Plato's Academy: 387 B.C.– 529

- Founded by Plato (427-347 B.C.) in 387 B.C.. After Roman empire collapsed in 476, was closed by Justinian as a pagan institution in 529.
- Had some of the greatest Greek mathematicians and philosophers in his academy: (i) Theaetetus (417–369 B.C.), Eudoxus (408–355 B.C.), (iii) Aristotle (384–322 B.C.).
- Over the door of his Academy apparently it said:

  Let no one ignorant of Geometry Enter Here.

- Curriculum (cf. Plato, in the *Republic*): arithmetic (i.e. number theory), plane and solid geometry, astronomy, and harmonics (music) (10 yrs) and dialectics (logic, grammar, rhetoric)(5 yrs). The geometry and number theory were codified later in the 13 books of Euclid ($\sim$ 300B.C.).

Some Notions Studied at Plato's Academy

- Platonic Solids (what are they? Who developed them? In which books of Euclid were they completely classified?)
- Plato's theory of Ideal Forms. Plato (in *Phaedo*): objects in real world are imperfect imitation of perfect forms. E.g. real lines (have breadth): only roughly approximate perfect forms. Equality: ideally it is transitive. In real world, copies of copies of copies of keys: don't work!
- Modern theory of Mathematical Platonism: what is it? Which mathematicians believe it?
- Astronomy: Plato believed stars, planets, etc. move around the earth in spheres: closest is the Moon, then the Sun, then Mercury, Venus, Mars, Jupiter, Saturn, then the stars.

Aristotle: 384-322 B.C.

- He studied (and later taught) at Plato's academy from the age of 18 until Plato's death. Was interested in a huge range of subjects and lectured extensively on them.
- He left Athens after Plato's death; did scientific research (e.g. marine biology). Tutored Alexander the Great.
- Aristotle returned to Athens in 335 B.C.to found his own school, the *Lyceum* where he continued his studies. He retired to an island in 323B.C., and he died the next year. It is known that he had two wives (the first died while he was at the Lyceum), and at least one son.
- Read MacTutor Biography of Aristotle.
- **Note:** Aristotle is one of the greatest scholars. We only discuss a tiny part of his work in mathematics and logic.

Aristotle's Magnitudes

- Aristotle divided the idea of "quantity" into two kinds:
  - Numbers, which have indivisible units. ($\sim$ Discrete)
  - Magnitudes, which are "infinitely divisible" ($\sim$ Continuous)

  Thus Magnitudes have no points: Aristotle (Book 6, Chap.1):

  > *. . . no continuum can be made up out of indivisibles, granting that the line is continuous and the point indivisible.*

- Completely different than the set-theoretic *arithmetization* of analysis done in the 19th century, basing mathematics on set theory: e.g. a line is a collection of (set-theoretic) points.
- Euclid (Book V)–essentially due to Eudoxus– develops the theory of order and comparability for *ratios of magnitudes*, via rationals; related to the modern theory of Dedekind cuts.

Aristotle: the father of logic

- Systematically developed and analyzed logic (Syllogistics).
- We interpret this in modern set-theoretic language. Let $\mathcal{U}$ be a universal set, $A, B \subseteq \mathcal{U}$, where $\overline{B}$ is the complement $\mathcal{U} \setminus B$.

All A's are B's	$\forall x(A(x) \rightarrow B(x))$	$A \subseteq B$
No A's are B's	$\forall x(A(x) \rightarrow \neg B(x))$	$A \subseteq \overline{B}$
Some A's are B's	$\exists x(A(x) \wedge B(x))$	$A \cap B \neq \emptyset$
Some A's are not B's	$\exists x(A(x) \wedge \neg B(x))$	$A \cap \overline{B} \neq \emptyset$

- Basic statements and inferences were not written in symbolic language. Formal algebraic/symbolic language first introduced by G. Boole , A. de Morgan (1847) ; quantifiers and modern symbolic logic by Frege (1879)
- See discussion in Anglin-Lambek, Chap. 23.

