

Université d'Ottawa  
Faculté de Génie,  
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Technologies de l'Information



University of Ottawa  
Faculty of Engineering,  
School of Information  
Technology and Engineering

ELG 4172 Digital signal processing

Professor Miodrag Bolic

Final Exam

This exam is **180 minutes** long.

- Simple calculators are allowed
- notes and textbooks not allowed (closed book exam)

Last name:

First name:

Student #:

Problem	Maximum	Score
1	15	
2	14	
3	13	
4	7	
5	15	
6	15	
7	8	
8	13	
Total	100	

Convolutions and summations:

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a} \quad |a| < 1$$

$$\sum_{k=0}^{n_1} a^k = \frac{1-a^{n_1+1}}{1-a} \quad a \neq 1$$

Fourier transform:

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega = \int_1^F X(f) e^{j2\pi f n} df$$

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$$

$$x(n-n_0) \leftrightarrow e^{-j\omega n_0} X(\omega)$$

$$x(k)(n) \leftrightarrow X(k\omega)$$

$$e^{j\omega_0 n} x(n) \leftrightarrow X(\omega - \omega_0)$$

$$\cos(\omega_0 n) x(n) \leftrightarrow \frac{1}{2} [X(\omega - \omega_0) + X(\omega + \omega_0)]$$

Continuous time sampling,  
A/D conversion:

$$x_p(t) = x_a(t) \times p(t)$$

$$x(n) = x_a(nT)$$

$$X_p(\Omega) = F_s \sum_{k=-\infty}^{\infty} X_a(\Omega - k\Omega_s)$$

$$X(\omega) = X_p\left(\frac{\omega}{T}\right) = X_p(\omega F_s)$$

D/A conversion

$$X_p(\Omega) = X(\Omega T) = X(\Omega / F_s)$$

$$X_0(\Omega) = X_p(\Omega) H_0(\Omega)$$

$$H_0(\Omega) = T \frac{\sin(\Omega T / 2)}{\Omega T / 2} e^{-j\Omega T / 2}$$

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of $z$
$\mu[n]$	$\frac{1}{1-z^{-1}}$	$ z  > 1$
$\alpha^n \mu[n]$	$\frac{1}{1-\alpha z^{-1}}$	$ z  >  \alpha $
$(r^n \cos \omega_0 n) \mu[n]$	$\frac{1 - (r \cos \omega_0) z^{-1}}{1 - (2r \cos \omega_0) z^{-1} + r^2 z^{-2}}$	$ z  > r$
$(r^n \sin \omega_0 n) \mu[n]$	$\frac{(r \sin \omega_0) z^{-1}}{1 - (2r \cos \omega_0) z^{-1} + r^2 z^{-2}}$	$ z  > r$

Property	Sequence	z-Transform	ROC
	$g[n]$ $h[n]$	$G(z)$ $H(z)$	$\mathcal{R}_g$ $\mathcal{R}_h$
Conjugation	$g^*[n]$	$G^*(z^*)$	$\mathcal{R}_g$
Time-reversal	$g[-n]$	$G(1/z)$	$1/\mathcal{R}_g$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(z) + \beta H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Time-shifting	$g[n-n_0]$	$z^{-n_0} G(z)$	$\mathcal{R}_g$ , except possibly the point $z=0$ or $\infty$
Multiplication by an exponential sequence	$\alpha^n g[n]$	$G(z/\alpha)$	$ \alpha  \mathcal{R}_g$
Differentiation of $G(z)$	$ng[n]$	$-z \frac{dG(z)}{dz}$	$\mathcal{R}_g$ , except possibly the point $z=0$ or $\infty$
Convolution	$g[n] \otimes h[n]$	$G(z)H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi j} \oint_C G(v)H(z/v)v^{-1} dv$	Includes $\mathcal{R}_g \mathcal{R}_h$

$$\text{Parseval's relation} \quad \sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi j} \oint_C G(v)H^*(1/v^*)v^{-1} dv$$

Note: If  $\mathcal{R}_g$  denotes the region  $R_{g-} < |z| < R_{g+}$  and  $\mathcal{R}_h$  denotes the region  $R_{h-} < |z| < R_{h+}$ , then  $1/\mathcal{R}_g$  denotes the region  $1/R_{g+} < |z| < 1/R_{g-}$  and  $\mathcal{R}_g \mathcal{R}_h$  denotes the region  $R_{g-} R_{h-} < |z| < R_{g+} R_{h+}$ .

