.g.'s: Traced UDC's

- PAC's:
  \[ \text{Rel}_+ : \sum_i R_i = \cup_i R_i \]
  \[ \text{Pf}_\equiv : \{ f_i \} \text{ summable} \]
  \[ \iff \text{ pairwise disjoint domains} \]
  \[ (\sum_i f_i)(x) = f_j(x) \quad x \in \text{Dom}(f_j) \]
  \[ = 1 \quad \text{else} \]

  SRel

- PInj: as for Pf_\equiv but pairwise disjoint domains & codomains.
Building a UDC from Hilbert Spaces

Let $\text{Hilb}$ be the category of Hilbert spaces and linear contractions (norm $\leq 1$). M. Barr defined a contravariant faithful functor $\ell_2: \text{Pinj}^{\text{op}} \to \text{Hilb}$ as follows:

**On Objects:** $X \mapsto \ell_2(X) =$ set of all complex valued functions $a$ on $X$ for which the (unordered) sum $\sum_{x \in X} |a(x)|^2$ is finite.

$\ell_2(X)$ is a Hilbert space with

- $\|a\| = (\sum_{x \in X} |a(x)|^2)^{1/2}$
- Inner product $< a, b > = \sum_{x \in X} a(x)\overline{b(x)}$ for $a, b \in \ell_2(X)$.

**On Maps** Given $f: X \to Y$ in $\text{Pinj}$,

$\ell_2(f): \ell_2(Y) \to \ell_2(X)$ is defined by

$$\ell_2(f)(b)(x) = \begin{cases} b(f(x)), & \text{if } x \in \text{Dom}(f); \\ 0, & \text{otherwise}. \end{cases}$$

So we get a correspondence $\text{partial inj. functions} \leftrightarrow \text{partial isometries in Hilb}$. 
Various isomorphisms:

\[ \ell_2(X \cup Y) \cong \ell_2(X) \oplus \ell_2(Y) \]

\[ \ell_2(X \times Y) \cong \ell_2(X) \otimes \ell_2(Y) \]

Define \( \mathbf{Hilb}_2 = \ell_2[\mathbf{PI} \mathbf{n} \mathbf{j}] \). More precisely:

define the subcategory \( \mathbf{Hilb}_2 \) of \( \mathbf{Hilb} \):

**Objects** \( \ell_2(X) \) for a set \( X \)

**Morphisms** \( u : \ell_2(X) \to \ell_2(Y) \) of the form \( \ell_2(f) \)
for some \( f : Y \to X \in \mathbf{P} \mathbf{I} \mathbf{n} \mathbf{j} \).

There are two \( \otimes \)-structures on \( \mathbf{Hilb}_2 \) induced from tensors on \( \mathbf{P} \mathbf{I} \mathbf{n} \mathbf{j} \)

\[ \ell_2(X) \otimes \ell_2(Y) \cong \ell_2(X \times Y) \]

\[ \ell_2(X) \oplus \ell_2(Y) \cong \ell_2(X \cup Y) \quad (\leftarrow \text{UDC tensor}) \]

Note: \( \ell_2(X) \oplus \ell_2(Y) \) is direct sum (biproduct) in \( \mathbf{Hilb} \) but only a tensor product in \( \mathbf{Hilb}_2 \), (otherwise \( X \cup Y \) would be coproduct in \( \mathbf{P} \mathbf{I} \mathbf{n} \mathbf{j} \), a contradiction.)
\textbf{Hilb}_2 is a traced UDC (with UDC structure induced from \textbf{Pinj})

\(\oplus\) is the tensor product, with unit \(\ell_2(\emptyset)\).

Consider a family \(\{\ell_2(f_i)\}_I \in \textbf{Hilb}_2(\ell_2(X), \ell_2(Y))\) with \(\{f_i\}_I \in \textbf{Pinj}(Y, X)\)

Define: \(\{\ell_2(f_i)\}\) is summable if \(\{f_i\}\) is summable in \textbf{Pinj} and in that case \(\sum_i \ell_2(f_i) = \text{def} \ell_2(\sum_if_i)\).

Clearly, \(\ell_2\) is an additive functor.

