

# Dynamic Auction: a Tractable Auction Procedure\*

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## Abstract

Dynamic auctions are trading mechanisms for discovering market-clearing prices and efficient allocations based on price adjustment processes. This paper studies the computational issues of dynamic auctions for selling multiple indivisible items. Although the decision problem of efficient allocations in a dynamic auction in general is intractable, it can be solved in polynomial time if the economy under consideration satisfies the condition of Gross Substitutes and Complements, which is known as the most general condition that guarantees the existence of Walrasian equilibrium. We propose a polynomial algorithm that can be used to find efficient allocations and introduce a double-direction auction procedure to discover a Walrasian equilibrium in polynomial time.

## Introduction

Auction processes have been a well-established research theme in economics and recently become an emerging research topic in AI due to a set of related computational challenges (Cramton *et al.* 2006). It is well-known that the problem of winner determination in a combinatorial auction is NP-complete (Rothkopf *et al.* 1998; Sandholm 2002). However, most of the discussions on the computational issues of combinatorial auctions are based on one-shot sealed-bid mechanisms. This paper aims to make a contribution towards the discussions on dynamic procedures of combinatorial auctions.

Dynamic auctions refer to any auction mechanisms based on price adjustment process (Gul and Stacchetti 2000; Ausubel 2006). A dynamic auction can be described as a rule for adjusting prices given buyers' demand correspondences (i.e., bids) and a rule for specifying an allocation (i.e., determining who gets the goods) (Gul and Stacchetti 2000). Assume that a seller wishes to sell a set of indivisible items to a number of buyers. The seller announces the current prices of the items and the buyers respond by reporting the bundles of items they wish to buy at the given prices. The seller then calculates the excess demand and increases or decreases the prices according to whether the excess demand is positive or negative. This iterative process continues until

all the selling items can be sold at the prices at which the demand is balanced.

There are a few reasons to believe that a dynamic auction mechanism can be "easier" than a one-shot combinatorial auction. On the one hand, from the buyer's perspective, a buyer only has to decide which items to buy at given prices at each round without requiring to consider different bundle-price combinations. Therefore the bidding language can be simpler. On the other hand, from the seller's point of view, the process of discovering market-clearing prices has been split into a sequence of decision-making rounds, therefore its complexity could be resolved to certain extent. However, as we will see, the core decision-making problems in dynamic auctions are proved to be computationally intractable if we do not apply any restrictions on buyers' value functions.

Different from one-shot combinatorial auctions, the main issue of a dynamic auction is whether the procedure can lead to an equilibrium state (Walrasian equilibrium) at which all the selling items are effectively allocated to the buyers (equilibrium allocation) and the price of each bundle of items gives the buyers their best values (equilibrium price). It has been observed that without certain assumptions on buyers' value functions, there is no guarantee for a dynamic auction to converge toward an equilibrium (Gul and Stacchetti 1999). Kelso and Crawford (1982) proposed a condition, named *gross substitutes* (GS), on buyers' value functions, which guarantees the existence of Walrasian equilibria. Gul and Stacchetti (1999) introduced two alternative conditions that are equivalent to GS and demonstrated that if no complementarities are allowed, these conditions are "almost" necessary to guarantee a Walrasian equilibrium. Sun and Yang (2006; 2009) extended the condition to *gross substitutes and complements* (GSC), which allows the present of complementarities. A set of procedures for finding Walrasian equilibria were developed by (Gul and Stacchetti 2000; Ausubel 2006) (for the economies with GS) and by (Sun and Yang 2009) (for the economies with GSC). However, none of these procedure is polynomial. In this paper, we present a dynamic auction procedure to discover Walrasian equilibria for any GSC economy in polynomial time.

The paper is organized as follows: the next section introduces the basic concepts and problems of the market model

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and show the problem of dynamic auction in general is NP-complete. Section 3 presents two key results on GSC: a demand correspondence under GSC is the base of a matroid. Sections 4&5 present a combinatorial optimization algorithm, which can be used to solve the problems of efficient allocation and price adjustment. Section 5 presents our auction algorithm and prove its correctness and convergence. Finally we conclude the work with a short remark on the related work.

## The market model

Consider a market situation where a seller wishes to sell a finite set of indivisible items to a finite number of buyers. Each buyer has a private value over each bundle of items. Formally, let  $E = (N \cup \{0\}, X, \{u_i\}_{i \in N})$  be an *economy*, where  $N = \{1, 2, \dots, n\}$  is the set of buyers, 0 represents the seller,  $X$  is the set of items and  $u_i$  the buyer  $i$ 's value function. We assume that the seller values each bundle of items at zero. Any subset  $A$  of items, i.e.,  $A \subseteq X$ , is called a *bundle*. We use  $|Y|$  to denote the number of elements in the set  $Y$ .

A *price vector*  $\mathbf{p}$  is a function  $\mathbf{p} : X \rightarrow \mathbb{R}^+$  that assigns a non-negative real number to each item in  $X$ . For each  $a \in X$ , we write  $p_a$ , instead of  $\mathbf{p}(a)$ , to indicate the price of item  $a$  under the price vector  $\mathbf{p}$ .

