Euclid's Elements 1

- Although many famous mathematicians in Plato's Academy, Euclid's elements ($\sim$ 300 B.C.) was so successful it superseded all previous writings (and they were no longer copied). Hence we don't have many of the older manuscripts (in addition to the burning of the library of Alexandria).
- Over 1000 editions of Euclid have appeared since 1482 (Eves).
- Modern editions of Euclid are based on Greek versions of Theon of Alexandria ($\sim$ 400), roughly 700 years after Euclid.
- Latin translations of Euclid came from Arab mathematicians based on early Greek versions. (We'll discuss this later)

Euclid's Elements 2

- The early Greek mathematicians and philosophers introduced the idea of Axiomatic Systems:
  - start with some precise language and careful definitions.
  - use a precise system of logic and reasoning,
  - start with a finite set of *Axioms* or *Postulates* (Euclid's terminology).
  - derive results formally and precisely.
- Although this used Aristotle's ideas of axiomatic theory, Aristotle's Syllogistic was too complicated for the geometers of the time: they used Stoics' Propositional Logic and theory of equality.

Euclid's Elements 3

- Euclid starts with *Common Notions*: axioms of equality (as a kind of equivalence relation). For example:

  > *Things equal to the same thing are equal to each other.*

- Note "things" for Euclid can be anything: triangles, circles, areas, volumes, magnitudes, numbers, etc. But they initially have to be of the same type (in any specific context), until we get to Eudoxus' theory of proportions in Book V, allowing us to compare ratios of different types.

Euclid's Elements 4: the Axioms

Euclid introduced 5 axioms (Postulates)

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe (i.e. draw) a circle with any centre and any radius.
4. All right angles are equal.
5. If a straight line, falling across two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines, if produced indefinitely, will meet on that side on which are the angles which are less than 2 right angles.

Euclid's Elements 5: the books I-IV

Following Eves and Anglin-Lambek.

1. Book I: 48 Propositions on triangles and congruences. At the end he concludes with proofs of Pythagorus's theorem and its converse. The ideas about compass and straightedge constructions starts.
2. Book II: 14 Propositions. Areas and geometric algebra. Again, following the Pythagoreans. There is some debate among historians if this is really about geometric algebra. (Prop.11: constructs the golden section)
3. Book III: 39 Propositions. On circles, chords, tangents. (Note: this leaves the realm of the Pythagoreans, who did not discuss circles).
4. Book IV: 16 Propositions.
   -Ruler and compass constructions in detail.
   - Inscribed polygons and regular polygons in circles.
   - Constructs a regular pentagon (using the earlier construction of the golden section).

Euclid's Elements 6 : the Books V-

1. Book V: Eudoxus's theory of proportions, resolving "scandal" of irrational quantities. Precursor to modern theory of Dedekind Cuts for constructing the reals from the rationals.
2. Book VI: Application of Eudoxus theory to geometry:
   - mean proportions.
   - geometric solutions of quadratics.
   - generalizations of Pythagorus' theorem.

Euclid's Elements 7 : the Books VII-IX

1. 102 Propositions
2. Elementary number theory. Beginnings of Euclidean Algorithms.
3. Pythagorean theory of proportions for numbers and $\mathbb{Q}$.
4. Book IX: Fundamental Theorem of Arithmetic. For example:
    - Prop. 20: "Prime numbers are more than any assigned multitude of prime numbers" .
    - The above says in modern language: "there is an infinity of primes". The Greeks did not allow this kind of statement. Why? What did they mean by their statement?

Euclid's Elements 8 : Books X-XIII

1. Book X: remarkable book on theory of irrationals (Incommensurables): thought to be due to Thaetetus, but polished and clarified by Euclid.
2. Deep results, proved and stated without any symbolic notation, on Pythagorean triples, formulas from Babylonian mathematics, etc.
3. Books XI–XIII: Solid geometry, ending (in Book XIII) in a complete construction of the 5 regular polyhedra.

The 3 Ancient Classical Problems

- Doubling the Cube
(i.e. finding $\sqrt[3]{2}$, i.e. given s, to solve $x^3 = 2s^3$.)
- Trisecting any given angle, for example 60 degrees
(i.e. $\pi/3$ radians).
- Squaring the circle: that is, construct a square equal in area to a given circle.

Know the history of these problems:

see MacTutor Website, link 4 on my website of Greek Mathematics.

- According to the famous geometer Pappus ($\sim$ 4th century), it was Plato who insisted on these restrictions ($>$ 600 years earlier!).
- What are the physical properties of Greek straightedges and compasses ?
- You should know how to do the following by straightedge and compass:
  1. to bisect a given angle;
  2. to find an orthogonal bisector of a line segment;
  3. draw a line through a given point parallel to a given line;
  4. to construct an equilateral triangle. (see Euclid, Book 1, Prop. 1).

What is allowed?

- In class and notes on Blackboard, we will go into great detail on the following questions:
  - Complete classification of rational operations.
  - The unsolvability of the Classical Greek Problems by ruler and straightedge, (using some elementary theory of quadratic extensions of $\mathbb{Q}$).
  - Constructing (and also not being able to construct) certain regular *n*-gons.