Lectures 1 and 2

MAT3100 – History of Mathematics

Notes by Weiyun Lu
Department of Mathematics & Statistics, University of Ottawa

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With Ah'mose's 2/n table added at the end, plus some remarks on Egyptian fractions. P. Scott

Contents

Preface

These are notes taken when I was a student in MAT3100: History of Mathematics, taught by Prof. Philip Scott at the University of Ottawa in Fall 2013. They were later revised and updated when I was the TA for the same course in Winter 2016.

This document comprises mostly of material from lectures, supplemented (occasionally) with outside research or thoughts. As such, there is no guarantee of their accuracy or completeness (and in fact, many dates may differ between sources in the literature because we are only sure of them up to being in some neighborhood of some year), but anyone reading this is welcome to use it as a reference.

The material of the first three chapters is divided (somewhat arbitrarily) into three main chapters corresponding to mathematics before, during, and after the time of the Greeks. This seemed to make organizational sense, as Greek mathematics make up the bulk of the middle of our story.

A note on notation: I will use an integer system to note dates, for instance, the year 300 B.C. will be written as -300 and the year 200 A.D. will be written as +200. This makes it (at least to me) more intuitive to compute dates, time intervals, etc...

Disclaimer: The exact contents of the course may change slightly from year to year. For this reason (as well as the possibility of typos), if you are currently a student in MAT3100, you should **not** take these notes to be an alternative to attending the lectures and you should **not** assume that reading these notes is a necessary and sufficient condition for doing well on exams. If you do, you do so at your own risk, and I cannot be held responsible for any loss of grades, loss of sanity, financial loss, emotional trauma, or health problems that occur as a result of using these notes.

Chapter 1

Prehistory to the common era

1.1 The stone age

The *stone age* is a prehistoric era, beginning roughly in -5000000. and ending in -3000. It is subdivided into three ages; the *palaeolithic era*, the *mesolithic era*, and the *neolithic era*.

Palaeolithic era

The palaeolithic era, or the *old stone age*, lasted from approximately -5000000 to -10000. This time was marked by the movement of hominans from Africa to Europe. During this time, an evolutionary split occurred. The group that moved to Europe would eventually meet an evolutionary dead end (i.e. the *neanderthals*), who died out long ago. Modern Europeans, may, in fact, have around 2 to 4 percent neanderthal DNA. The group that stayed in Africa, on the other hand, would eventually evolve into the modern man.

Neanderthals & homo sapiens

Contrary to what many children's Saturday cartoons would have one believe, the neanderthals were actually quite intelligent. They had large brains, stood around 5 ft. to 5 ft. 3, and were 2 to 3 times as strong as the homo sapiens. However, they were not as fast and so were not able to chase after animals as efficiently, and naturally preferring cold climates, they were not able to adapt to the gradually warming climate. They did not live in societies. The last known remains of the neanderthals were found in a cave in Gibraltar, dating to about 30,000 years ago. Neanderthal art and paintings dating to this time have also been found.

Compared to the neanderthals, homo sapiens were faster (though weaker), but they formed societies with specialized roles (e.g. hunters). They were better at abstraction and held a stronger command of language. While it is not known whether the neanderthals died out naturally by their inability to adapt or whether the homo sapiens ended them in a violent conflict, ultimately the homo sapiens outcompeted the neanderthals for food and tools (i.e. whilst the neanderthals learned to use rocks, homo sapiens were able to craft tools such as spears and hammers, and were able to make clothes).

Ishango bone

The *Ishango bone* was found around -25000 in Africa. No number system yet existed, but this bone contained notches which have led to different mathematical theories as to their purpose. Some believe it to be the earliest table of prime numbers, while others believe it was man's first pass at charting the lunar cycle. Some progressive-thinking women, however, claim that men have no use for such things and that it was a woman who did this for the purposes of tracking menstrual cycles—and so according to them, the first mathematician would have been female.

