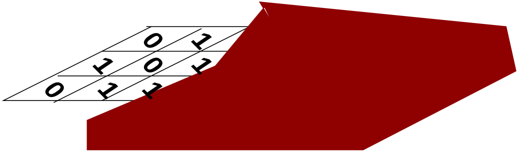
Covering Arrays and Generalizations

Lucia Moura
School of Information Technology and Engineering
University of Ottawa
lucia@site.uottawa.ca

UPC Seminar, November 2006



Covering array examples

array on alphabet $\{0,\ldots,g-1\}$; k columns;
find covering array with minimum n.

Examples: $g=3, k=4$

K =

	1	2	3	4
1	0	0	0	0
2	0	1	1	0
3	0	2	2	2
4	1	0	2	1
5	1	1	0	2
6	1	2	1	0
7	2	0	1	2
8	2	1	2	0
9	2	2	0	1
N= 10	0	0	1	1

K =

	1	2	3	4
1	0	0	0	0
2	0	1	1	2
3	0	2	2	1
4	1	0	2	2
5	1	1	0	1
6	1	2	1	0
7	2	0	1	1
8	2	1	2	0
N= 9	2	2	0	2

	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

(OAs, $k-2$ MOLS, lower bound)

Covering array definition

Definition: Covering Array

A *covering array* with k factors, g levels for each factor and size n, denoted by $CA(n; k, g)$, is an $n \times k$ array with symbols from $[0, g-1]$ such that for every pair of columns, every ordered pair in $[0, g-1]^2$ is covered at least once.

Objective: given k and g find a covering array with mininum size n.

$$CAN(k, g) = \min\{n : \ there\ exists\ a\ CA(n; k, g)\}.$$

Example: $g = 2, k = 4$:

K =

	1	2	3	4
1	1	1	1	1
2	1	0	0	0
3	0	1	0	0
4	0	0	1	0
N= 5	0	0	0	1

Is this optimal?

Application: component interaction testing

Testing pairwise interaction of factors.

components of a hardware system

Operating System (linux / windows)
Web browser (netscape / mozilla)
File format (pdf / postscript)
Printer (hp / epson)

Factors:

	Operating System	Web browser	File format	Printer	binary outcome:
test1	1	1	1	1	PASS/FAIL
test2	1	0	0	0	PASS/FAIL
test3	0	1	0	0	PASS/FAIL
test4	0	0	1	0	PASS/FAIL
test5	0	0	0	1	PASS/FAIL

factors influencing publication sucess

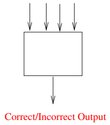
Topic (superconnectivity / Cayley graphs)
Professor (Anna Llado' / Oriol Serra)
Student (Jordi Moragas /Amanda Montejano)
Meeting place (UPC / tapas bar)

Factors:

	Topic	Professor	Student	Meeting place	binary outcome:
paper1	1	1	1	1	ACCEPTED/REJECTED
paper2	1	0	0	0	ACCEPTED/REJECTED
paper3	0	1	0	0	ACCEPTED/REJECTED
paper4	0	0	1	0	ACCEPTED/REJECTED
paper5	0	0	0	1	ACCEPTED/REJECTED

inputs to software or combinatorial circuits

test1	1	1	1	1
test2	1	0	0	0
test3	0	1	0	0
test4	0	0	1	0
test5	0	0	0	1



covering array:

1	1	1	1
1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1

exhaustive testing: 2^4=16 tests

pairwise testing: 5 tests

Assumption: failures come from the interaction of 2 or less factors.
All 2−way interactions are covered.
May not detect bad 3−way interactions – example: . Oriol Serra Jordi Moragas tapas bar

Advertisement: buy covering arrays!

- AETG (Telcordia): http://aetgweb.argreenhouse.com/
  (web service) price per year: US$6,000 - US$16,000
- TestCover.com: http://www.testcover.com/
  (web service) license price per year: US$1,200
- CaseMaker: http://www.casemakerinternational.com/
  (GUI software) price not in their web page
- Pro-test (SigmaZone):
  http://www.sigmazone.com/protest.htm
  (GUI software) license: US$399
- Other tools: IBM Intelligent Test Case Handler, CATS, OATS, IPO, TConfig, TCG (NASA), AllPairs, Jenny, ReduceArray2, DDA, Test Vector Generator, OA1, CTE-XL, PICT (Microsoft), rdExpert.