## Bidder's valuation

We assume that each buyer  $i$  has an integer value function, i.e.,  $u_i : 2^X \rightarrow \mathbb{Z}^+$ , which assigns each bundle  $A$  of items an integer  $u_i(A)$  (in the unit of money) as the buyer's valuation to the bundle with  $u_i(\emptyset) = 0$ . Following (Gul and Stacchetti 2000), we assume that each buyer has a quasilinear preference. We also assume that each buyer's value function is monotonic, i.e., for all  $A \subseteq B \subseteq X$ ,  $u_i(A) \leq u_i(B)$ .

In addition, the following standard notions will be used throughout the paper (Gul and Stacchetti 2000; Ausubel 2006; Sun and Yang 2009):

- *Indirect utility*:  $V_i(\mathbf{p}) = \max_{A \subseteq X} (u_i(A) - \sum_{a \in A} p_a)$ .
- *Demand correspondence*:  $D_i(\mathbf{p}) = \arg \max_{A \subseteq X} (u_i(A) - \sum_{a \in A} p_a)$ .
- *Minimum demand correspondence*:  $D_i^*(\mathbf{p}) = \{A \in D_i(\mathbf{p}) : |A| \leq |B| \text{ for all } B \in D_i(\mathbf{p})\}$ .
- *Lyapunov function*:  $L(\mathbf{p}) = \sum_{a \in X} p_a + \sum_{i \in N} V_i(\mathbf{p})$ .

In the sequent, whenever we consider a single buyer, we drop the subscript of the buyer from all the notations we introduced above for the sake of simplicity.

## Efficient allocations and Walrasian equilibria

Given an economy  $E = (N \cup \{0\}, X, \{u_i\}_{i \in N})$ , the purpose of auction is to allocate the items in  $X$  to the buyers. An *allocation* of  $X$  is a function  $\pi : N \cup \{0\} \rightarrow 2^X$  such that  $\pi(i) \cap \pi(j) = \emptyset$  for all  $i, j \in N \cup \{0\}, i \neq j$  and  $\bigcup_{i \in N \cup \{0\}} \pi(i) = X$ . Note that  $\pi(0)$  represents the unsold items. We say an allocation  $\pi^*$  of  $X$  is *efficient* if  $\pi^*(0) = \emptyset$

and  $\sum_{i \in N} u_i(\pi^*(i)) \geq \sum_{i \in N} u_i(\pi(i))$  for every allocation  $\pi$  of  $X$ . We let  $R(N) = \sum_{i \in N} u_i(\pi^*(i))$ .

**Definition 1** A Walrasian equilibrium of the economy  $E$  is a pair  $(\mathbf{p}, \pi)$ , where  $\mathbf{p}$  is a price vector and  $\pi$  is an allocation of  $X$  such that  $\pi(0) = \emptyset$  and  $\pi(i) \in D_i(\mathbf{p})$  for all  $i \in N$ .

If  $(\mathbf{p}, \pi)$  is a Walrasian equilibrium of  $E$ , we call  $\mathbf{p}$  an equilibrium price vector and  $\pi$  an equilibrium allocation. Obviously any Walrasian equilibrium allocation is efficient.

## Dynamic auction: an informal discription

Assume that a seller sells a set of indivisible items to a number of buyers using the following procedure.

- (1) Initially set the price vector  $\mathbf{p}$  to a starting price vector  $\mathbf{p}^0$ .
- (2) Ask each buyer  $i$  to report her demand  $D_i(\mathbf{p})$ .
- (3) The seller makes a decision to the following problems:
  - i. determine if an efficient allocation exists. If yes, stop.
  - ii. determine which items have excess demand (positive or negative). Reset the prices of the items and go to (2).

Obviously the complexity of the above procedure mainly lies in two decision-making problems in step (3), which are referred to as the *Efficient Allocation Problem* (EAP) and the *Price Adjustment Problem* (PAP), respectively.

**Proposition 1** The efficient allocation problem is NP-complete.

With respect to PAP, the following result is given by (Sun and Yang 2009).

**Proposition 2** (Sun and Yang 2009) Given an economy  $E$ ,  $\mathbf{p}$  is a Walrasian equilibrium price vector if and only if it is a minimizer of the Lyapunov function  $L$  with its value  $L(\mathbf{p})$  equal to  $R(N)$ .

According to Gul and Stacchetti (1999, Lemma 6) and the above Proposition 1, minimizing Lyapunov function is also an NP-hard problem.