Mesolithic era

The mesolithic era, or the *middle stone age*, lasted from approximately -10000 to -7000. This period in human history was marked by a hunter-gatherer society with a migratory lifestyle (i.e. they moved with the animals that they hunted). This meant that there was no time for building libraries or sitting and thinking, which limited their economic development. Housing had to be portable, and so existed largely in the form of tents and teepees. Societal roles were developed, (e.g. some men would be hunters, some women would tend to children). No cities yet existed, though basic arithmetic was being developed for mostly practical purposes, such as:

- Keeping track of the size of the tribe.
- Keeping track of the number of children (and wives or husbands?) that one had.
- Dividing the share of the hunt.

Neolithic era

The neolithic era, or the *late stone age*, lasted from approximately -7000 to -3000. During this period, agricultural societies began to form along with villages, owns, and social structure. Farmers planted seeds, cultivated crops, and domesticated animals. Science and mathematics began to develop. Some people (e.g. priests) did not have to work directly for their survival. They counted (lunar calendar, tides, etc...) and were supported by farmers for this work.

Metals (bronze, iron) were developed towards the end of this period, and much better tools became available. Mathematics was developed largely for practical reasons, including:

- Trade and commerce (i.e. exchanging resources like food and wood).
- Allotments of fields, grains.
- Weather/calendar predictions. (Someone who could accurately predict major events would suddenly find themselves in a position of power.)
- Atmospheric events.

Hence mathematics was used in the development of cities and advanced cultures. City dwellers were called “civilis” in Latin—this is the root word for “civilization”. As written records still did not exist, numbers were manifested and passed on in:

- Scratches in rocks, sticks, and bones.
- Oral tradition (i.e. songs).
- Knotted strings with intricate patterns.

Counting herds

In ancient times, if a farmer wished to know which of two flocks of sheep contained more sheep, he would send sheep in pairs through a set of posts, thus constructing a one-to-one correspondence, and thus the herd with leftover sheep would be the larger one. This idea was later generalized to infinite sets—Galileo observed that the set of positive integers and its subset of even positive integers could be put into one-to-one correspondence even though one was a proper subset of the other! This greatly confused him, but has now become the modern definition of “equal cardinality”.

1.2 Bases of notation

We count today in base 10. For instance, the number 105 represents $5 \cdot 10^0 + 0 \cdot 10^1 + 1 \cdot 10^2$. This leads to the following general definition:

Definition 1.2.1 (Base b, placeholder notation)**.** Let b be a positive integer. Then, numbers in base b are written in the form

$$a_0 \cdot b^0 + a_1 \cdot b^1 + \cdots + a_n \cdot b^n,$$

or in *placeholder notation*, in the form

$$(a_n a_{n-1} \ldots a_0)_b,$$

for some $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in \{0, 1, \ldots, b-1\}$.

Theorem 1.2.2. *Let b be a positive integer which is at least 2. Then, any positive integer a can be written uniquely in base b:*

$$a = \sum_{i=0}^{n} a_i b^i = a_0 + a_1 b + a_2 b^2 + \cdots + a_n b^n,$$

where $0 \leq a_i \leq b-1$ for all i.

Proof. This is a consequence of the *division algorithm* (see, for instance, [KR98, pp. 9-10]) which guarantees existence and uniqueness of integers q, r with $0 \leq r < b$ such that $a = qb + r$. We simply divide a by b, take the remainder to be a_0, divide the quotient q by b again, taking this remainder to be a_1, and iterate this process until we obtain a quotient of 0. This technique will be demonstrated explicitly in examples later in this section. □

We now describe some common bases that have been used throughout history:

(a) With $b = 2$, we have the *binary* system. Its uses:

- The language of computers today.
- The Chinese *Book of Changes*, around -1200 B.C., which had two symbols - yin for 0 (female) and yang for 1 (male). It was thought that 34, or $(100010)_2$ was an important number, representing progress and change.
- Used in Chinese puzzle books; in particular, the game of Nim. See [AL98, Chapter 2] for details.
- The Hindus, around -800, used binary to classify meters in verse.

(b) With $b = 5$, we have the *quinary* system. This corresponds to the fingers on one hand. Its uses:

- Historically used by some South American Indian tribes.
- German peasant calendars until the +1800s.
- A traditional measurement of heights of (race) horses, still in use today.
- In Japan, 5 is considered a sacred number and in buddhism, 5 represents the primal forces: wood, fire, earth, metal, water.