Covering array generalizations important for applications

- **mixed alphabet sizes:** factors may have different number of possible values
  example: 2 OSs, 3 browses, 4 file formats, 10 printers.

Covering array generalizations important for applications

- **mixed alphabet sizes:** factors may have different number of possible values
  example: 2 OSs, 3 browses, 4 file formats, 10 printers.
- **forbiden configurations:** some combinations may be forbidden (not supported)
  example: (linux,internet explorer) not allowed.

Covering array generalizations important for applications

- **mixed alphabet sizes:** factors may have different number of possible values
  example: 2 OSs, 3 browses, 4 file formats, 10 printers.
- **forbiden configurations:** some combinations may be forbidden (not supported)
  example: (linux,internet explorer) not allowed.
- **higher strength:** want to test 3-way interaction of factors
  example: failures may occur due to bad 3-way combinations (Oriol, Jordi, Tapas bar)

Covering array generalizations important for applications

- **mixed alphabet sizes:** factors may have different number of possible values
  example: 2 OSs, 3 browses, 4 file formats, 10 printers.
- **forbiden configurations:** some combinations may be forbidden (not supported)
  example: (linux,internet explorer) not allowed.
- **higher strength:** want to test 3-way interaction of factors
  example: failures may occur due to bad 3-way combinations (Oriol, Jordi, Tapas bar)
- **mixed strength:** certain factors need higher srength
  example: all pairwise + (student,professor,meeting place)

Covering array generalizations important for applications

- **mixed alphabet sizes:** factors may have different number of possible values
  example: 2 OSs, 3 browses, 4 file formats, 10 printers.
- **forbiden configurations:** some combinations may be forbidden (not supported)
  example: (linux,internet explorer) not allowed.
- **higher strength:** want to test 3-way interaction of factors
  example: failures may occur due to bad 3-way combinations (Oriol, Jordi, Tapas bar)
- **mixed strength:** certain factors need higher srength
  example: all pairwise + (student,professor,meeting place)
- **covering arrays on graphs:** certain combinations don't need interactions tested example: all pairwise interactions except (topic, professor), (meeting place, student)

Covering array: summary of results

- general lower bound: $CAN(k, g) \geq g^2$

Covering array: summary of results

- general lower bound: $CAN(k, g) \geq g^2$
- $CAN(k, g) = g^2$ iff $\exists OA(k, g, \lambda = 1)$ iff $\exists (k-2)$ MOLS of order g. Only possible for $k \leq g + 1$.

Covering array: summary of results

- general lower bound: $CAN(k, g) \geq g^2$
- $CAN(k, g) = g^2$ iff $\exists OA(k, g, \lambda = 1)$ iff $\exists (k-2)$ MOLS of order g. Only possible for $k \leq g + 1$.
- $CAN(k, g) \leq CAN(k + 1, g)$

Covering array: summary of results

- general lower bound: $CAN(k,g) \geq g^2$
- $CAN(k,g) = g^2$ iff $\exists OA(k,g,\lambda=1)$ iff $\exists (k-2)$ MOLS of order g. Only possible for $k \leq g+1$.
- $CAN(k,g) \leq CAN(k+1,g)$
- $CAN(k,g) \leq CAN(k,g+1)$

Covering array: summary of results

- general lower bound: $CAN(k, g) \geq g^2$
- $CAN(k, g) = g^2$ iff $\exists OA(k, g, \lambda = 1)$ iff $\exists (k-2)$ MOLS of order g. Only possible for $k \leq g + 1$.
- $CAN(k, g) \leq CAN(k + 1, g)$
- $CAN(k, g) \leq CAN(k, g + 1)$
- $CAN(k, g = 2)$ has been solved.

