Cockett-Lack Restriction

Categories, Semigroups and Topologies

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Many have written about categories of partial maps

- Di Paola and Heller, 1986
- Carboni, 1987
- Robinson and Rosolini, 1988
- Curien and Obtulowicz, 1989
- Jay, 1990
- Mulry, 1992
- Fiore, 1996

For $\mathcal{C}$ a category, $\mathcal{M}$ a stable system of monics, the **partial morphism category**

$$Par(\mathcal{C}, \mathcal{M})$$

has morphisms equivalence classes $[m, f]$ with $X \xleftarrow{m} \xrightarrow{f} Y$ and $m \in \mathcal{M}$. Composition is via pullback as usual.

The “domain of definition” of $[m, f]$ can be modelled as the **restriction endomorphism** $[m, m]$.

Cockett and Lack's idea was to make restriction a primitive.

Robin Cockett and Stephen Lack, Restriction Categories I: Categories of Partial Maps, *Theoretical Computer Science* 279, 2002, 223-259.

A **restriction category** is an "abstract category of partial morphisms", being a category with a restriction operator

$$f : X \to Y \quad \mapsto \quad \overline{f} : X \to X$$

satisfying the axioms $R.1, \ldots, R.4$ on the board.

Note that a full subcategory of a restriction category again is one.

For a restriction category $\mathcal{C}$ denote by $\mathcal{R}(\mathcal{C})$ the set of **restriction idempotents**

$$\mathcal{R}(\mathcal{C}) \;=\; \{x : x = \overline{x}\} \;=\; \{\overline{f} : f \text{ a morphism}\}$$

The restriction itempotents $X \to X$ form a semilattice by (D), $R.2$, (A).

A **split** restriction category has the property that all restriction idempotents split.

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[dr, "\overline{f}"'] & Z \arrow[d, "m"] \arrow[dr, "id"] & \\
 & X \arrow[r, "e"'] & Z
\end{tikzcd}$$

In that case, the set of all m as above form a stable system of monics.

Theorem (Cockett & Lack) $Par(\mathcal{C}, \mathcal{M})$ is a split restriction category. For every restriction category $\mathcal{C}$ there exists $\mathcal{D}$, $\mathcal{M}$ for which $\mathcal{C}$ is a full restriction category of $Par(\mathcal{D}, \mathcal{M})$

Proof Idea: Let $\mathcal{E}$ be the itempotent completion of $\mathcal{C}$ splitting $\mathcal{R}(\mathcal{C})$. $\mathcal{E}$ is a restriction category: $e_1 \xrightarrow{f} e_2$ has restriction $\overline{f}e_1$.

In any restriction category, f is **total** if $\overline{f} = id$.

Take $\mathcal{D}$ to be the total morphisms of $\mathcal{E}$. Take $\mathcal{M}$ as the monics that arise in the splittings of restriction itempotents in $\mathcal{E}$. (Though $\overline{f}$ is not total, all monics are total).

The embedding is

$$f \quad \mapsto \quad [X \xleftarrow{m} \xrightarrow{m} X \xrightarrow{f} Y]$$

where m is the monic in the splitting of $\overline{f}$.

Via the Yoneda embedding of the previous construction, one sees further that $\mathcal{C}$ is a full restriction category of

$$Par(\mathbf{Set}^{\mathcal{D}^{op}}, \mathcal{N})$$

for a suitably-chosen stable system of monics $\mathcal{N}$.

Summary

- Restriction categories have captured partial morphism categories.

- There is no use of universal properties in the axioms. Any full subcategory continues to be a restriction category.

Earlier work by some theoretical programmers had a different emphasis.

- The logic is classical (Boolean)

- But programs can have nondeterministic behavior.

Edsger Dijkstra, *A Discipline of Programming*, Prentice-Hall, 1976:

> "In this book –and that may turn out to be one of its distinctive features– I shall treat nondeterminancy as the rule and determinacy as the exception . . ."

Dijkstra's **guards** are precisely restrictions.

Ernie Manes, *Predicate Transformer Semantics*, Cambridge University Press, 1992.

Boolean Categories

(B.l) $X + Y$, initial 0

(B.2) Coproduct injections is stable system of monics

(B.3) Coproduct injections pull back binary coproducts

(B.4) Except for 0, coproduct injections in $X + X$ are different

Coproduct-injection subobjects are called **summands**.

Theorem The poset $Summ(X)$ of summands of X forms a Boolean algebra.