(c) With $b = 10$, we have the *decimal* system. This corresponds to the fingers on two hands. Its uses:

- The common decimal notation used today.
- Also used by ancient Egyptians, though with backwards (to us) positional notation!

(d) With $b = 12$, we have the *duodecimal* system. This is handy, as 12 has many divisors. The digits are represented by

$$\{0, 1, \ldots, 9, t, e\}.$$

Its uses:

- Twelve inches in a foot.
- Twelve eggs in a dozen.

(e) With $b = 16$, we have the *hexadecimal* system. The digits are represented by

$$\{0, 1, \ldots, 9, a, b, c, d, e, f\}.$$

Its uses:

- Various fields of engineering, where sophisticated algorithms to perform computations in hexadecimal have been developed.

- UNICODE, URLs (hex pairs).

(f) With $b = 20$, we have the *vigesimal* system. This corresponds to the fingers and toes on both hands and feet. Its uses:

  - Historically used by American Indians and Mayans.
  - A relic in modern French: 80 is *quatre-vingt*, or "four twenties". An exception to this is in Switzerland, where *huitante* or *octante* are used instead.
  - Similarly, the *score* in English corresponds to twenty. The earliest use is Psalm 90:10 in the King James version of the Bible: "The days of our lives are three score and ten years", meaning that the average human life span was 70 years. The most famous use is perhaps Abe Lincoln's Gettysburg address "Four score and seven years ago, our founding fathers etc etc..."

(g) With $b = 60$, we have the *sexigesimal* system. This was used by the Babylonians and also spills into certain areas of modern use. We will discuss this system further in the next section.

When unspecified, we assume that our numbers are in base 10.

Example 1.2.3. Consider $(12t)_{12}$. We wish to express this number in decimal notation. We have that

$$\begin{aligned}(12t)_{12} &= t \cdot 12^0 + 2 \cdot 12^1 + 1 \cdot 12^2 \\ &= 10 + 24 + 144 \\ &= 178.\end{aligned}$$

Example 1.2.4. Let's convert $(3t1e)_{12}$ to decimal notation.

$$\begin{aligned}(3t1e)_{12} &= e \cdot 12^0 + 1 \cdot 12^1 + t \cdot 12^2 + 3 \cdot 12^3 \\ &= 11 + 12 + 10 \cdot 12^2 + 3 \cdot 12^3 \\ &= 6647.\end{aligned}$$

Example 1.2.5. Now consider the opposite process. Suppose that we wish to write 198 in base 4. We have that

$$\begin{aligned}198 \div 4 &= 49 \text{ (remainder 2)}, \\ 49 \div 4 &= 12 \text{ (remainder 1)}, \\ 12 \div 4 &= 3 \text{ (remainder 0)}, \\ 3 \div 4 &= 0 \text{ (remainder 3)}.\end{aligned}$$

And so, reading backwards, we have that 198 is equivalent to $(3012)_4$. Note that in each successive line, we are essentially dividing the original number by successive powers of 4, stopping once we reach a quotient of 0, to find the value of each digit in placeholder notation.

Exercise 1.2.6. Find 2011 in base 6.

Base b addition & multiplication tables

Just as the addition and multiplication tables in base 10 that we all fondly remember from kindergarten, the same tables can be written for any base. Note that this is *not* the same thing as mod b arithmetic.

Example 1.2.7. In base 4 ,we have "digits" $\{0, 1, 2, 3\}$ giving the following addition table:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	10
2	2	3	10	11
3	3	10	11	12

and the following multiplication table:

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	10	12
3	0	3	12	21

1.3 Egyptian mathematics

Some of the early development of mathematics past the era of prehistory occurred in Egypt and Mesopotamia (inhabited first by the Sumerians, until they were conquered by the Babylonians in -1900). The Egyptians and the Babylonians, though in geographic proximity, had very little contact due to separation by the Nile river, so there are differences in ancient Egyptian and ancient Babylonian mathematics. Furthermore, Egypt was more of a unified country, whilst Babylon was comprised of city states who were constantly at war with each other. Hence, even within the states of Babylon, mathematics developed differently.