## Gross substitutes and complements condition

As we have mentioned in the introduction, even if we would have devised a tractable procedure for dynamic auctions, we cannot guarantee that the procedure converges to a Walrasian equilibrium without restrictions on buyers' value functions (Gul and Stacchetti 1999). However, Gul and Stacchetti (2000) showed that if an economy satisfies Gross Substitutes (GS), then a Walrasian equilibrium exists. Note that under GS, no complementarity among items is allowed<sup>1</sup>. The problem of designing dynamic auction mechanisms for economies with complementarities has been a major challenge in auction theory (Sun and Yang 2006; 2009). A large amount of real-world problems, such as the sale of computers and software packages to customers, the allocation of workers and machines to firms, the assignment of takeoff and landing slots to airlines, and so on, require

<sup>1</sup>Roughly speaking, two items are substitutable to a buyer if the combination of two does not give her extra value; otherwise, they are complementary.

such a mechanism. To solve the problem, Sun and Yang (2006) extended GS into the following condition.

**Definition 2** (Sun and Yang 2006)<sup>2</sup> A value function  $u$  of a buyer satisfies the condition of *gross substitutes and complements* (GSC) w.r.t. a partition  $(X^1, X^2)$  of  $X$  if for any price vectors  $\mathbf{p}$  and  $\mathbf{q}$  with  $p_a \leq q_a$  for all  $a \in X^j$  ( $j = 1$  or  $2$ ) and  $p_a \geq q_a$  for all  $a \in (X^j)^c$ , and any bundle  $A \in D(\mathbf{p})$ , there exists a bundle  $B \in D(\mathbf{q})$  such that  $\{a \in A \cap X^j : q_a = p_a\} \subseteq B$  and  $\{a \in A^c \cap (X^j)^c : q_a = p_a\} \subseteq B^c$ . Here  $A^c$  represents the complement of  $A$ .

It is easy to see that GS is a special case of GSC when either  $X^1$  or  $X^2$  is empty. To understand GSC, it is better to gain an intuition of GS. GS means that if the prices of items were increased, the buyer would still want to buy the items the prices of which have not increased. GSC then says that if all the selling items can be divided into two categories, say software and hardware, increasing the prices of items in one category and decreasing the prices of items in the other category would not affect the demand of other items which prices remain the same.

We say an economy  $E = (N \cup \{0\}, X, \{u_i\}_{i \in N})$  has the GSC property w.r.t. a partition  $(X^1, X^2)$  of  $X$  if each buyer's value function satisfies GSC w.r.t.  $(X^1, X^2)$ .

Sun and Yang (2009) shows that any economy that satisfies GSC has a Walrasian equilibrium. In addition, Sun and Yang devised a dynamic auction procedure with double tracks (prices of the items in  $X^1$  increase and prices of the items in  $X^2$  decrease), which can discovery a Walrasian equilibrium. Unfortunately the procedure is not polynomial. In this paper we will propose a new dynamic procedure which is polynomial and also guarantees to converge to a Walrasian equilibrium. Before we proceed, let us present a few technique lemmas.

**Lemma 1** Suppose  $u$  has the GSC property and fix a price vector  $\mathbf{p}$ . For any  $A \in D(\mathbf{p})$ ,  $B \in D^*(\mathbf{p})$ , if  $U \subseteq B \setminus A$ , there exists a bundle  $T \subseteq A \setminus B$  such that  $(B \setminus U) \cup T \in D^*(\mathbf{p})$ .

Gul and Stacchetti (2000) showed that if an economy satisfies GS, then a Walrasian equilibrium exists. The key technique they used is matroid theory (Schrijver 2004). We will use the same technique to develop our algorithm.

A *matroid* is a pair  $(X, I)$  where  $X$  is a finite set and  $I$  is a set of subsets of  $X$  such that (i).  $\emptyset \in I$ , (ii)  $A \subseteq B \in I$  implies  $A \in I$ , and (iii)  $A, B \in I$  and  $|A| < |B|$  implies that there is an  $a \in B \setminus A$  such that  $A \cup \{a\} \in I$ . A set  $A \in I$  is called a *base* if  $A$  is maximal in  $I$  with respect to set inclusion. See (Schrijver 2004) for more details.

**Lemma 2** Suppose  $u$  has the GSC property. For any price vector  $\mathbf{p}$ , the pair  $(X, \bigcup_{B \in D^*(\mathbf{p})} 2^B)$  is a matroid, where  $2^B$  is the power set of  $B$ . Moreover, the set of bases of the matroid is  $D^*(\mathbf{p})$ .

One of the important properties of a matroid we will frequently use is: for any  $A, B \in D^*(\mathbf{p})$ , if  $a \in A \setminus B$ , then  $(A \setminus \{a\}) \cup \{b\} \in D^*(\mathbf{p})$  for some  $b \in B \setminus A$  ((Schrijver 2004) Theorem 39.6).

<sup>2</sup>In (Sun and Yang 2006), GSC was stated in a different way which is equivalent to this form (Sun and Yang 2009).

## Graph representation of demand situations

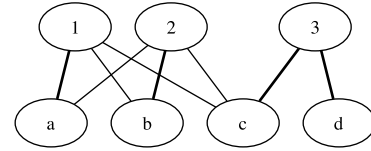
In this section, we show that the problem of finding efficient allocations can be converted into a pure combinatorial optimization problem.