Partial fraction expansion

Simple poles:

$$G(z) = \sum_{\ell=1}^N \left( \frac{\rho_{\ell}}{1-\lambda_{\ell} z^{-1}} \right) \quad \rho_{\ell} = (1-\lambda_{\ell} z^{-1})G(z) \Big|_{z=\lambda_{\ell}}$$

Multiple poles

Let the pole at  $z = \nu$  be of multiplicity  $L$  and

the remaining  $N-L$  poles be simple and at

$$z = \lambda_{\ell}, \text{ for } 1 \leq \ell \leq N-L$$

$$G(z) = \sum_{\ell=0}^{M-N} \eta_{\ell} z^{-\ell} + \sum_{\ell=1}^{N-L} \frac{\rho_{\ell}}{1-\lambda_{\ell} z^{-1}} + \sum_{i=1}^L \frac{\gamma_i}{(1-\nu z^{-1})^i}$$

$$\gamma_i = \frac{1}{(L-i)!(-\nu)^{L-i}} \frac{d^{L-i}}{d(z^{-1})^{L-i}} \left[ (1-\nu z^{-1})^L G(z) \right] \Big|_{z=\nu}, \quad 1 \leq i \leq L$$

## Fixed Window Functions

- **Rectangular window** -  $\Delta_{ML} = 4\pi/(2M + 1)$   
 $A_{s\ell} = 13.3 \text{ dB}$ ,  $\alpha_s = 20.9 \text{ dB}$ ,  $\Delta\omega = 0.92\pi/M$
- **Hann window** -  $\Delta_{ML} = 8\pi/(2M + 1)$   
 $A_{s\ell} = 31.5 \text{ dB}$ ,  $\alpha_s = 43.9 \text{ dB}$ ,  $\Delta\omega = 3.11\pi/M$
- **Hamming window** -  $\Delta_{ML} = 8\pi/(2M + 1)$   
 $A_{s\ell} = 42.7 \text{ dB}$ ,  $\alpha_s = 54.5 \text{ dB}$ ,  $\Delta\omega = 3.32\pi/M$
- **Blackman window** -  $\Delta_{ML} = 12\pi/(2M + 1)$   
 $A_{s\ell} = 58.1 \text{ dB}$ ,  $\alpha_s = 75.3 \text{ dB}$ ,  $\Delta\omega = 5.56\pi/M$

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## Adjustable Window Functions

- **Kaiser Window** -

$$w[n] = \frac{I_0\{\beta\sqrt{1-(n/M)^2}\}}{I_0(\beta)}, \quad -M \leq n \leq M$$

where  $\beta$  is an adjustable parameter and  $I_0(u)$  is the modified zeroth-order Bessel function of the first kind:

$$I_0(u) = 1 + \sum_{r=1}^{\infty} \left[ \frac{(u/2)^r}{r!} \right]^2$$

- Note  $I_0(u) > 0$  for  $u > 0$
- In practice  $I_0(u) \cong 1 + \sum_{r=1}^{20} \left[ \frac{(u/2)^r}{r!} \right]^2$

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## Fixed Window Functions

- **Hann:**

$$w[n] = 0.5 + 0.5 \cos\left(\frac{2\pi n}{2M+1}\right), \quad -M \leq n \leq M$$

- **Hamming:**

$$w[n] = 0.54 + 0.46 \cos\left(\frac{2\pi n}{2M+1}\right), \quad -M \leq n \leq M$$

- **Blackman:**

$$w[n] = 0.42 + 0.5 \cos\left(\frac{2\pi n}{2M+1}\right) + 0.08 \cos\left(\frac{4\pi n}{2M+1}\right)$$

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## Adjustable Window Functions

- $\beta$  controls the minimum stopband attenuation of the windowed filter response

- $\beta$  is estimated using

$$\beta = \begin{cases} 0.1102(\alpha_s - 8.7), & \text{for } \alpha_s > 50 \\ 0.5842(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21), & \text{for } 21 \leq \alpha_s \leq 50 \\ 0, & \text{for } \alpha_s < 21 \end{cases}$$

- Filter order is estimated using

$$N = \frac{\alpha_s - 8}{2.285(\Delta\omega)}$$

where  $\Delta\omega$  is the normalized transition bandwidth

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Types of Linear Phase FIR filters