Some historical thoughts on the origins of mathematics were:

- The Greeks believed, according to [AL98], that mathematics started with the Egyptians (though we now know that it was developed independently and simultaneously by the Babylonians as well).
- Aristotle (-384 to -322) thought the priests had enough leisure time to devote to mathematics, and this is how it began.
- Herodotus (-484 to -425) believed annual flooding of the Nile required resurveying of property. The surveyors used ropes to determine the property lines.
- Democritus (-460 to -370), the first man to entertain the notion of the atom, called Egyptian mathematicians "rope stretchers".

Agriculture arose in the Nile valley around -5000. The first dynasty of Egypt arose around -3100—the rulers were called pharaohs, considered to be between mortals and gods. A powerful bureaucratic class existed, consisting of priests, officials, and scribes. The Egyptian calendar of 12 months per year and 365 days per year has survived to modern use. Two styles of writing were developed:

- Hieroglyphics for monuments; these were carved in large stones and pyramids.
- A different system, since hieroglyphics were not portable, called *hieratic*. It was used for documents, made of *papyrus*, and was a cursive language written using quills/brushes and ink.

When the Greeks eventually conquered the Egyptian civilization (Alexander the Great, -332), these writing styles disappeared. When King Tutankhamen's tomb was found in the +1920s, the problem of how to read these writings arose. Jean Champollion (+1790 to +1832) translated the *Rosetta Stone* (-196), which contained the same text in 3 languages (Greek, hieroglyphics, hieratics). This work took him around 20 years. He eventually found a bijection between the characters and modern language, and this in fact was the beginning of modern ideas in linguistics/translation.

There were two documents of mathematics:

- The *Moscow papyrus* (-1850, approximately the time of Abraham in the bible), is a 15-foot long scroll (but only 3 inches wide).
- The *Rhind papyrus* (-1650, written by the Egyptian scribe Ahmose, bought by a Scotsman named Rhind—so, seemingly named for the wrong person). Ahmose said he copied it from a document that was 200 years ago, and so was probably around the same age as the Moscow papyrus.

These were essentially Schaum's Outlines of ancient Egypt, comprising a number of exercises (presumably for students), in a "this is what to do" style without rigorous explanations.

Recall that the Egyptians used base 10 in reverse. The number $(a_n a_{n-1} \ldots a_0)_{10}$ for them was written $a_0 a_1 \ldots a_n$ (sometimes vertically on the scrolls). Symbols they used were:

- 1 - vertical stroke,
- 10 - hook,
- 10^2 - coiled rope,
- 10^3 - lotus flower,
- 10^4 - finger.

Egyptian arithmetic

Addition was done by concatenation (merging together) sequences, then carrying over when there was more than 10 of one symbol. They did not have a notion of 0, they would simply leave a blank space. This is more or less the same system we use to add, albeit with more cumbersome notation.

The Egyptians multiplied by using base 2 (i.e. doubling). If they wished to multiply 12 by 12, they would repeatedly double as such:

$$\begin{aligned} 1 - 12&,\\ 2 - 24&,\\ *4 - 48&,\\ *8 - 96&. \end{aligned}$$

Thus, since $12 = 4 + 8$, adding 48 and 96 gave the result, 144. This works, in principle, because these numbers can all be written uniquely in base 2! They had many tables (in this case, a 2-times table), which they likely memorized, from which this algorithm could be used.

Now if we multiply 26 by 33:

$$\begin{aligned} 1 - 33&,\\ *2 - 66&,\\ 4 - 132&,\\ *8 - 264&,\\ *16 - 528&. \end{aligned}$$

As $26 = 2 + 8 + 16$, we add $66 + 264 + 528$ to get the desired answer.

What is 184/17? We want to solve $17x = 184$. How the Egyptians did this was, just as with multiplication, repeated doubling:

$$\begin{aligned} 1 - 17&,\\ *2 - 34&,\\ 4 - 68&,\\ *8 - 136&. \end{aligned}$$

Since $2 + 8 = 170$, we have the answer of 10 with remainder 14. One possible way they may have done this (we don't know since they wrote their answers without explaining the reasoning) would first to be to subtract the final number, $184 - 136 = 48$, and then asking how to divide 48, upon which we subtract 34 from 48 and realize that the number we obtain is smaller than 17.