### Demand situations and NX graphs

Given an economy  $E = (N \cup \{0\}, X, \{u_i\}_{i \in N})$ , we call  $\mathcal{D} = (D_i)_{i \in N}$  a *demand situation* of  $E$  if  $D_i \subseteq 2^X$  for all  $i \in N$ . Obviously for any price vector  $\mathbf{p}$ ,  $(D_i(\mathbf{p}))_{i \in N}$  is a demand situation of  $E$ .

We can represent a demand situation with a bipartite. Let  $G^{NX} = (N \cup X, E^{NX})$  be a graph where  $E^{NX} = \{(i, a) : i \in N \text{ and } a \in A \text{ for some } A \in D_i\}$ . We call  $G^{NX}$  the *NX graph* of the demand situation.

**Example 1** Let  $N = \{1, 2, 3\}$  and  $X = \{a, b, c, d\}$ . The demand correspondences of the buyers at price  $\mathbf{p}$  are  $D_1 = \{\{a\}, \{b, c\}\}$ ,  $D_2 = \{\{a, b\}, \{c\}\}$ ,  $D_3 = \{\{c\}, \{c, d\}\}$ .



One may think that it would be “more accurate” if we represent a demand situation by using an AND-OR graph because obviously an NX graph cannot represent the full information of its corresponded demand situation (for instance, the graph does not recognize the difference between  $D_1$  and  $D_2$ ). However, we will find that the NX graph representation is more convenient and useful if the demand situation is a collection of matroids.

### Quasi-matching

Let  $G^{NX} = (N \cup X, E^{NX})$  be the NX graph of a demand situation  $\mathcal{D} = (D_i)_{i \in N}$ . A set  $M$  of edges in  $G^{NX}$  is a *matching* if all edges in  $M$  are pairwise disjoint (no endpoint in common).  $M$  is said to be a *quasi-matching* in  $\mathcal{D}$  if

1. for any  $(i, a), (j, b) \in M$ ,  $a = b$  implies  $i = j$ , that is, no endpoint in  $X$  is in common.
2. for each  $i \in N$ , there is a  $B \in D_i$  such that  $\{a \in X : (i, a) \in M\} \subseteq B$ .

It is not hard to see that a quasi-matching determines an allocation of  $X$ . We write  $\pi^M$  to denote the allocation that is determined by  $M$ , that is,

$$\pi^M(i) = \{a \in X : (i, a) \in M\} \text{ for all } i \in N, \\ \pi^M(0) = X \setminus \bigcup_{i \in N} \pi^M(i).$$

**Example 2** Consider the demand situation in Example 1 again. The bold lines in the graph shows a quasi-matching, determines the allocation:  $\pi(0) = \emptyset$ ,  $\pi(1) = \{a\}$ ,  $\pi(2) = \{b\}$  and  $\pi(3) = \{c, d\}$ .

A quasi-matching  $M$  is said to be *maximum* if (i) it contains a maximum cardinality matching of  $G^{NX}$  and (ii) if  $M'$  is a quasi-matching,  $M \subseteq M'$  implies  $M = M'$ . Note that condition (i) will play an important role, which guarantees that each buyer can be allocated at least one item if it is possible meanwhile there is a demand from this buyer.



## Maximum quasi-matching algorithm

In this section, we develop a polynomial algorithm to solve the problem of efficient allocation. We will see also in the next section that with this algorithm the problem of price adjustment can also be solved in polynomial time.

The standard technique in graph theory to find maximum matchings is augmenting. We will use the same technique to find a maximum quasi-matching. In stead of finding alternating paths in the NX graph, we need to extend the NX graph into a directed graph to specify alternating paths.

### Extended NX graph

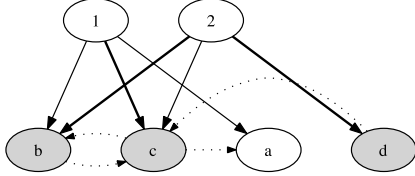
Given a demand situation  $\mathcal{D} = (D_i)_{i \in N}$  and a buyer  $i$ , we say that  $x$  is *substitutable* by  $y$  w.r.t.  $A$ , denoted by  $[A, x, y]_i$ , if  $x \in A$ ,  $y \in X \setminus A$  and there is a bundle  $B \in D_i$  such that  $A \setminus \{x\} \cup \{y\} \subseteq B$ .

Let  $M$  be a quasi-matching in the NX graph  $G^{NX} = (N \cup X, E^{NX})$ . We extend  $G^{NX}$  into a digraph  $G^M = (N \cup X, E^{NX} \cup (\bigcup_{i \in N} E_i^M))$  by adding the following arcs (directed edges) to  $G^{NX}$ :

$$E_i^M = \{(a, b) : a \in \pi^M(i), b \in X \setminus \pi^M(i) \text{ and } [\pi^M(i), a, b]_i\}$$

Meanwhile, each edge  $(i, a) \in E^{NX}$  is converted to a directed edge with the direction from  $N$  to  $X$ .