Quasi injections and projections are the \(\ell_2\) images of quasi projections and injections in \textbf{Pinj}.

\textbf{Hilb}_2 is traced. Given

\[ u : \ell_2(X) \oplus \ell_2(U) \to \ell_2(Y) \oplus \ell_2(U) \]

\[ \text{Tr}(u) = \text{def} \ell_2(\text{Tr}^U_{Y,X}(f)) \]

where \(u = \ell_2(f)\) with \(f : Y \uplus U \to X \uplus U \in \textbf{Pinj}\).

\textbf{Pinj} and \textbf{Hilb}_2 form GoI situations.
GoI Situations:

1. $TT \lessdot T (e, e')$ (Comult.)
   \[
   T T U \xleftarrow{e} \xrightarrow{e'} T U
   \]

2. $Id \lessdot T (d, d')$ (Dereliction)
   \[
   U \xleftarrow{d} \xrightarrow{d'} T U
   \]

3. $T \otimes T \lessdot T (c, c')$ (Contraction)
   \[
   T U \otimes T U \xleftarrow{c} \xrightarrow{c'} T U
   \]

4. $\mathcal{K}_I \lessdot T (w, w')$ (Weakening).
   \[
   I \xleftarrow{w} \xrightarrow{w'} T U
   \]

5. $U \in C$, a reflexive object,
   
   (a) $U \otimes U \lessdot U (j, k)$
   \[
   U \otimes U \xleftarrow{j} \xrightarrow{k} U
   \]

   (b) $I \lessdot U$

   (c) $TU \lessdot U (u, v)$.
   \[
   T U \xleftarrow{u} \xrightarrow{v} U
   \]
Pinj is GoI situation

\[ U = \mathbb{N}, \quad T(\cdot) = \mathbb{N} \times \langle \cdot \rangle \]

- \( T \) is additive, mon. functor
- \( \mathbb{N} \cup \mathbb{N} \xrightarrow{\langle \rangle} \mathbb{N} \)
  - \( j(1,n) = 2n \)
  - \( j(2,n) = 2n+1 \)

\[ k(n) = \begin{cases} \mathbb{S}(1, n/2) & \text{even} \\ \mathbb{S}(2, n+1/2) & \text{n odd} \end{cases} \]

- \( T(\mathbb{N}) \triangleleft \mathbb{N} \) i.e. \( \mathbb{N} \times \mathbb{N} \triangleleft \mathbb{N} \) (Cantor)

etc. (retracts are monoidal...)

\[ \mathbb{L}_2[\text{Pinj}] \text{ is also a GoI situation} \]
Stochastic Rel's

Objects: \((X, \Sigma_X)\) \(\omega/\tau\)-alg

Arrows: Stochastic Kernels

\(f: X \times \Sigma_Y \longrightarrow [0,1]\)

1. \(\forall B \in \Sigma_Y, \ f(-, B)\) measurable
2. \(\forall x \in X, \ f(x, -): \Sigma_Y \rightarrow [0,1]\)

is subprobability meas. \(f(x, y) \leq 1\)

Composition: \((g \circ f)(x, C) = \int g(y, C) \, d\{f(x, y)\}\)

Forma PAC, \(\mathcal{U} = \mathbb{N}^\mathbb{N}\) (Baire Space)

\(T, (X, \Sigma_X) = (N \times X, \Sigma_{N \times X})\).
Matrix Notations

\[ A \otimes B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \]

e.g., if \( s = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \),

\[ \sigma = s \otimes \ldots \otimes s \quad (m \text{ times}) \]

\[ \begin{bmatrix} s & & \\ & s & \circ \\ & & s \end{bmatrix} \]

\[ \begin{bmatrix} s & & \\ & s & \circ \\ & & s \end{bmatrix}_{2 \times 2 \times m} \]
\[
\begin{pmatrix}
\Delta \\
\Gamma
\end{pmatrix}
\rightarrow \mathcal{U}^{n+2m} 
\]

where \( \sigma : \mathcal{U}^{2m} \rightarrow \mathcal{U}^{2m} = \otimes \cdots \otimes (\text{m-times}) \)

\[ [\Delta] : \mathcal{U} \otimes \mathcal{U} \xrightarrow{\text{twist}} \mathcal{U} \otimes \mathcal{U} \]

\[ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathcal{S} \]