Type I	Type II	Type III	Type IV
$a_n = a_{N-n}$ even order	$a_n = a_{N-n}$ odd order	$a_n = -a_{N-n}$ even order	$a_n = -a_{M-n}$ odd order
$H_1(z) = z^{-N} H_1(z^{-1})$	$H_2(z) = z^{-N} H_2(z^{-1})$	$H_3(z) = -z^{-N} H_3(z^{-1})$	$H_4(z) = -z^{-N} H_4(z^{-1})$

N-th order analog lowpass Butterworth filter:  $|H_a(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}$

Poles of the transfer function  $H(s)$  of N-th order analog lowpass Butterworth filter

:  $p_\ell = \Omega_c e^{j\pi(N+2\ell-1)/2N}$ ,  $1 \leq \ell \leq N$

The system function of the N-order Butterworth filter is:  $H(s) = \frac{\Omega_c^N}{(s - p_1)(s - p_2)\dots(s - p_N)}$

Bilinear transform:  $s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$

## DFT

Type of Property	Length- $N$ Sequence	$N$ -point DFT
	$g[n]$ $h[n]$	$G[k]$ $H[k]$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G[k] + \beta H[k]$
Circular time-shifting	$g[(n - n_o)_N]$	$W_N^{kn_o} G[k]$
Circular frequency-shifting	$W_N^{-k_o n} g[n]$	$G[(k - k_o)_N]$
Duality	$G[n]$	$N g[(-k)_N]$
$N$ -point circular convolution	$\sum_{m=0}^{N-1} g[m]h[(n - m)_N]$	$G[k]H[k]$
Modulation	$g[n]h[n]$	$\frac{1}{N} \sum_{m=0}^{N-1} G[m]H[(k - m)_N]$

Parseval's relation  $\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$

## Trigonometry:

$$\begin{aligned} \sin(a + b) &= \sin a \cos b + \cos a \sin b \\ \sin(a - b) &= \sin a \cos b - \cos a \sin b \\ \cos(a + b) &= \cos a \cos b - \sin a \sin b \\ \cos(a - b) &= \cos a \cos b + \sin a \sin b \\ \sin a \sin b &= 1/2[\cos(a - b) - \cos(a + b)] \\ \cos a \cos b &= 1/2[\cos(a - b) + \cos(a + b)] \\ \sin a \cos b &= 1/2[\sin(a + b) + \sin(a - b)] \\ \cos a \sin b &= 1/2[\sin(a + b) - \sin(a - b)] \\ 1 - \cos 2a &= 2 \sin^2 a \\ 1 + \cos 2a &= 2 \cos^2 a \\ e^{j\theta} &= \cos \theta + j \sin \theta \end{aligned}$$

**Question 1 (a) 9 points, b), d) 2 points, c) 3 points, total 15)**

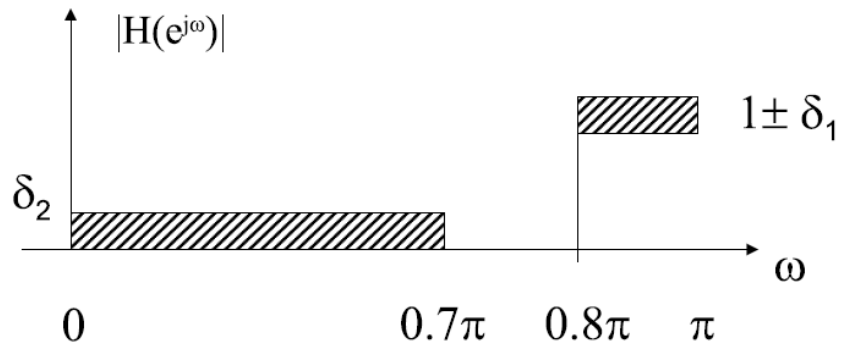
A causal system is described as follows:

$$H(z) = \frac{1 - \sqrt{2}z + z^2}{0.81 - 0.9\sqrt{2}z + z^2}$$

- a) Trace the pole-zero plot in the Z-plane for this system. Indicate the region of convergence
- b) Is this system stable? Explain
- c) Sketch the frequency response of this system and illustrate the key characteristics, using the pole-zero plot as a reference. Do not calculate  $H(e^{j\omega})$ .
- d) Is this system FIR or IIR? Why?

