MAT3100: Base and Number Systems

P. Scott   Fall, 2013

In Lecture 1 we discussed number systems and bases. References: Eves, Chap.1, Anglin-Lambek, Chap.2, my class notes.

Let $\mathbb{N}^+ = \{1, 2, 3, 4, ...\}$ be the positive natural numbers. Let $b \geq 2$ be a number in $\mathbb{N}^+$.

Fact: Every number $a \in \mathbb{N}^+$ can be uniquely written in *base b* (sometimes called *radix b*)

$$a = \sum_{i=0}^{n} a_i b^i = a_0 + a_1 b + a_2 b^2 + \cdots + a_n b^n. \tag{1}$$

where $0 \leq a_i \leq b - 1$, for $i = 0, 1, \cdots, n$. We use the notation

$$a = (a_n a_{n-1} a_{n-2} \cdots a_2 a_1 a_0)_b \tag{2}$$

This is familiar *place* or *positional* notation: a_0 is the number of units (or b^0's), a_1 is the number of b's, a_2 is the number of b^2's, etc. *We will always assume our numbers are written in positional notation, unless stated otherwise.*

If $b = 10$, this is the usual decimal notation we use daily. **Note:** the number $a \in \mathbb{N}^+$ is always assumed written in the usual base 10 (positional) notation, unless we state otherwise.

In class, we discussed (or will discuss) various base systems: $b = 2, 5, 10, 12, 16, 20, 60$. Make sure you know the names and a bit of the history of these systems (Base 60 will be discussed as part of the Babylonian tradition).

Conversion between base 10 and other systems

Given a number a in base b assumed written according to (2), just use (1) to convert it to base 10. For example: $(1102)_5 = 2{\cdot}5^0 + 0{\cdot}5^1 + 1{\cdot}5^2 + 1{\cdot}5^3 = 2 + 25 + 125 = 155$ (in usual base 10). In class, we discussed $(12t)_{12}$ in the base 12 system (with base 12 "digit symbols" $\{0, 1, 2, \cdots, 9, t, e\}$), where the symbol t denotes 10 and e denotes 11 (in usual base 10).

$$\begin{aligned} (12t)_{12} &= t{\cdot}12^0 + 2{\cdot}12^1 + 1{\cdot}12^2 \\ &= t + 24 + 144 = 10 + 24 + 144 = 178 \quad \text{(base 10)} \end{aligned}$$

In the other direction, to find a base b representation of a positive integer a, we must solve $a = (x)_b$ for some sequence of base-b "digits" x.

As an example, what is a base 4 representation of 198? The method is to repeatedly divide by 4, and get a sequence of remainders. When you read these remainders in reverse order, this gives the base b "digits" in the representation :

$$\begin{array}{ll} 4|\underline{198} & \\ \;4|\underline{49} & \text{rem } 2 \\ \;\;4|\underline{12} & \text{rem } 1 \\ \;\;\;4|\underline{3} & \text{rem } 0 \\ \;\;\;\;\;0 & \text{rem } 3 \end{array}$$

gives the base 4 representation $(3012)_4 = 198$. Exercise: check this is correct, by using equations (1),(2) above.

Exercises: Using repeated division and remainders as above, (i) find a base 6 representation of 2011, (ii) find a base 12 representation of 6647, where the base-12 digits are $\{0, 1, ..., 9, t, e\}$.