Axiom: \( \vdash A, A^\perp \)  \( m = 0, n = 2 \)
Tensor/Cut interpretation:

\[ \Gamma' [\Delta', \Gamma', A] \quad \Gamma'' [\Delta'', A^\perp, \Gamma''] \quad \text{cut} \]

\[ \Gamma [\Delta', \Delta'', A, A^\perp], \Gamma', \Gamma'' \]

\( \tau \cdot ( [\pi'] \otimes [\pi''] \otimes [\text{cut}] ) \)
\[ \Gamma, \Gamma', ??A, ??A \]

\[ \Pi : \vdash \Gamma, \Gamma', ??A \]

\[ \left[ \Pi \right] = \left( 1_{\tau} \otimes (u \cdot (c_u \otimes \nu \nu)) \otimes 1_{\Delta} \right) \]

\[ \left[ \Pi' \right] \cdot \left( 1_{\tau} \otimes (u \otimes \omega c'_u \otimes \nu \otimes 1_{\Delta}) \right) \]

idea: ??A: U "really" is ??A: TU
\[
\begin{align*}
\frac{\Gamma'(\Delta), \Gamma', A}{\Delta, \Gamma', \Delta, \overline{A}} &= \text{Dereliction} \\
\end{align*}
\]

\[
\begin{align*}
[\Pi] &= \left(1_{\mathcal{F}} \otimes u \otimes 1_{\Delta'}\right) \cdot [\Pi'] \cdot \\
&\left(1_{\mathcal{F}} \otimes d'_{\mathcal{U}} v \otimes 1_{\Delta}\right)
\end{align*}
\]
Example of GoI Semantics

Let $\Pi$ be the following proof:

\[
\begin{array}{c}
\vdash A, A^\perp \\
\vdash [A, A^\perp], A^\perp, A \\
\end{array}
\begin{array}{c}
\text{(cut)}
\end{array}
\]

Then the GoI semantics of this proof is given by

\[
[\Pi] = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & \text{Id} \\
\text{Id} & 0
\end{bmatrix}_{4 \times 4}
\]

Hence, $m=1$ and $S = S = \begin{bmatrix}
0 \\
1
\end{bmatrix}$.
**Dynamics**

Let $\Pi$ be a proof of $\vdash [\Delta], \Gamma$. Consider

\[ \Pi^n \xrightarrow{1_\Pi} \Pi^m \xrightarrow{1_\Pi} \frac{1}{\sigma} \xrightarrow{1_\Pi} \Pi^n \]

**Execution Formula**

\[
\text{Ex}(([[\Pi]], \sigma) = \text{def } \text{Tr} \left( (1_{\Pi^n} \otimes \sigma) [\Pi^n] \right) \\
= \Pi_{11} + \sum_{n \geq 0} \Pi_{12} (\sigma^n \Pi_{22}) (\sigma^n \Pi_{12})
\]

in any traced UDC
Example

Recall \[ \vdash A^\perp, A \vdash A^\perp, A \]

\[ \vdash [A, A^\perp], A^\perp, A \]

\[ \begin{bmatrix} \pi \end{bmatrix} = \begin{bmatrix} Q & 0 \\ \frac{1}{iQ} & I_2 \end{bmatrix} \quad ; \quad \sigma = s \]

\[ E_x([\pi \pi], \sigma) = \begin{aligned} &\text{def} \\
&\text{Tr} \left( (1 \otimes \sigma) [\pi \pi] \right) \end{aligned} \]

\[ \begin{aligned} &\text{Tr} \left( \begin{bmatrix} \begin{bmatrix} I_2 & Q \\ Q & s \end{bmatrix} \end{bmatrix} \left( \begin{bmatrix} Q & I_2 \\ I_2 & Q \end{bmatrix} \right) \right) \\
&\cdots \quad = \pi_{ii} + \sum_{\eta_{20}} \pi_{i_2} (\sigma \pi_{22})^n (\sigma \pi_{21}) \end{aligned} \]

\[ \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \sum_{\eta_{20}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left[ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right] \end{bmatrix} \]

\[ = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = [I \vdash A^\perp, A] = [n.f. \text{ of } \mathcal{F}] \]
The Main Idea: Cut-Elim.

is computation, so $[[\Pi]]$ should be given by an algorithm:

- Run $\mathsf{Ex}([[\Pi]], \sigma)$. It should terminate in finitely many steps.
- It terminates in a datum i.e. a cut-free proof.