**Question 2 (total 14 points)**

Design a high-pass FIR filter, that is find  $h[n]$ , using the windowing design technique that satisfies approximately the specifications shown in Figure 1. There is no need to evaluate the numerical value of each coefficient, an analytical expression is sufficient.



$$\delta_1 = \delta_2, 20 \log \delta_2 = -72 \text{dB}$$

**Figure 1**

**Question 3 ( a) 10 points, b)3 points, total 13)**

An analog filter of the form:

$$H(s) = \frac{s}{s^2 + w_r^2} = \frac{s}{s^2 + 25}$$

has a resonance frequency at  $w_r = 5$ .

- a) Using bilinear transform, design an IIR filter which resonates at  $\omega_r = \pi/4$ , using the above filter as a basis.
- b) Trace the pole-zero plot for the obtained digital filter.

**Question 4 (total 7 points)**

Consider the system of Fig 2, where the input continuous-time signal  $x_a(t)$  has a band-limited spectrum  $X_a(j\Omega)$ , as sketched in Figure 3 (left), and is being sampled at the Nyquist rate. The discrete-time processor is an ideal lowpass filter with a frequency response  $H(e^{j\omega})$ , as shown in Figure 3(right), and has a cutoff frequency  $\omega_c = \Omega_m T / 3$ , where  $T$  is the sampling period. Sketch as accurately as possible the spectrum  $Y_a(j\Omega)$  of the output continuous-time signal  $y_a(t)$ .

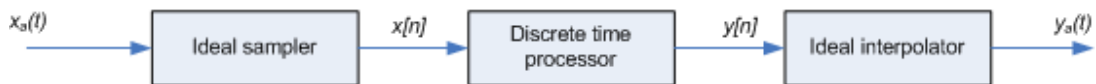


Figure 2

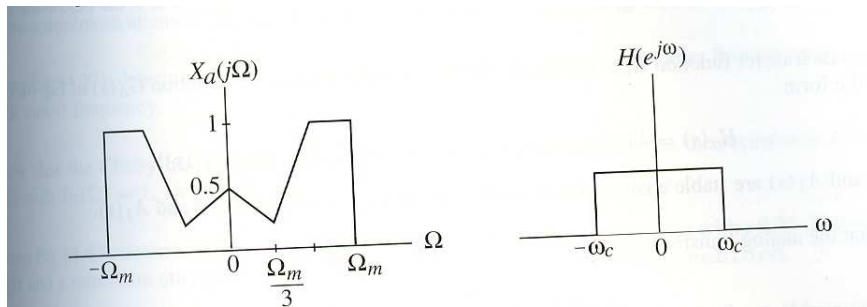


Figure 3



**Question 5 - Please select either question 5A or 5B**

**5A ( a) 11 points, b) 4 points, total 15)**

A signal has a bandwidth  $[-70, 70]$  kHz and useful frequency in the range  $[-15, 15]$  kHz. We wish to do spectral analysis of this signal using a system operating at  $f_s=50$ kHz. An anti-aliasing filter is available with the characteristics shown in Figure 4.

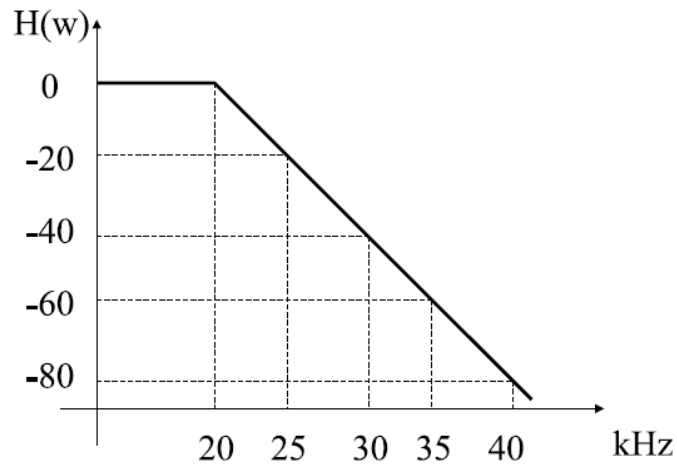


Figure 4

Also, an A/D converter with 10 bits and a quasi-perfect low-pass digital filter, with  $\omega_c=17$ kHz are available.

- Develop a solution and present the block-diagram
- Describe the physical spectrum (frequency axis in kHz) for each step that follows digitalization.