The pullback of $X \xrightarrow{f} Y \longleftarrow 0$ is the **kernel** of f, $Ker(f) \to X$.

$Dom(f) = (Ker(f))'$.

f is **total** if $Ker(f) = 0$.

$Dom(f)$ is the largest summand restricted to which f is total.

f is **undefined** if f factors through 0.

To define restrictions requires *canonical* undefined maps.

In a Boolean category, these are provided by "projection systems" which correspond bijectively to maximal Boolean subcategories with zero maps. Let us fix one of these so that

> We now work in a Boolean category with a zero object.

Here's how restrictions are defined in a Boolean category with zero:

$$
\begin{tikzcd}
Ker(f) \arrow[r, "i'"] \arrow[dr, "0"'] & X \arrow[d, "\overline{f}"] & Dom(f) \arrow[l, "i"'] \arrow[dl, "i"] \\
& X &
\end{tikzcd}
$$

Fact: $R.1$, $R.2$, $R.3$ hold. Restriction idempotents split.

What is the situation with $R.4$?

Proposition Restrictions $X \to X$ form a Boolean algebra (with $a \wedge b = ab = ba$) isomorphic to $Summ(X)$.

Proof Idea

$$A \xrightarrow{i} X \longleftarrow A' \quad \mapsto \quad a = \bar{i}$$

$$a = \bar{a} : X \to X \quad \mapsto \quad A = eq(id_X, a)$$

In our Boolean category with zero, $f : X \to Y$ is *deterministic* if

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[d, "f"] & P' \arrow[l] \arrow[d] \\ Q \arrow[r] & Y & Q' \arrow[l] \end{tikzcd}$$

$\forall\, Y = Q + Q' \;\; \exists\, X = P + P'$ and a commutative diagram as above.

Deterministic maps form a Boolean subcategory.

Toward an interpretation of Axiom R.4

Theorem In a Boolean category with zero, $(R.4)$ holds for $f : X \to Y$, i.e. for all $g : Y \to Z$, $\overline{g}f = f\overline{gf}$ if and only if f is deterministic.

Thus a Boolean category with zero is a restriction category if and only if all morphisms are deterministic.

Thus, for each Boolean category with zero, the determinisic morphisms constitute a restriction category.

Toward restrictions for semigroups

Semigroup theorists should be interested in restriction!

Let's start with some basic semigroup stuff.

Let S be a semigroup. $a \in S$ is **regular** if $\exists\, x \in S$ with $axa = a$.

S is **regular** if all of its elements are regular.

An **inverse** of a is x with $axa = a$ and $xax = x$.

Example Let $S = A \times B$ with $(a, b)(c, d) = (a, d)$. Then S is a semigroup in which each element is inverse to all elements.

Every regular element has an inverse.

An **inverse semigroup** is a semigroup in which each element a has a unique inverse a^{-1}.

Inverse semigroups are equationally definable:

$$\begin{array}{rcl} x(yz) & = & (xy)z \\ (x^{-1})^{-1} & = & x \\ (xy)^{-1} & = & y^{-1}x^{-1} \\ xx^{-1}yy^{-1} & = & yy^{-1}xx^{-1} \end{array}$$

Example Any group.

Example Injective partial functions $X \to X$.

Proposition A semigroup is an inverse semigroup if and only if it is regular and any two idempotents commute.

Vagner-Preston Theorem If S is an inverse semigroup then

$$S \xrightarrow{\lambda} \mathbf{Pfn(S,S)}, \quad a \mapsto \lambda_a$$

$$\lambda_a\, x \;=\; \begin{cases} ax & \text{if } x \in a^{-1}aS \\ \perp & \text{otherwise} \end{cases}$$

is an injective semigroup homomorphism. Each λ_a is an injective partial function.

Books on inverse semigroups

- Petrich, 1984 (674 pages)
- Lipscomb, 1996
- Lawson, 1998 "Self-similarities are examples of what we term partial symmetries...".

There has been literature on semigroups with $x \mapsto x^\star$ satisfying

$$\begin{aligned} (xy)^\star &= y^\star x^\star \\ x^{\star\star} &= x \\ xx^\star x &= x \end{aligned}$$

Inverse semigroups are “abstract injective $\mathbf{Pfn(X, X)}$”.

What plays the role of “abstract $\mathbf{Pfn(X, X)}$”?

Restriction algebras!