Example 1.3.1. Problem 69 of the Rhind papyrus: "Find the number of ro of flour in a loaf of bread, given 1120 ro = 80 loaves." So, using the method of doubling, this was solved by

$$\begin{aligned} &1 - 80, \\ *&2 - 160, \\ *&4 - 320, \\ *&8 - 640. \end{aligned}$$

Thus, the answer is 14.

Egyptian fractions

Definition 1.3.2 (Unit fraction)**.** A *unit fraction* is a fraction of the form $1/n$, $n \in \mathbb{N}^+$.

The Egyptians had a curious tradition with fractions. With the exception of $\frac{2}{3}$, they insisted that all fractions must be written as a sum of distinct unit fractions. They would write $\frac{1}{n}$ as a circle with n underneath it - we will represent this as $\overline{n}$. For instance:

$$\begin{aligned} \frac{2}{9} &= \frac{1}{6} + \frac{1}{18}, \\ \frac{19}{8} &= 2 + \frac{3}{8} = 2 + \frac{1}{4} + \frac{1}{8}. \end{aligned}$$

They had a rather impressive $2/n$ table found in the works of Ahmose, necessary for the doubling method. For instance:

$$\begin{aligned} \frac{2}{7} &= \frac{1}{4} + \frac{1}{128}, \\ \frac{2}{97} &= \frac{1}{56} + \frac{1}{679} + \frac{1}{776}. \end{aligned}$$

A modern open problem (Paul Erdos): If n is odd, $n > 4$, prove $\frac{4}{n}$ can be written as a sum of 3 distinct unit fractions.

Exercise 1.3.3. Check Erdos' conjecture for the cases $n = 5, 7, 9$ by finding distinct natural numbers n_1, n_2, n_3 such that $\frac{4}{n} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}$.

Example 1.3.4. Problem 3 of the Rhind papyrus: Divide 6 loaves of bread among 10 men. How much does each man get? We know obviously how to do this, as

$$\begin{aligned} 10l &= 6, \\ l &= \frac{6}{10}, \\ l &= \frac{1}{2} + \frac{1}{10}. \end{aligned}$$

Their approach was more interesting; they give an approximation, then modify it. Since 6 doesn't divide 10, but 5 does, we take one loaf away, cut the 5 loaves in half and give half a loaf to each man. Finally, they slice the remaining loaf and divide it evenly among the ten men. Ahmose would then check this with the doubling method.

$$
\begin{gathered}
\text{Men} - \text{Loaves of Bread} \\
1 - \overline{2} + \overline{10} \\
\ast 2 - 1 + \overline{5} \\
4 - 2 + \overline{3}\ \overline{15} \ast\ast \\
\ast 8 - 4 + \overline{\overline{3}}\ \overline{10}\ \overline{30} \ast\ast\ast
\end{gathered}
$$

$\ast\ast$ - look up in the $\frac{2}{n}$ table to find $\frac{2}{5} = \frac{6}{15} = \frac{5}{15} + \frac{1}{15} = \overline{3}\ \overline{15}$.
$\ast\ast\ast$ - $\overline{\overline{3}} = \frac{2}{3}$, and we have $\frac{2}{15} = \frac{4}{30} = \frac{3}{30} + \frac{1}{30} = \overline{10}\ \overline{30}$.

Example 1.3.5. Consider multiplication; $(1\ \overline{2}\ \overline{4}) \cdot (\overline{2}\ \overline{14})$. The modern method:

$$
\begin{aligned}
\left(1 + \frac{1}{2} + \frac{1}{4}\right) \cdot \left(\frac{1}{2} + \frac{1}{4}\right) &= \left(\frac{1}{2} + \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4}\right) + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{4}\right) \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{14} + \frac{1}{28} + \frac{1}{56}.
\end{aligned}
$$