**Example 3** Let  $N = \{1, 2\}$  and  $X = \{a, b, c, d\}$ . Consider a demand situation  $\mathcal{D} = (D_i)_{i \in N}$  where  $D_1 = \{\{a\}, \{b\}, \{c\}\}$  and  $D_2 = \{\{b, c, d\}\}$ . Assume that  $M = \{(1, c), (2, b), (2, d)\}$ .



The figure shows the extended NX graph of the demand situation w.r.t.  $M$  (bold arrows represent the quasi-matching, shaded nodes stand for the allocated items and dotted arrows show the extended arcs).

For the readers who are familiar with matroid theory, it is easy to find the similarity between the extended NX graphs and the graphs of matroid union (see (Schrijver 2004) Section 42.3). However, augmenting a quasi-matching is more complicated than augmenting a matroid union.

A path  $P = (a_0, a_1, \dots, a_k)$  in the extended NX graph  $G^M$  is called an *M-augmenting path* if

1.  $a_j \in X$  for all  $j = 0, 1, \dots, k$ .
2. there is an  $i_0 \in N$  such that  $\pi^M(i_0) \cup \{a_0\} \subseteq B$  for some  $B \in D_{i_0}$ .
3.  $a_k \in \pi^M(0)$ .

We then augment  $M$  along  $P$  via the following procedure:

1. Set  $M' := M$ .
2. For each  $j = k, \dots, 1$ , set  
 $M' := (M' \setminus \{(i_j, a_{j-1})\}) \cup \{(i_j, a_j)\}$   
 where  $i_j$  is the unique buyer such that  $a_{j-1} \in \pi^{M'}(i_j)$ .

3.  $M' := M' \cup \{(i_0, a_0)\}$ .

The outcome  $M'$  of the augmenting process is denoted by  $M \bowtie P$ .

Note that the key difference between augmenting a quasi-matching and augmenting a matroid union is that to augment a matroid union, we only need to find a new element that can be added to the union while to augment a quasi-matching, we not only need to allocate an unallocated item but also have to redirect the existing allocation to swap items between different bundles in a demand correspondence. The idea of the above procedure is: in order to allocate a unallocated item  $a_k$ , if all buyers who demand this item have received a full bundle, we rearrange the allocation through a sequence of swapping so that a buyer who has not received a full bundle can get an item after a few swaps. This procedure is done backward along the augmenting path. Note that the special property of matroid allows us to switch between different bundles in a demand correspondence by swapping a single element (see (Schrijver 2004) Section 39.5).

Consider the demand situation and the quasi-matching in Example 3. There is an M-augmenting path  $P = (c, a)$ . After the M-augmentation by  $P$ , the quasi-matching becomes  $\{(1, a), (2, c), (2, b), (2, d)\}$ , which determines an efficient allocation.

**Theorem 1** Let  $\mathcal{D} = (D_i)_{i \in N}$  be a demand situation such that for all  $i \in N$ ,  $D_i$  is the set of bases of a matroid on  $X$ . For any quasi-matching  $M$  and any M-augmenting path  $P$ ,  $M \bowtie P$  is a quasi-matching and  $|M \bowtie P| = |M| + 1$ .

### The algorithm

After the standard setting of the augmenting technique, we are now ready to present our algorithm of calculating a maximum quasi-matching in an NX graph.

#### MAXIMUM QUASI-MATCHING ALGORITHM

*Input:* A demand situation  $\mathcal{D} = (D_i)_{i \in N}$ .

*Output:* A maximum quasi-matching  $M$ .

- (1) Construct the NX graph  $G^{NX}$  of  $\mathcal{D}$ .
- (2) Calculate a maximum matching  $M^0$  in  $G^{NX}$  and set  $M := M^0$ .
- (3) Set  $S_M := S_1 \cup \dots \cup S_n$ ;  $T_M := \pi^M(0)$ , where  
 $S_i := \{a \in X \setminus \pi^M(i) : \pi^M(i) \cup \{a\} \subseteq A \text{ for some } A \in D_i\}$
- (4) Generate the extend NX graph  $G^M$  as specified above.
- (5) For each  $a \in T_M$ , apply BFS to find a shortest M-augmenting  $P$  from  $S_M$  to  $a$  in  $G^M$ . If none exists, stop.
- (6) Set  $M := M \bowtie P$  and go to step (3).

The algorithm is a combination of the algorithm for cardinality bipartite matching and the algorithm for maximum matroid union (see (Schrijver 2004) Chapters 16 & 42). The complexity of computing maximal bipartite matching in the NX graph is in  $O(|N \cup X|^2)$  or even less. The complexity of computing maximum matroid union is in  $O(|N \cup D \cup X|^3)$ , where  $D = \bigcup_{i \in N} D_i$ . Therefore the complexity of calculating maximum quasi-matching is in  $O(|N \cup D \cup X|^3)$ . The reader is invited to practice the algorithm using Example 3.