**Lemma (Associativity of Cut)**

(Gives Soundness, i.e. Church-Rosser)

\[
\mathsf{Ex}([[\Pi]], 1 \otimes 2) = \mathsf{Ex}(\mathsf{Ex}([[\Pi]], \sigma), \tau) \]

Where $\{[[\sigma]], [\tau]\}$

**PE: Properties of Trace.**
Types

- Let \( f, g \in \mathcal{C}(U, U) \). We say \( f \) is orthogonal to \( g \) (\( f \perp g \)) if \( gf \) is nilpotent.

(Recall \( \mathcal{O}_{U} \subseteq \mathcal{C}(U, U) \) : \( \perp \) makes sense & \( \perp \) is symmetric.)

Let \( X \subseteq \mathcal{C}(U, U) \).

\[ X^\perp = \{ f \in \mathcal{C}(U, U) \mid f \perp X \} \]

(Where \( f \perp X \equiv f \perp g, \forall g \in X \))

\[ \text{Type} \equiv \text{st. } X = X^{\perp \perp} \]
**MELL Formulas as Types**

**formula A $\vdash \Theta \Rightarrow \Theta A$ a type**

- $\alpha \mapsto X$
- $\alpha^\perp \mapsto X^\perp$
- $B \otimes C \mapsto Y^{\perp\perp}$

  **where** $Y = \{ j(a \otimes b) \& | a \in \Theta B \}$

  **with** $U \otimes U \overset{\&}{\underset{j}{\leftrightarrow}} U$

- $B \& C \mapsto Z^\perp$

  **where** $Z = \{ j(a \& b) k | a \in \Theta B^\perp \}$
A = \{ !B \} \quad \rightarrow \quad Y^{+2}
A = \{ ?B \} \quad \rightarrow \quad Z^{+}

where \quad Y = \left\{ u T(a) v \mid a \in \Theta B \right\}

\[
\begin{array}{c}
U \xrightarrow{a} U \\
U U \xrightarrow{T} U I_a \xrightarrow{T} U U \xrightarrow{u} U
\end{array}
\]

and \quad Z = \left\{ u T(a) v \mid a \in (\Theta B)^{+} \right\}
Data & Algorithms

Let $I = A_1, \ldots, A_n$. \( \Theta I = \Theta A_1, \ldots, \Theta A_n \).

A datum of type $\Theta I$

\[= M : U^n \rightarrow U^n \quad \text{s.t.} \]

for any $\beta_i \in \Theta(A_i^\perp)$,

\[(\beta_1 \otimes \cdots \otimes \beta_n)^\perp \perp M\]

An algorithm of type $\Theta I$

\[= M : U^{n+2m} \rightarrow U^{n+2m} \quad \text{(for some } m \text{ s.t. } \sigma : U^{2m} \rightarrow U^{2m}) \]

\[\text{s.t. } \text{Ex}(M, \sigma) \text{ is finite sum} \]

& datum of type $\Theta I$. Here

\[\text{Ex}(M, \sigma) = \text{Tr}_{U^n, U^n} \left( (1 \otimes \sigma)M \right)\]
Example:
\[ \Gamma = \delta, \quad \Theta \Gamma = X \]

\[ M : U \to U \text{ is datum of type } X \]

iff \[ \forall \beta \in X^+, \quad \beta \perp M \]

i.e. \[ M \in X^+X = X \]

\[ M : U^{1+2m} \to U^{1+2m} \text{ is algorithm of type } X \] (for some \( \sigma : U^{2m} \to \))

iff \[ U \xrightarrow{M} U \xrightarrow{\sigma} U \]

\[ = \text{Tr}(\sigma \otimes \sigma)M \]

so finite \( \varepsilon \in X^+X = X \)
Using technical lemmas on nilpotence one obtains:

**Characterization lemma.** Consider $M : U^n \to U^n$, $a : U \to U$.

$M = (M_{ij})_{n,m}$ is a datum of type $\Theta(A, \Gamma) \iff$ for any $a \in \Theta A^\perp$, $a M_{ii}$ is nilpotent

$\& \text{Tr} (S^{-1}_{\Gamma, A} (a \otimes \text{id}_{\mathfrak{A}})^* M S_{\Gamma, A}) \in \Theta(\Gamma)$

where $S_{\Gamma, A} : \mathfrak{F} \otimes \mathfrak{A} \to \mathfrak{F} \otimes \Gamma$
Theorem (Girard). Let \( \Pi \) be a proof of \( \vdash [\Delta], \Gamma \) in MELL. Then

1. \( [\Pi] \) is an algorithm of type \( \Pi \Gamma \); in particular \( \text{Ex}([\Pi], \sigma) \) is a finite sum.

2. If \( \Pi \geq \Pi' \) by Cut-Elim and \( ? \) does not occur in \( \Gamma \) then \( \text{Ex}([\Pi], \sigma) = \text{Ex}([\Pi'], \tau) \).

3. If \( \Pi' \) is n.f. of \( \Pi \) then \( \text{Ex}(\Pi, \sigma) = \text{Ex}(\Pi', \sigma) = [\Pi'] \).
4. In Hilb₂, we get Giraud's original execution formula:

\[ E_X(\mathbb{H}, \sigma) = \left(1 - \tilde{\sigma}^2\right) \sum_{n=0}^{\infty} \mathbb{H}_n (\tilde{\sigma}(\mathbb{H}))^n (1 - \tilde{\sigma}^2) \]

where \( \tilde{\sigma} = O_n \otimes \sigma = \left(\frac{O_n}{\sigma}\right)_{n+2m} \)

and \((A)_{n \times n} = \text{the } n \times n \text{ submatrix of } A\).
Example of the proof in (1):

By induction on proofs.

Axiom: \( \vdash A, A^\perp \rightarrow \top \)

Show \( E_X(\top, 0) = \top \) is a datum of type \( \Theta \).

\( \forall a \in \Theta A, b \in \Theta A, \)
\( M = \top \top (a \otimes b) = \left[ \begin{array}{cc} a & b \\ 0 & 0 \end{array} \right] \) must be nilpotent.

E.g. if \( n \) even
\( M^n = \left[ \begin{array}{cc} (ba)^{n/2} & 0 \\ 0 & (ab)^{n/2} \end{array} \right] \)

But \( a \perp b \). \( \therefore \) \( ba, ab \) are nilpotent. \( \therefore \) so is \( M \).
Denotational Models from GoI

Let \((G, T, U)\) be a \textsc{udc-goI} situation. Define a category \(\mathcal{C}(G)\) as follows:

- Objects = Types (i.e. subsets \(A \subseteq \mathcal{G}(U, U)\) s.t. \(A^{++} = A\)).

- Arrows = \(A \xrightarrow{f} B\) is a morphism \(f \in \mathcal{G}(U, U)\) s.t. \(\forall a \in A, f \circ a \in B\) where \(f \circ a = \text{Tr}_{U} \mathcal{U} \left( S(a \otimes I) (k_f) S \right)\)
Motivation: a morphism = cut-free proof of \( \vdash A^+, B \) = datum of type \( \Theta(A^+, B) \).

Identity: \( \mu \sum_k \)

\[
\begin{array}{c}
U \xrightarrow{\mu} U \otimes U \xrightarrow{S} U \otimes U \xrightarrow{\mu} U
\end{array}
\]

(Intuition: Cut-free proof of \( \vdash A^+, A \))

Composition: consider

\[
\begin{array}{c}
\vdash A^+, B \\
\vdash B^+, C
\end{array}
\]

\[
\vdash \left[ B, B^+ \right], A^+, C
\]

& Run EX.
This gives a cat-free proof

\[ \theta \vdash A^+, C \quad (i.e. \ datum \ of \ type \ \Theta(A^+, C)) \]

In terms of \( \text{Int}^+(\theta) = \beta(\theta) \)

\[
\begin{align*}
A^+ & \quad B \\
\downarrow & \quad \downarrow \quad \downarrow \\
A^- & \quad B \\
B^1 & \quad C
\end{align*}
\]

\[ g_{of} = \int \text{Tr} \left[ \begin{array}{c} A \\
\end{array} \right]_{4 \times 4} \]

where \( A \) is complicated \( 4 \times 4 \) matrix
Then: let $(G, I, U)$ be a GoI situation & suppose $U \otimes U \cong U$ (i.e.). Then
$\Theta(G)$ is *-aut. category without units.