A **restriction algebra** is a semigroup equipped with a unary operation $x \mapsto \overline{x}$ which satisfies axioms $(R.1, \ldots, R.4)$

Thus restriction algebras constitute an equationally definable class of universal algebras.

Example The endomorphisms of any object in a restriction category.

Example Let $\mathcal{C}$ be a restriction category. Let S be the morphisms of $\mathcal{C}$ together with a new element 0. Then S is a restriction algebra if

$$xy = \begin{cases} xy & \text{if } x \neq 0 \neq y,\ cod(y) = dom(x) \\ 0 & \text{otherwise} \end{cases}$$

$$\overline{x} = \begin{cases} \overline{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

The Robinson-Rosolini P-categories / Cockett copy categories produce a restriction category by

$$\overline{f} \quad = \quad A \xrightarrow{\triangle} A \otimes A \xrightarrow{f \otimes 1} B \otimes A \xrightarrow{! \otimes 1} I \otimes A \cong A$$

whose endomorphism monoids are restriction algebras.

Not an example semigroup theorists would rush to.

Example Let S be a semigroup, $a \in S$. Define $\overline{x} = a$. This is a restriction algebra if and only if a is a unit for S.

Example Let S be a left cancellative semigroup which is not a monoid. Then no restriction operator exists to make S a restriction algebra. (Proof: $\overline{x}\,\overline{y} = \overline{x}\,\overline{x}\,\overline{y} \Rightarrow \overline{x}\,\overline{y} = \overline{y}$. As restriction itempotents commute, the same proof gives $\overline{x}\,\overline{y} = \overline{y}$. Now use the previous example.)

Example Every meet semilattice $xy = x \wedge y$ is a restriction algebra if $\overline{x} = x$. We say a restriction algebra "is a semilattice" if it is of this form.

Exercise for you: Show that the center

$$Z(S) \quad = \quad \{x \in S : \forall y \in S \; xy = yx\}$$

is a restriction subalgebra.

Hint: Use all four axioms.

Proposition (Cockett and Lack) Every inverse semigroup is a restriction algebra with $\overline{x} = x^{-1}x$. Inverse semigroups are a full coreflective subcategory of restriction algebras with the coreflection $I(S)$ of S given by

$$\{x \in X : \exists a \in S \text{ with } xa = \overline{a},\ ax = \overline{x}\}$$

$I(S)$ is analogous to the group of units of monoid.

By a **partially ordered semigroup** we mean a semigroup with a partial order such that

$$x \leq y \quad \Rightarrow \quad \forall a \, \forall b \;\; axb \leq ayb$$

Every restriction algebra is a partially ordered semigroup if $x \leq y$ means $y\overline{x} = x$.

Restriction algebra homomorphisms are monotone.

“Vagner-Preston Theorem” for restriction algebras If S is a restriction algebra then

$$S \xrightarrow{\lambda} \mathbf{Pfn(S,S)}, \quad a \mapsto \lambda_a$$

$$\lambda_a x = \begin{cases} ax & \text{if } \overline{a}x = x \\ \perp & \text{otherwise} \end{cases}$$

is an injective restriction algebra homomorphism mapping $I(S)$ to injective partial functions.

This recaptures the classical theorem for inverse semigroups.

When S is a monoid with $\overline{x} = 1$, get usual Cayley theorem.

Every meet semilattice can be embedded in a Boolean algebra.

Corollary Every small restriction category $\mathbf{C}$ is isomorphic to a restriction subcategory of **Pfn**.

Proof idea Cockett and Lack obtained this also. But the same constructions as "Vagner-Preston" give a more direct proof. Discover the details by regarding such a category as a restriction algebra as per earlier example.

Form itempotent completion $\hat{\mathrm{C}}$ of $\mathbf{C}$ so objects are restriction itempotents and maps $\alpha : e \to f$ satisfy $f\alpha e = \alpha$. Then

$$\hat{\mathbf{C}} \xrightarrow{\psi} \mathbf{Pfn}, \quad \psi e \ = \ \{t : et = t\}$$

$$\psi(e \xrightarrow{\alpha} f)t \ = \ \begin{cases} \alpha t & \text{if } \overline{\alpha} f t = t \\ \perp & \text{otherwise} \end{cases}$$

But, to paraphrase Marshall Stone,

One must topologize!

Let $\mathcal{T}$ be a topology of open sets on X.

For $A \subset X$ write the closure of A as

$\overline{A}$ no, wait, that's restriction.