Covering array: summary of results

- general lower bound: $CAN(k, g) \geq g^2$
- $CAN(k, g) = g^2$ iff $\exists OA(k, g, \lambda = 1)$ iff $\exists(k-2)$ MOLS of order g. Only possible for $k \leq g+1$.
- $CAN(k, g) \leq CAN(k+1, g)$
- $CAN(k, g) \leq CAN(k, g+1)$
- $CAN(k, g = 2)$ has been solved.
- For general g: direct and recursive constructions.

Covering array: summary of results

- general lower bound: $CAN(k,g) \geq g^2$
- $CAN(k,g) = g^2$ iff $\exists OA(k,g,\lambda=1)$ iff $\exists (k-2)$ MOLS of order g. Only possible for $k \leq g+1$.
- $CAN(k,g) \leq CAN(k+1,g)$
- $CAN(k,g) \leq CAN(k,g+1)$
- $CAN(k,g=2)$ has been solved.
- For general g: direct and recursive constructions.
- Non-constructive asymptotic result known for fixed g:

$$CAN(k,g) \sim \frac{g}{2} \log k, \quad \text{as } k \to \infty$$

Covering array optimization questions

Fix g.

Minimizing n for fixed k (number of tests)

$$CAN(k,g) = \min\{n : \textit{ there exists a } CA(n;k,g)\}.$$

Maximizing k for fixed n (number of factors)

$$CAK(n,g) = \max\{k : \textit{ there exists a } CA(n;k,g)\}.$$

Relationship between min-max problems

$$CAN(k,g) = \min\{n : CAK(n,g) \geq k\}.$$

Direct construction via Orthogonal arrays

Definition: Orthogonal Array

An *orthogonal array* with k factors, g levels for each factor, denoted by $OA(k, g)$, is an $g^2 \times k$ array with symbols from a $[0, g-1]G$ such that for every pair of columns, every ordered pair in $[0, g-1]^2$ is **appears at exactly once**.

- For g a prime power, there exists g mutually orthogonal Latin squares which gives an $OA(g+1, g)$ orthogonal array. (equivalent to the existence of a projective plane of order g). Construction uses finite fields.

Direct construction via Orthogonal arrays

Definition: Orthogonal Array

An *orthogonal array* with k factors, g levels for each factor, denoted by $OA(k, g)$, is an $g^2 \times k$ array with symbols from a $[0, g-1]G$ such that for every pair of columns, every ordered pair in $[0, g-1]^2$ is **appears at exactly once**.

- For g a prime power, there exists g mutually orthogonal Latin squares which gives an $OA(g+1, g)$ orthogonal array. (equivalent to the existence of a projective plane of order g). Construction uses finite fields.
  - For g prime power, $CAN(k, g) = g^2$ for all $k \leq g+1$.

Direct construction via Orthogonal arrays

Definition: Orthogonal Array

An *orthogonal array* with k factors, g levels for each factor, denoted by $OA(k, g)$, is an $g^2 \times k$ array with symbols from a $[0, g-1]G$ such that for every pair of columns, every ordered pair in $[0, g-1]^2$ is **appears at exactly once**.

- For g a prime power, there exists g mutually orthogonal Latin squares which gives an $OA(g+1, g)$ orthogonal array. (equivalent to the existence of a projective plane of order g). Construction uses finite fields.
  - For g prime power, $CAN(k, g) = g^2$ for all $k \leq g+1$.
- For g not a prime power, use the larger knwon number of MOLS:

Direct construction via Orthogonal arrays

Definition: Orthogonal Array

An *orthogonal array* with k factors, g levels for each factor, denoted by $OA(k, g)$, is an $g^2 \times k$ array with symbols from a $[0, g-1]G$ such that for every pair of columns, every ordered pair in $[0, g-1]^2$ is **appears at exactly once**.

- For g a prime power, there exists g mutually orthogonal Latin squares which gives an $OA(g+1, g)$ orthogonal array. (equivalent to the existence of a projective plane of order g). Construction uses finite fields.
  - For g prime power, $CAN(k, g) = g^2$ for all $k \leq g+1$.
- For g not a prime power, use the larger knwon number of MOLS:
  - $CAN(k, 6) = 36$ for $k = 1, 2, 3$, but $CAN(4, 6) > 36$.