Note that if  $S_M \cap T_M \neq \emptyset$ , each element in the intersection is a shortest path.

The following theorem shows the correctness of the algorithm.

**Theorem 2** Let  $\mathcal{D} = (D_i)_{i \in N}$  be a demand situation such that for all  $i \in N$ ,  $D_i$  is the set of bases of a matroid on  $X$ .  $M$  is a maximum quasi-matching if and only if there is no  $M$ -augmenting path from  $S_M$  to  $T_M$  in  $G^M$ .

## The algorithm for dynamic auctions

In this section, we present an algorithm for dynamic auction under GSC. Our algorithm is designed in spirit of (Sun and Yang 2009)'s double track auction (DTA) procedure. However, it is more general than DTA because it can be used for ascending or descending with or without complementarities. The difference is that our algorithm converges to any Walrasian equilibrium price vector, not necessarily the smallest one.

### DOUBLE-DIRECTION AUCTION (DDA) ALGORITHM

- (1) Announce an initial price vector  $\mathbf{p}^0$ .
- (2) At round  $t$ , invite each buyer  $i$  to submit her demand correspondence  $D_i(\mathbf{p}^t)$ .
- (3) Calculate a maximum quasi-matching  $M$  in the extended NX graph of the current demand situation. Let  $\pi^M$  be the allocation determined by  $M$ .

- (4) If  $\pi^M$  is a Walrasian equilibrium allocation, stop; otherwise, adjust price vector  $\mathbf{p}^{t+1}$  as follows:

$$p_a^{t+1} := \begin{cases} p_a^t + 1, & \text{if } a \in U; \\ p_a^t - 1, & \text{if } a \in V; \\ p_a^t, & \text{otherwise,} \end{cases} \quad \text{where}$$

$$U = \bigcup \{ \pi^M(j) : j \in N \& \bigcup_{B \in D_j(\mathbf{p}^t)} B \subseteq X \setminus \pi^M(0) \}$$

$$V = \pi^M(0) \setminus \bigcup_{i \in N} \bigcup_{B \in D_i(\mathbf{p}^t)} B,$$

- (5) Got to step (2) for next round.

To understand the price adjustment mechanism (step (4)), notice that  $\bigcup_{B \in D_i(\mathbf{p})} B$  represents all the items that buyer  $i$  has expressed interest to buy.  $X \setminus \pi^M(0)$  represents the items that have been allocated by  $M$ . So  $\bigcup_{B \in D_i(\mathbf{p}^t)} B \subseteq X \setminus \pi^M(0)$  means that all the items buyer  $i$  is interested in have been allocated. As a result, all the items in the set  $U$  are demanded and already allocated. Most importantly, as we will shown in the following theorem, if  $M$  does not lead to an efficient allocation, the set  $U$  is non-empty, which indicates a positive excess demand. One the other hand, since  $\pi^M(0)$  contains all the items that have not been allocated by  $M$ , the set  $V$  specifies the items that have not been allocated and no buyer is interested to buy. The pricing policy is then increasing the prices of all the items in  $U$  and decreasing the prices of the items in  $V$ .

**Theorem 3** Suppose that  $E = (N \cup \{0\}, X, \{u_i\}_{i \in N})$  is an economy that has the GSC property w.r.t. the common partition  $(X^1, X^2)$  of  $X$ . Set the initial price vector  $\mathbf{p}^0$  as

follows:

$$p_a^0 := \begin{cases} 0, & \text{if } a \in X^1; \\ M, & \text{if } a \in X^2. \end{cases}$$

where  $M > \max_{i \in N} u_i(X)$ . Then the above double-direction auction procedure converges to a Walrasian equilibrium in finite rounds. The overall complexity of the procedure is in  $O(|N \cup D \cup X|^4)$ , where  $D = \max_t \bigcup_{i \in N} D_i(\mathbf{p}^t)$ .

Note that GSC plays the most important role in the result because it guarantees that the minimization of Lyapunov function has integer solutions.

## Conclusion and related work

We have presented a dynamic auction procedure for discovering Walrasian equilibrium in an economy that satisfies the GSC property. The computational issues on dynamic auctions are discussed. Firstly, we have shown that the efficient allocation problem in dynamic auction in general is NP-complete. Secondly, we have developed an algorithm calculating a maximum quasi-matching. The algorithm solves both of the problems of efficient allocation and price adjustment in polynomial time. Finally, we have introduced a double-direction auction algorithm and have shown that the algorithm converges to a Walrasian equilibrium with GSC economy also in polynomial time.

Computational issues on combinatorial auctions have been studies intensively in the AI literature. Most of the existing results can be found in (Cramton *et al.* 2006). The term of dynamic auction has been used in the literature for different settings. The background related to our setting can be found in (Milgrom 2004; Ausubel 2006). Specifically, Lehmann *et al.* (2006) discussed the computational issues of dynamic auctions with GS and submodular economies. To the best of our knowledge, the computational issues of dynamic auction in the economies with complementarities have not been sufficiently explored.