$A^\star$ no, wait, that's the free monoid

$A\hat{}$

A function f is continuous

$$\Leftrightarrow \; \forall A \, f(A\hat{}) \subset (fA)\hat{}$$
$$\Leftrightarrow \; \forall A \, \forall B \, A\hat{} = B\hat{} \;\Rightarrow\; (fA)\hat{} = (fB)\hat{}$$

A **pospace** is a topological space in which any intersection of open sets is open.

Let **PoSp** be the category of pospaces and continuous maps.

Proposition (Lorrain, 1969) The category **PreO** of sets with reflexive and transitive relation and monotone maps is isomorphic over **Set** to **PoSp**.

$$\begin{array}{rcl} x \leq y & \Leftrightarrow & y \in \{x\}\hat{} \ (\textit{specialization order}) \\ \text{open set} & = & \text{lower set} \\ \text{closed set} & = & \text{upper set} \\ A\hat{} & = & \uparrow A \end{array}$$

A **right topological semigroup** is $(X, \cdot, \mathcal{T})$ with $(X, \cdot)$ a semigroup and $(X, \mathcal{T})$ a topological space such that

$$\forall x \in X \ \rho_x y \ = \ yx \text{ is continuous}$$

- Use rts for right topological semigroup

- Use rtm for right topological monoid

The forgetful functor from monoids to semigroups has a left adjoint $S \mapsto S^1$.

Here $S^1 = S + \{1\}$ with $x1 = x = 1x$.

The same is true for rtm and rts (let 1 be an isolated point).

Proposition Let X be rts. Let $\mathcal{C}$ be the family of closed subsets of X. Then

$$S = X^1 \times \mathcal{C}$$

is a restriction algebra if

$$\begin{aligned} (x, C)(y, D) &= (xy, (Cy)\widehat{\ } \cup D) \\ \overline{(x, C)} &= (1, C) \end{aligned}$$

Call this the **full restriction algebra** of X.

Observation Every semigroup X is a subsemigroup of a restriction algebra S whose restriction itempotents form a Boolean algebra.

For let S be the full restriction algebra of X where X has the discrete topology. Use the embedding $x \mapsto (x, X)$

Let X be any semigroup. **Green's left order** is

$$x \leq_{\mathcal{L}} y \;\;\Leftrightarrow\;\; x \in X^1 y$$

Being reflexive and transitive, this induces the pospace

$$A\hat{} \;=\; \uparrow A \;=\; \{y : \exists x \in S^1 \, xy \in A\}$$

and S is rts because $x = zy \;\;\Rightarrow\;\; xa = z(ya)$, i.e., right translations are monotone.

Predecessors in Semigroup theory:

- Scheiblich, 1973
- Munn, 1974
- Schein, 1975

Theorem (Cockett and Lack) The free restriction algebra generated by a semigroup X is the sub-restriction algebra of the full one $X^1 \times \mathcal{C}$, $\mathcal{C} =$ closed sets of the $\mathcal{L}$-topology, of all

$$(x, \{a_1, \ldots, a_n\}\hat{}\,),\ \ x \neq 1 \ \Rightarrow\ x \in \{a_1, \ldots, a_n\}\hat{}$$

The inclusion of the generators is

$$x \mapsto (x, \{x\}\hat{}\,)$$

The literature on topological semigroups is primarily about the Hausdorff case, often compact Hausdorff.

Here's a rich supply of compact Hausdorff topological restriction algebras.

Start with a compact Hausdorff monoid M. Let $\mathcal{C}$ be the "hyperspace" of closed subsets of M with the "finite topology" (see Vietoris 1923, and Michael, 1951). Then $\mathcal{C}$ is compact Hausdorff.

The full restriction algebra $M \times C$ is then a compact Hausdorff restriction algebra.

Ai yai yai, another new structure

A **band with restriction** is a semigroup xy with unary $x \mapsto \overline{x}$ satisfying $(R.1), (R.2), (R.3)$ as well as axiom (α) on the board.

Extremal example A semilattice $xy = x \wedge y$ with $\overline{x} = x$. This is the only example if a unit exists –consider axiom (α) with $x = 1$.

Extremal example A left zero semigroup $xy = x$ with $\overline{x} = e$ any fixed e

Observation A restriction algebra satisfies (α) if and only if $\overline{x} = x$, in which case it is a semilattice.

We are interested in bands with restriction because there is a forgetful functor over **Set** from restriction algebras to bands with restriction. Given a restriction algebra S with multiplication xy and restriction $\overline{x}$,

$$x * y \;=\; x\overline{y}$$

with the same restriction gives a band with restriction.