Direct construction via Orthogonal arrays

Definition: Orthogonal Array

An *orthogonal array* with k factors, g levels for each factor, denoted by $OA(k, g)$, is an $g^2 \times k$ array with symbols from a $[0, g-1]G$ such that for every pair of columns, every ordered pair in $[0, g-1]^2$ is **appears at exactly once**.

- For g a prime power, there exists g mutually orthogonal Latin squares which gives an $OA(g+1, g)$ orthogonal array. (equivalent to the existence of a projective plane of order g). Construction uses finite fields.
  - For g prime power, $CAN(k, g) = g^2$ for all $k \leq g+1$.
- For g not a prime power, use the larger knwon number of MOLS:
  - $CAN(k, 6) = 36$ for $k = 1, 2, 3$, but $CAN(4, 6) > 36$.
  - $CAN(k, 10) = 100$ for $k = 1, 2, 3, 4$, but $CAN(5, 10)? = 100$.

Recursive construction: Blocksize recursive construction

CA(4,3)

	1	2	3	4
1	0	0	0	0
2	0	1	1	2
3	0	2	2	1
4	1	0	2	2
5	1	1	0	1
6	1	2	1	0
7	2	0	1	1
8	2	1	2	0
9	2	2	0	2

size=9

CA(3,3) with 3 disjoint rows:

0	0	0
1	1	1
2	2	2

OD(3,3)

0	1	2
0	2	1
1	0	2
1	2	0
2	0	1
2	1	0

size=6

CA(12=4*3,3)

1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	2	0	1	1	2	0	1	1	2
0	2	2	1	0	2	2	1	0	2	2	1
1	0	2	2	1	0	2	2	1	0	2	2
1	1	0	1	1	1	0	1	1	1	0	1
1	2	1	0	1	2	1	0	1	2	1	0
2	0	1	1	2	0	1	1	2	0	1	1
2	1	2	0	2	1	2	0	2	1	2	0
2	2	0	2	2	2	0	2	2	2	0	2
0	0	0	0	1	1	1	1	2	2	2	2
0	0	0	0	2	2	2	2	1	1	1	1
1	1	1	1	0	0	0	0	2	2	2	2
1	1	1	1	2	2	2	2	0	0	0	0
2	2	2	2	0	0	0	0	1	1	1	1
2	2	2	2	1	1	1	1	0	0	0	0

size=9+6=15

CA(N1,k1,g)+OD(N2,k2,g)= CA(N1+N2,k1*k2,g)

Algorithmic construction: Greedy method

Greedy method used in the AETG system (D. Cohen, Dalal, Fredman and Patton (1997)):
"Choose one test at a time. At each stage select a test that covers the maximum number of uncovered pairs."

- Good news: for fixed g, CA size is proportional to $\log k$.
- Bad news: to pick a test covering the maximum number of uncovered tests is NP-complete, so the authors use a heuristic for test selection which does not guarantee the logarithmic growth.

The DDA (determinisitc density algorithm) by M. Cohen, Colbourn and Turban (2004):

- greedy method that runs in polynomial time;
- for fixed g, CA size is proportional to $\log k$; this is based on a guarantee that each selected test cover the *average* number of uncovered tests.

CAs with $g=2$ are extremal set systems

1	1	1	1	1
2	1	0	0	0
3	0	1	0	0
4	0	0	1	0
5	0	0	0	1

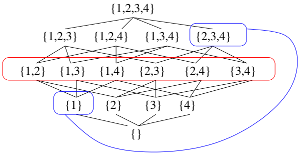
	set system S	complement: C
column1	{1,2}	{3,4,5}
column2	{1,3}	{2,4,5}
column3	{1,4}	{2,3,5}
column4	{1,5}	{2,3,4}

base set = {1,2,3,4,5}

- S must be pairwise intersecting: pair $(1,1)$ is covered.
- C must be pairwise intersecting: pair $(0,0)$ is covered.
- each of S and C must have the Sperner property: pairs $(0,1)$ and $(1,0)$ covered.