## Appendix: Proof of Theorems

(Due to space limitation, we omitted the proof of two lemmas.)

**Proof of Proposition 1:** To show that the problem EAP is in NP, pick up a bundle  $B_i$  from each buyer's demand correspondence and check if it is an allocation of  $X$ .

To show that EAP is NP-complete, we prove that EXACT COVER, one of Karp's 21 NP-complete problems, polynomially transforms to EAP. Given a collection  $S$  of subsets of  $X$ , we duplicate  $S$  as the demand correspondence of each buyer, i.e.,  $D_i(\mathbf{p}) = S$  for all  $i \in N$ . If there is an exact cover  $S^* \subseteq S$  such that  $|S^*| = n$ , it is an efficient allocation of the economy. By varying  $n$  from 1 to  $|X|$ , we can discover an exact cover if it exists.  $\square$

**Proof of Theorem 1:** We prove the statement by induction on the length of  $P$ . In the case when  $P$  is a single point  $\{a_0\}$ , by the definition of M-augmenting path, there is a bundle  $B \subseteq D_{i_0}$  such that  $\pi^M(i_0) \cup \{a_0\} \subseteq B$ . It implies that after the augmentation by  $P$ ,  $\pi^{M'}(i_0) \subseteq B$ . On the other hand, we know  $a_0 \in \pi^M(0)$ . After removing  $a_0$  from  $\pi^M(0)$  to  $\pi^M(i_0)$ , the allocation is still valid. Therefore  $M'$  is a quasi-matching. Obviously  $|M'| = |M| + 1$  for  $(i_0, a_0) \notin M$ .

Now assume that the statement holds for any  $M$ -augmenting path which length is no more than  $k - 1$ . Consider an  $M$ -augmenting path  $P = (a_0, a_1, \dots, a_k)$ . Since there is an arc from  $a_0$  to  $a_1$ , by the construction of the extended NX graph, there is a unique buyer  $i_1 \in N$  such that  $a_0 \in \pi^M(i_1)$ ,  $a_1 \in X \setminus \pi^M(i_1)$  and  $[\pi^M(i_1), a_0, a_1]_{i_1}$ . Consequently, there is a bundle  $B \in D_{i_1}$  such that  $(\pi^M(i_1) \setminus \{a_0\}) \cup \{a_1\} \subseteq B$ . Since  $a_0 \in \pi^M(i_1)$ ,  $(i_1, a_0) \in M$ . We let  $\hat{M} = M \setminus \{(i_1, a_0)\}$  and  $\hat{P} = (a_1, \dots, a_k)$ . Then  $\hat{P}$  is an  $\hat{M}$ -augmenting path in  $G^{\hat{M}}$  (note that the only differences between  $G^M$  and  $G^{\hat{M}}$  are the arcs that related to node  $a_0$ ). By the inductive assumption,  $\hat{M} \bowtie \hat{P}$  is a quasi-matching and  $|\hat{M} \bowtie \hat{P}| = |\hat{M}| + 1 = |M|$ . Let  $\hat{M}' = \hat{M} \bowtie \hat{P}$  and  $M' = M \bowtie P$ . Because  $a_0$  is not in path  $\hat{P}$ , the first  $k - 1$  steps in the constructions of  $\hat{M}'$  and  $M'$  are exactly the same. At the last step of the construction of  $\hat{M}'$ , we have  $(i_1, a_1) \in \hat{M}'$  (if it is not the case, we can enforce it to be true). Since  $(i_1, a_0)$  has been removed from  $\hat{M}$ , the only difference between  $\hat{M}'$  and  $M'$  is  $(i_0, a_0)$ , which belongs to  $M'$  not the other. Obviously  $M'$  is still a quasi-matching because there is a bundle  $B \in D_{i_0}$  such that  $\pi^M(i_0) \cup \{a_0\} \subseteq B$ . It implies that  $\pi^{M'}(i_0) \subseteq B$ . Meanwhile we have  $|M'| = |\hat{M}'| + 1 = |M| + 1$ .  $\square$

**Proof of Theorem 2:** To show sufficiency, assume that there is an  $M$ -augmenting path  $P$  from  $S_M$  to  $T_M$  in  $G^M$ . By Theorem 1,  $M \bowtie P$  is a quasi-matching and  $|M \bowtie P| > |M|$ . According to the augmentation procedure,  $M \bowtie P$  contains a maximal matching in the NX graph if  $M$  does. Thus  $M$  is not maximal, a contradiction.