The example of partial functions $X \to X$ shows that a great deal of information is lost.

A **band** is a semigroup in which each element is itempotent.

Three extremal cases are

- Left zero semigroup: $xy = x$
- Right zero semigroup: $xy = y$
- Rectangular band: $axa = a$

The varieties of left zero and right zero semigroups are isomorphic to **Set**, the algebras of the identity monad. Rectangular bands are the algebras of $\mathbf{id} \times \mathbf{id}$.

Let $\mathcal{V}_0$ be the class of all semigroups for which $\overline{x}$ exists yielding a band with restriction. Let $\mathcal{V}$ be the variety generated by $\mathcal{V}_0$

Theorem $\mathcal{V}$ is the variety of all left normal bands:

$$\begin{aligned} x^2 &= x \\ axy &= ayx \end{aligned}$$

Proof Idea C. F. Fennemore, 1971 classified all varieties of bands. Consider the band with restriction $\{0, \alpha, a\}$ with

$$\begin{array}{rcccl} 0x & = & x & = & 0 \\ aa\alpha & = & aa & = & a \\ \alpha\alpha & = & \alpha a & = & \alpha \end{array}$$

and with restriction

$$\overline{0} = 0, \; \overline{\alpha} = \overline{a} = a$$

Example The free left normal band generated by $\{a, b\}$ has multiplication table

	a	b	ab	ba
a	a	ab	ab	ab
b	ba	b	ba	ba
ab	ab	ab	ab	ab
ba	ba	ba	ba	ba

No $x \mapsto \overline{x}$ exists making this a band with restriction.

Let $\mathcal{W}$ be any class of semigroups. A semigroup S is a **semilattice of type** $\mathcal{W}$ if there exists a semilattice L and a surjective semigroup homomorphism $\psi : S \to L$ such that each $\psi^{-1}(e)$ (obviously a subsemigroup of S is in $\mathcal{W}$.

Thus S is partitioned into subsemigroups $S_e = \psi^{-1}(e)$ with $S_e S_f \subset S_{ef}$.

Example every semilattice is a semilattice of groups.

Theorem (Clifford 1941, McLean 1954) Every band is a semilattice of rectangular bands.

The following strengthening is due to Clifford:

Let L be a meet semilattice and let

$$F : (L, \leq)^{op} \to Semigroups$$

be a functor. Let

$$S = \coprod_{e \in L} Fe$$

Then S is a semigroup with multiplication

$$x \in Fe,\ y \in Ff \ \ \mapsto \ \ xy \ = \ F_{e,ef}(x)F_{f,ef}(y)$$

a product in the semigroup F(ef).

Such S is a **strong semilattice** of the semigroups Fe.

A band is **normal** if $axya = ayxa$. Note that every rectangular band is normal.

Left normal (recall $axy = ayx$) is *stronger* than normal.

Theorem (Yamada and Kimura 1958) The normal bands are precisely the strong semilattices of rectangular bands.

Corollary The left normal bands are precisely the strong semilattices of left zero semigroups.

We can now characterize $\mathcal{V}_0$, the class of semigroups of bands with restriction.

A semilattice of semigroups $\psi : S \to L$ is **split** if ψ is split epic in the category of semigroups.

Theorem A semigroup has the structure of a band with restriction if and only if it is a split strong semilattice of left zero semigroups.

A band with restriction is a partially ordered semigroup via

$$x \leq y \text{ if } yx = x$$

Notice that the order on a restriction algebra is exactly this order on its underlying band with restriction.

Homomorphisms of bands with restriction are monotone.

Theorem The category of restriction algebras and monotone maps is cartesian closed. The category of bands with restriction and monotone maps is cartesian closed.

Theorem (Linton 1966) The variety of bands with restriction is a symmetric monoidal closed category.

For any partially ordered semigroup S, the **negative cone** $N(S)$ is defined by

$$N(S) \;=\; \{x \in S : \forall y \; xy \leq y, \; yx \leq y\}$$

For any semigroup S, let its center be denoted $Z(S)$. If S is a restriction algebra or a band with restriction, let the set of restriction item-potents of form $\overline{x}$ be denoted $R(S)$.

Proposition For bands with restriction,

$$N(S) \;=\; Z(S) \;\subset\; R(S)$$

For restriction algebras,

$$N(S) \;=\; R(S)$$

You did it!

You got through

56 slides!

Quiz tonight at 3 AM