Their method would have been the reverse process of the doubling method:

$$
\begin{gathered}
1 - \overline{2}\ \overline{14}, \\
\overline{2} - \overline{4}\ \overline{28}, \\
\overline{4} - \overline{8}\ \overline{56}.
\end{gathered}
$$

Example 1.3.6. Problem 69 of the Rhind Papyrus: Calculate $\frac{80}{3.5}$. The Egyptians would also sometimes, at the drop of a hat, multiply by 10 along with doubling. Ahmose would set up a table of 3.5s, the right column being reciprocals:

$$
\begin{array}{rr}
1 - 3\ \overline{2} & 6 - 21 \\
10 - 35 & \ast\ast\ \overline{21} - \overline{6} \\
\ast 20 - 70 & \ast\ast\ \overline{7} - \overline{2} \\
\ast 2 - 7 & \\
\ast \overline{\overline{3}} - 2 + \overline{3} &
\end{array}
$$

Taking the three $\ast$s, we have a total of $79\frac{1}{3}$. Taking the two $\ast\ast$s, we have $\frac{2}{3}$. To get 80, we have a total of $22\ \overline{\overline{3}}\ \overline{21}\ \overline{7}$.

The following example illustrates the *method of false position*.

Example 1.3.7. "A number and its seventh make 19". In modern notation,

$$x + \frac{1}{7}x = 19.$$

First, Ahmose guesses 7. Together, we get 8 which is less than 19. Then he multiplies by $\frac{19}{8}$ to get the right answer:

$$x = \frac{19}{8} \cdot 7 = \frac{133}{8}.$$

Checking with doubling tables:

$$\begin{aligned} 1 &- 8, \\ 2 &- 16, \\ 4 &- 32, \\ 8 &- 64, \\ 16 &- 128. \end{aligned}$$

Noting that at 16, we are pretty close to 133 (with a remainder of 5), so the answer is

$$\begin{aligned} 16 + \frac{5}{8} &= 16 + \frac{4}{8} + \frac{1}{8} \\ &= 16\ \overline{2}\ \overline{8}. \end{aligned}$$

Egyptians computed areas of rectangles, triangles, and trapezoids, as well of volumes of truncated pyramids. Oddly enough, they did not calculate the volumes of complete pyramids despite the fact that they were building them all the time.

Example 1.3.8. How would we calculate $\frac{5}{13}$ divided by 12? First, we have

$$\frac{5}{13} = \frac{2 \cdot 2 + 1}{13} = 2\left(\frac{2}{13}\right) + \frac{1}{13}.$$

Then,

$$\begin{aligned} \frac{2}{13} &= \frac{1}{8} + \frac{1}{52} + \frac{1}{104}, \\ 2\left(\frac{2}{13}\right) &= \frac{1}{4} + \frac{1}{26} + \frac{1}{52,} \\ 2\left(\frac{2}{13}\right) + \frac{1}{13} &= \frac{1}{13} + \frac{1}{4} + \frac{1}{26} + \frac{1}{52}. \end{aligned}$$

Then we multiply this number by $\frac{1}{12}$.

Exercise 1.3.9. Prove for any natural number m that

$$\frac{2}{2m+1} = \frac{1}{m+1} + \frac{1}{(m+1)(2m+1)}.$$

Example 1.3.10. Problem 50 of the Rhind papyrus: a round field has diameter 9; what is its area? Take away $\frac{1}{9}$ of the diameter. Remainder is 8. Multiply 8 times 8. Answer: 64. So, what Ahmose was saying was:

$$\begin{aligned} A &= \left(d - \frac{d}{9}\right)^2 \\ &= \left(\frac{8}{9}d\right)^2 \\ &= \frac{64}{81}d^2. \end{aligned}$$

Since we know

$$\begin{aligned} A &= \pi \left(\frac{d}{2}\right)^2 \\ &= \pi \cdot \frac{d^2}{4}, \end{aligned}$$

the approximation used was

$$\frac{256}{81} = \pi = 3.16049\ldots,$$

which is remarkably good!

1.4 Babylonian mathematics

We now turn our attention to the more interesting (in the sense that there was much conflict, as well as more interesting mathematics) area known as Mesopotamia. It contained two large rivers, the Tigris and the Euphrates. At the south where the two rivers met lay the major city of Ur.