To show necessity, let  $(X, \mathcal{I})$  be the union of the matroids  $\{(X, \bar{D}_i)\}_{i \in N}$  (see Lemma 2). For any maximal quasi-matching  $M$ , let  $\pi^M$  be the allocation determined by  $M$ . It turns out that  $I = \bigcup_{i \in N} \pi^M(i) \in \mathcal{I}$ . If  $I$  is not maximal in  $\mathcal{I}$ , there is  $a \in \pi^M(0)$  and  $i \in N$  such that  $\pi^M(i) \cup \{a\} \subseteq \bar{D}_i$ . This means that  $M \cup \{(i, a)\}$  is a quasi-matching, which contradicts the fact that  $M$  is maximal. This is to say that any maximal quasi-matching determines a maximal matroid union. According to Theorem 42.4 (Schrijver 2004), for any  $b \in T_M$ , there is no augmenting path from  $S_M$  to  $b$  in the matroid union graph. Since an augment path in the matroid union graph uniquely corresponds to an  $M$ -augmenting path in the extended NX graph, we conclude that there does not exist an  $M$ -augmenting path from  $S_M$  to  $T_M$  in  $G^M$ .  $\square$

**Proof Theorem 3:** Suppose that the auction procedure proceeds to a stage at which a Walrasian equilibrium has not been reached. Let  $\mathbf{p}^t$  be the price vector at round  $t$  and  $\mathbf{p}^{t+1}$  the updated price vector as specified at step (4) of the above algorithm. According to (Sun and Yang 2009), the difference of Lyapunov function values at prices  $\mathbf{p}^t$  and  $\mathbf{p}^{t+1}$  is:

$$L(\mathbf{p}^t) - L(\mathbf{p}^{t+1}) = \sum_{i \in N} \left( \min_{B \in D_i(\mathbf{p}^t)} \sum_{a \in B} (p_a^{t+1} - p_a^t) \right) - \sum_{a \in X} (p_a^{t+1} - p_a^t) \quad (1)$$

According to the price adjustment mechanism (step (4) in the algorithm), the above equation leads to:

$$\begin{aligned} L(\mathbf{p}^t) - L(\mathbf{p}^{t+1}) &= \left( \sum_{i \in N} \min_{B \in D_i(\mathbf{p}^t)} (|B \cap U| - |B \cap V|) \right) - (|U| - |V|) \\ &= \left( \sum_{i \in N} \min_{B \in D_i(\mathbf{p}^t)} |B \cap U| \right) - (|U| - |V|) \\ &= \left( \sum_{i \in N} \min_{B \in D_i(\mathbf{p}^t)} |B \cap U| - |U| \right) + |V| \end{aligned}$$

If  $U = \emptyset$  but  $V \neq \emptyset$ , the Lyapunov function has a positive reduction after the price adjustment, i.e.,  $L(\mathbf{p}^t) - L(\mathbf{p}^{t+1}) > 0$ .

Next we prove that if  $V = \emptyset$  and  $M$  does not determine an

equilibrium allocation, then  $U \neq \emptyset$ . Let  $M$  be the current maximum quasi-matching. We say  $M$  to be *saturated* to buyer  $i$  if  $\pi^M(i) \in D_i(\mathbf{p}^t)$ .  $M$  is *saturated* if it is saturated to all  $i \in N$ . Since  $M$  does not lead to an equilibrium allocation, there is at least one buyer to whom  $M$  is not saturated, that is  $M$  is not *saturated*. If  $U$  is empty, for all  $i \in N$  there is  $B \in D(\mathbf{p}^t)$  and  $b \in B$  such that  $b \in \pi^M(0)$ . In such a case,  $M$  must be saturated to  $i$  because otherwise  $(i, b)$  can be added to  $M$  through augmentation. Therefore  $M$  is saturated, which is a contradiction.

Finally we prove that if  $U \neq \emptyset$ , then

$$\sum_{i \in N} \min_{B \in D_i(\mathbf{p}^t)} |B \cap U| - |U| > 0 \quad (2)$$

To this end, let  $N' = \{i \in N : \bigcup_{B \in D_i(\mathbf{p}^t)} B \subseteq \bigcup_{i \in N} \pi^M(i)\}$ .

It turns out that  $U = \bigcup_{i \in N'} \pi^M(i)$ . Furthermore, we have

$$\begin{aligned} \sum_{i \in N'} \min_{B \in D_i(\mathbf{p}^t)} |B \cap U| &\geq \sum_{i \in N'} \min_{B \in D_i(\mathbf{p}^t)} |B \cap U| = \\ \sum_{i \in N'} \min_{B \in D_i(\mathbf{p}^t)} |B| &> \sum_{i \in N'} |\pi^M(i)|. \end{aligned}$$

The last inequality is due to  $M$  is not saturated to at least one buyer in  $N'$ . We conclude that equation (2) holds, which implies that the Lyapunov function has a positive reduction after the price adjustment. Since the Lyapunov function is bounded and its solutions of minimization are integer vectors ((Sun and Yang 2009) Theorem 3), the algorithm converges to its minimizer no more than  $m|X|$  rounds, where  $m = \max_{i \in N} u_i(X)$ . Since the complexity for solving EAP and PAP is  $O(|N \cup D \cup X|^3)$ , and  $O(|N \cup D \cup X|^2)$  respectively, the overall complexity of the algorithm is in  $O(|N \cup D \cup X|^4)$  (assume that  $m$  is a constant).  $\square$

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