Aristotle's Syllogistics

- Aristotle discussed various fundamental forms of logical inference (syllogisms). These were studied in detail by the medieval scholastics. These include four inference forms:

$$(1)\quad \frac{\begin{array}{c}\text{All } A\text{'s are } B\text{'s}\\ \text{All } B\text{'s are } C\text{'s}\end{array}}{\therefore \text{ All } A\text{'s are } C\text{'s}} \qquad (2)\quad \frac{\begin{array}{c}\text{No } A\text{'s are } B\text{'s}\\ \text{All } C\text{'s are } A\text{'s}\end{array}}{\therefore \text{ No } C\text{'s are } B\text{'s}}$$

$$(3)\quad \frac{\begin{array}{c}\text{All } A\text{'s are } B\text{'s}\\ \text{Some } C\text{'s are } A\text{'s}\end{array}}{\therefore \text{ Some } C\text{'s are } B\text{'s}} \qquad (4)\quad \frac{\begin{array}{c}\text{No } A\text{'s are } B\text{'s}\\ \text{Some } C\text{'s are } A\text{'s}\end{array}}{\therefore \text{ Some } C\text{'s are not } B\text{'s}}$$

- Modern Interpretation of Inference (3):

$$A \subseteq B, \text{ and } C \cap A \neq \emptyset \Rightarrow C \cap B \neq \emptyset.$$

- Discussed in great detail deepest questions of language and formal logic (today we'd say the theory of quantifiers: $\forall, \exists$). He clearly knew $\neg\forall x A(x) \equiv \exists x \neg A(x)$.
- Much of Aristotelian logic is based on the *Law of Excluded Middle* (sometimes called *Law of Excluded Third*) : For all propositions p, $(p \vee \neg p)$ holds: either p is true or $\neg p$ is true.
- Aristotle of course analyzed standard inferences:

$$\frac{\begin{array}{c}\text{All men are mortal}\\ \text{Socrates is a man}\end{array}}{\therefore \text{ Socrates is mortal}}$$

- Aristotle also knew about subtleties of modal logic: for example the logic of "possibility" and "necessity" (e.g. sentences such as "it is possible that ...." or "it is necessary that" .)

Aristotle: more meta-theory

- In class we discussed briefly Aristotle's famous analysis of

  There will be a sea battle tomorrow.

  –he discusses whether this sentence has a meaningful truth value at all. There is also a link to this on the webpage, at `http://www2.drury.edu/cpanza/aristotleseabattle.html`.
- We discussed Aristotle's observation of failure of substitution of equals for equals in modal contexts (e.g. I may believe $a = b$. But in fact (in the real world), suppose b is really c. That does not mean I believe $a = c$. No! Which examples were discussed in class, and Anglin-Lambek, and in Wei-Lu's notes?
- Aristotle's philosophy of math was heavily constructivist or even finitist. What does that mean? Examples he gave?

Stoics I

- Aristotle discussed syllogisms, quantifiers, and subtleties of logic in philosophy and natural language.
- Practical mathematicians wanted simpler inference forms. These laws were codified about 75–100 years later by the Stoics, e.g. Chrysippus (280-206B.C.) in terms of simple inferences. They certainly knew the idea of truth tables. Truth tables were later mentioned in the early 20th century by the philosopher Wittgenstein (1889–1951).
  The Stoics discussed, the following inference principles (Stanford Encyclopedia, Katz's book):
  1. if p then q; p; therefore q (modus ponens).
  2. if p then q; not-q; therefore not-p (modus tollens).
  3. it is not the case that both p and q; p; therefore not-q.
  4. either p or q; not-p; therefore q .(alternate syllogism)
  5. if p then q; if q then r; therefore if p then r (hypothetical syllogism).

Propositional Calculus

- Rather than Aristotle's complicated system, Greek mathematicians wanted simple propositional calculus and equational logic (substituting equals for equals).
- Propositional Calculus was introduced towards the end of the 19th century. Start with a set of Propositional Letters $p_1, p_2, p_3 \cdots$, which we write $p, q, r, \cdots$, etc. and connectives $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$.
- We interpret formulas as $\mathbf{T}, \mathbf{F}$, according to the *truth table rules.*
- It is easy to translate simple arguments and decide their validity in a mechanical manner, using truth tables.

Propositional Calculus II

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	**T**	**F**	**T**	**T**	**T**	**T**
T	**F**	**F**	**F**	**T**	**F**	**F**
F	**T**	**T**	**F**	**T**	**T**	**F**
F	**F**	**T**	**F**	**F**	**T**	**T**

See my notes on History of Logic on Blackboard.