Sumerians and Akkadians

Up to about -3500, this area was inhabited by the Sumerians. They were quite advanced, even by modern standards—they even had a postal system! Many invasions occurred in this area, roughly modern Iraq. Around -2300, the Akkadians had also attacked and taken over, but since they were more backwards they simply absorbed the Sumerian culture.

One thing the Akkadians did have, however, was a writing system. They wrote in *cuneiform* on clay tablets—while the data on our iPhones, iPods, and iWhatevers may be lost in n years, these tablets, baked in ovens, are quite durable and have survived the test of time. Around -2100, the Sumerians drove the Akkadians out. Finally around -1900, the Babylonians overthrew them, naming their capital city Babylon.

The Egyptian 2/*n* table, the recto table of the Ahmes (Rhind) papyrus

The Egyptian concept of fraction requires that any fraction be represented as a sum of unit fractions without any repetitions, except 2/3 which was allowed. Thus, for example, our common fraction 2/5 would be treated as a problem, not as an answer. The problem is to divide 2 by 5; the answer would be any sum of unit fractions without repetition. One answer is 1/3 + 1/15, the preferred answer. Another possible answer would be 1/4 + 1/10 + 1/20, but that's a more complicated answer having both more terms and larger denominators. Note that 1/5 + 1/5 would not be an answer because 1/5 is repeated.

The Egyptian algorithms for multiplication and division are based on addition, subtraction, and doubling. Therefore, one ingredient necessary to compute products and quotients involving fractions is a table of doubles of unit fractions. It's also necessary for addition since when adding two sums of unit fractions, some particular unit fraction might occur twice.

The back (recto) of the most important Egyptian mathematical papyrus, the Ahmes, or Rhind, papyrus, includes a table of doubles of unit fractions. We can call it a 2/*n* table. Here it is, transcribed into modern numerals. Note that only the denominators are listed in this transcription. In one column appears the denominator of the unit fraction to be doubled, and in the next column appear the denominators of the unit fractions for that double.

5	3 15	39	26 78	71	40 568 710
7	4 28	41	24 246 328	73	60 219 292 365
9	6 18	43	42 86 129 301	75	50 150
11	6 66	45	30 60	77	44 308
13	8 52 104	47	30 141 470	79	60 237 316 790
15	10 30	49	28 196	81	54 162
17	12 51 68	51	34 102	83	60 332 415 498
19	12 76 114	53	30 318 795	85	51 255
21	14 42	55	30 330	87	58 174
23	12 276	57	38 114	89	60 356 534 890
25	15 75	59	36 236 531	91	70 130
27	18 54	61	40 244 488 610	93	62 186
29	24 58 174 232	63	42 126	95	60 380 570
31	20 124 155	65	39 195	97	56 679 776
33	22 66	67	40 355 536	99	66 198
35	30 42	69	46 138	101	101 202 303 606
37	24 111 296				

At first glance, the only apparent regularity in the table occurs for denominators divisible by 3, and for those the rule is:

$$\frac{2}{3n} = \frac{1}{2n} + \frac{1}{6n}$$

Upon further analysis, you can perceive other principles used in constructing the table.

(taken from a table of Prof. David Joyce, Clark University)

Egyptian Fractions: Some remarks.

Students often ask: how can we find Egyptian fractions? The answer is:

(1) Trial and error

(2) Using Ah'mose's 2/n table on our website.

(3) Using a few of the formulas mentioned in class or various books on reserve.

Example: (i) obviously any number of the form $3/n = 2/n + 1/n$. You can then use Ah'mose's 2/n table — provided the final answer you get has distinct unit fractions.

(ii) $3/7 = 2/7 + 1/7 \underset{\text{Ah'mose}}{=} \overline{4}\ \overline{28}\ \overline{7}$

(iii) $5/13 = 4/13 + 1/13 = 2(2/13) + 1/13$. Using Ah'mose's table above:

$5/13 = 2(1/8 + 1/52 + 1/104) + 1/13$

$= \overline{4}\ \overline{26}\ \overline{52}\ \overline{13}$ in Egyptian notation.