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**5B ( a), b), c) 5 points each, total 15)**

For the following system:

$$H(z) = \frac{1 + 0.875z^{-1}}{(1 - 0.7z^{-1})(1 + 0.2z^{-1} + 0.9z^{-2})}$$

- Show the direct form II structure of  $H(z)$ .
- Draw a cascade structure using first and second order direct form II structures.
- Draw a parallel structure using first and second order direct form II structures.

**Question 6 ( a) 10 points, b) 5 points, total 15)**

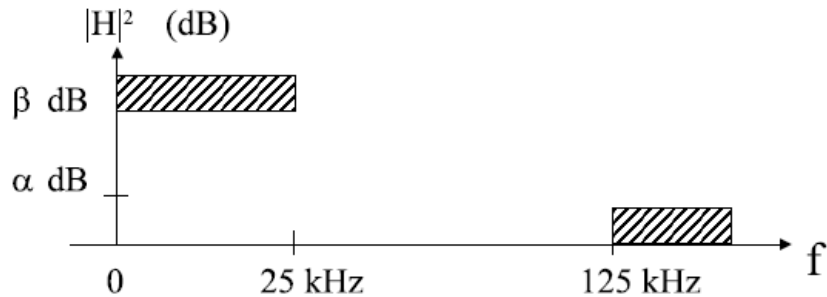
a) For the following sequence  $x(n) = \{1,2,3,4,3,2\}$  ,  $n = 0,1,2,3,4,5$  and the following FIR filter  $h(n) = \{4,5\}$ ,  $n = 0,1$  , compute the output  $y(n)$  for  $n = 0,1,2,3,4,5$  **using the DFT-based overlap and save technique**, with blocks of  $L = 3$  samples.

b) To verify your answer in a), compute the time domain convolution  $y(n) = x(n) * h(n)$  for  $n = 0,1,2,3,4,5$

**Question 7 - Please select either question 7A or 7B**

**7A (total 8 points)**

A Chebyshev analog filter was created to satisfy the following requirements:



The solution has the pole located at

$$S_{1,2} = (r_1 \cos 5\pi/8, \pm r_2 \sin 5\pi/8)$$

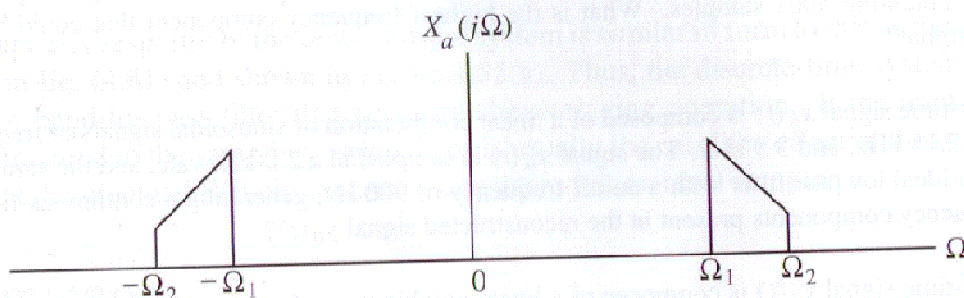
$$S_{3,4} = (r_1 \cos 7\pi/8, \pm r_2 \sin 7\pi/8)$$

Use the impulse invariance technique to create a digital filter, that is, find  $H(z)$  for this filter. Keep  $r_1$  and  $r_2$  as variables. Do not calculate them.

---

**7B (total 8 points)**

A continuous time signal  $x_a(t)$  has a band-limited spectrum  $X_a(j\Omega)$  as indicated in the figure bellow. Determine the smallest sampling frequency  $F_T$  that can be employed to sample  $x_a(t)$  so that it can be fully recovered from its sampling version  $x[n]$  for the following band-edges  $\Omega_1 = 160\pi$  and  $\Omega_2 = 250\pi$ . Sketch the Fourier transform of the sampled version  $x[n]$  obtained by sampling  $x_a(t)$  at the smallest sampling rate  $F_T$  and the frequency response of the ideal reconstruction filter needed to fully recover  $x_a(t)$  for each case.



**Question 8 ( a), b)4 points each, c) 5 points, total 13)**

(For the full mark, show derivations or justifications)

a) Determine Z transform and ROC of  $y[n] = (n+1)\alpha^n \mu[n]$ ? What is DTFT of  $y[n]$ ?

b) Is the following system linear:  $y[n] = x[n] \sin(n\pi/2)$ ?

c) Is the following system time-invariant:  $y[n] = \sum_{k=-\infty}^n x[k]$ ?