MAT 3100: Homework 2

Prof. P.J. Scott, Feb.1, 2016

Due: Tuesday, Feb. 9, by 4:00

(in the submission box for MAT3100, at the entrance of the Math Building)

1. (Study problem: not to hand in)

   (i) Verify that 1184 and 1210 are amicable numbers. (This new pair of "small" amicable numbers was only noticed in 1866 by a sixteen-year-old Italian student Nicolo Paganini, more than 2,360 years after Pythagorus !)

   (ii) Let a, b be positive integers. Prove that $2^a - 1$ divides $2^{ab} - 1$. Hence it follows that $2^m - 1$ can only be prime when m is prime. (This will arise in class discussing Euclid's theorem for perfect numbers).

2. (i) Recall the Greek's geometric interpretations of algebraic identities. Prove geometrically that $(2q + 1)^2$ has the form $4s + 1$.

   (ii) Use (i) to prove that every integer square leaves remainder 0 or 1 on division by 4.

   (iii) Prove that if (a, b, c) is a Pythagorean triple, then a and b cannot both be odd. (use (ii)).

3. (a) Suppose p is a prime. Prove $\sqrt{p}$ is irrational.

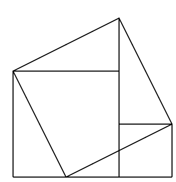
   (Hint: Recall, a number p is prime if its only divisors are 1 and p. We write $m \,|\, n$ to say "m divides n". Hence

   $$m \,|\, n \text{ if and only if } \textit{for some integer } k > 0, n = km$$

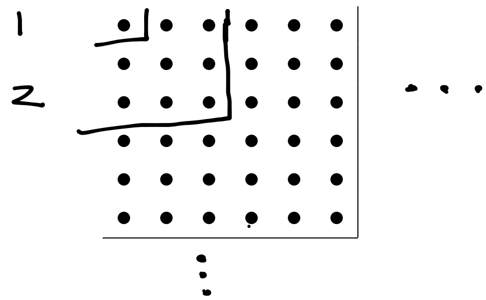
   Follow the proof in class that $\sqrt{2}$ is irrational. We prove it by contradiction. So suppose p is a prime and that $\sqrt{p} = m/n$ is rational, where $gcd(m, n) = 1$. You can use the following **Fact:** for positive integers a, b, if $p \,|\, ab$ then $p \,|\, a$ or $p \,|\, b$.

   (b) Prove that $log_{10}2$ is irrational. (Recall from calculus that $log_{10}(-)$ is the inverse function to $10^{(-)}$.)

4. Consider the following diagram, due to Henry Perigal (1873). By adding extra figures, rearranging and comparing, show how this gives another proof of Pythagorus's Theorem. Be precise (for example, just because our drawing makes two triangles look identical, doesn't prove it!).



5. Prove that for each $n \geq 1$, the sum of the first n cubes is the square of sum of the first n numbers. Prove it two ways: (i) Mathematical Induction and (ii) Stare at the following array, and give a geometric proof:



6. Prove the converse of Pythagorus's theorem: if a triangle has sides of lengths a, b, c and if $a^2 + b^2 = c^2$ then the triangle is a right triangle.

   Hint: Here is a simple method, using Cartesian coordinates and algebra. Draw the following triangle: three points P, Q, R with coordinates $P(0,0)$, $Q(x,y)$ and $R(b,0)$. Suppose the length of PQ is a, the length of QR is c. Obviously, the length of PR is b. Suppose $a^2 + b^2 = c^2$. The goal is to prove triangle PQR is a right triangle. Draw a perpendicular from Q to the line PR. Call the point of intersection X. What are the coordinates of X? Find algebraic relationships between the sides of the two right triangles PQX and XQR. (You may assume Pythagorus's theorem is already proved).

7. In the figure given in class (from Diophantus) used to construct rational points on a circle[1] consider the line $y = t(x+1)$, where t is rational, and passing through two points on the unit circle (around the origin O): the point $P(-1,0)$ and the point $R(x,y)$, say in the first quadrant. Consider the two angles given by RP and RO with the x-axis. Call them ψ and θ, respectively. Show that $\psi = \theta/2$.

8. Extra Credit:

   Generalize the proof in (3)(b) to: $log_a(b)$ is irrational if a and b are positive integers and one of them contains a prime factor not contained in the other. Here $log_a(\ )$ is the inverse function to $a^{(\ )}$.

[1] [taken from Stillwell, in my edition it's on p.7]