Tensor, Par, etc. given by operations on types.

Trouble with units: $A \triangleleft A \otimes I$, but not iso.
Denotational Models of MELL

- **MLL:** *-aut. category
  
  \((\mathcal{C}, \otimes, I, s, (-)^\perp)\)

- **Exponentials:** \(! : \mathcal{C} \to \mathcal{C}\)

- **Monoidal n.t.'s**
  
  \(\text{def}_A : !A \to A\)

  \(\delta_A : !A \to !!A\)

  \(\text{Weak}_A : !A \to I\)

  \(\text{Con}_A : !A \to !A \otimes !A\)

- \((!, \text{def}, \delta)\) commonoid

- \((!, \text{weak}_A, \text{con}_A)\) coComm. comon.

  \(\text{Weak}_A, \text{Con}_A : \text{co-} \alpha \text{-alg maps}, \delta_A \text{ map}\)
Theorem: \((\mathcal{C}, T, !)\) UDC-
GoI situation. Define

\[ ! : \mathcal{O}(\mathcal{C}) \to \mathcal{O}(\mathcal{C}) \text{ by} \]

\[ !A = \{ u | T(a)u ! | a \in A \} \]

\[ u \xrightarrow{a} v \]

\[ u \xrightarrow{T} Tu \xrightarrow{u} Tu \xrightarrow{u} u \]

Suppose \( u \otimes u = u \), \( Tu = u \).

\[ \cdot (T, d', e') \text{ comonad} \]

\[ \cdot (TA, w_A', c_A') \text{ comm comm.} \]

\[ \cdot e_A' \text{ is map of comm.} \]

\[ \cdot w_A', c_A' : \text{ maps of coalgs} \]

Then \((\mathcal{O}(\mathcal{C}), !) = \text{ denotational model} \) MELL.
GoI/Int Construction $\mathcal{G}(-)$

$\mathcal{I} = \text{T.M.c.}$

$\mathcal{G}(\mathcal{C})$ : \textbf{objects} = \((A^+, A^-)\) \qquad \uparrow \quad \text{player} \quad \text{opponent}

\text{Arrows:} \quad (A^+, A^-) \xrightarrow{f} (B^+, B^-)

\begin{align*}
A^+ &\quad \cdots & A^- &\quad \cdots & A^+ \\
B^- &\quad \cdots & B^+ &\quad \cdots & B^-
\end{align*}

\[ A^+ \otimes B^- \rightarrow A^- \otimes B^+ \]

\text{Comp}^* : \text{Symmetric feedback}

\begin{tikzpicture}
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw (0.5,0.5) node {$A^+$} -- (1.5,0.5) node {$A^-$};
\draw (0.5,0) node {$B^-$} -- (1.5,0) node {$B^+$};
\draw (0,0.5) node {$B^-$} -- (0,1.5) node {$B^+$};
\draw (1,0.5) node {$A^+$} -- (1,1.5) node {$A^-$};
\end{tikzpicture}
Thm (JSV/Ab.) If $C$ is traced Symm. monoidal cat, $\mathcal{G}(C)$ is compact closed.

(2-categorically: Compact-closure of $C$)

Let $Gl =$ double gluing

Prop: There is a faithful $(-)^\perp$-preserving embedding $F: \mathcal{O}(C) \rightarrow Gl(\mathcal{G}(C))$
Future Directions

- GoI 2: Non-converging algs
  (untyped I-calc / PCF)
  - uses more topological info
  on operators alg
- GoI 3: Uses additive & additive
  proof nets
- GoI 4: (last month): Von Neumann
  algebras: $\text{Ex}(f, \tau) \nleq f \circ b$ (not necessarily
  coming from proof)
- Quantum GoI?