Sperner theorem for set systems

A system of subsets of an n-set has the *Sperner property* if no two subsets in the system are comparable.



Sperner's Theorem (1928)

If $\mathcal{A}$ has the Sperner property, then $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

The upper bound is only acchieved by the set of all $(\lfloor \frac{n}{2} \rfloor)$-subsets of the n-set, or by its (subsetwise) complement.

Erdos-Ko-Rado theorem for set systems

A system of subsets of an n-set is (pairwise) *intersecting* if every two subsets in the system have nonempty intersection.
Examples:
$(n = 5)$ $\mathcal{A} = \{\{1,2,3\},\{1,4,5\},\{2,3,4\},\{2,4,5\},\{3,4,5\}\}$
$(n = 6)$ $\mathcal{B} = \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,3,4\},$
$\{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},\{1,5,6\}\}$

Erdos-Ko-Rado Theorem (1961)

Let $\mathcal{A}$ be an intersecting system of subsets of an n-set, such that each subset has cardinality at most k.
If $n \geq 2k$, then $|\mathcal{A}| \leq \binom{n-1}{k-1}$.

Moreover, if $n > 2k$, then equality holds if and only if $\mathcal{A}$ is a k-uniform trivially intersecting system.

Optimal construction for binary alphabet

Pick all $\lfloor n/2 \rfloor$-subsets of $[1, n]$ that contain a common element.

n odd:

1	2	3	4	5
1	1	0	0	0
1	0	1	0	0
1	0	0	1	0
1	0	0	0	1

n even:

1	2	3	4	5	6
1	1	1	0	0	0
1	1	0	1	0	0
1	1	0	0	1	0
1	1	0	0	0	1
1	0	1	1	0	0
1	0	1	0	1	0
1	0	1	0	0	1
1	0	0	1	1	0
1	0	0	1	0	1
1	0	0	0	1	1

Note: the arrays are transposed here $(k \times n)$.
Both A and $\overline{A}$ are intersecting and Sperner.

The binary covering array theorem

Theorem (Katona 1973, Kleitman and Spencer 1973)

$CAK(n, t = 2, g = 2) = \binom{n-1}{\lfloor n/2 \rfloor - 1}$. *Moreover, this bound is attained by a* $\lfloor n/2 \rfloor$*-uniform trivially* 1*-intersecting set system.*

Proof: Let $\mathcal{A}$ be the set system corresponding to a CA.

- **(Case 1)** n **even.**

The binary covering array theorem

Theorem (Katona 1973, Kleitman and Spencer 1973)

$CAK(n, t = 2, g = 2) = \binom{n-1}{\lfloor n/2 \rfloor - 1}$. *Moreover, this bound is attained by a* $\lfloor n/2 \rfloor$*-uniform trivially* 1*-intersecting set system.*

Proof: Let $\mathcal{A}$ be the set system corresponding to a CA.

- **(Case 1)** n **even.**
- $\mathcal{A}^* = \{A, \overline{A} : A \in \mathcal{A}\}$ is Sperner.

The binary covering array theorem

Theorem (Katona 1973, Kleitman and Spencer 1973)

$CAK(n, t = 2, g = 2) = \binom{n-1}{\lfloor n/2 \rfloor - 1}$. *Moreover, this bound is attained by a* $\lfloor n/2 \rfloor$*-uniform trivially* 1*-intersecting set system.*

Proof: Let $\mathcal{A}$ be the set system corresponding to a CA.

- **(Case 1)** n **even.**
- $\mathcal{A}^* = \{A, \overline{A} : A \in \mathcal{A}\}$ is Sperner.
- Sperner's theorem implies $|\mathcal{A}^*| \leq \binom{n}{n/2}$.

The binary covering array theorem

Theorem (Katona 1973, Kleitman and Spencer 1973)

$CAK(n, t = 2, g = 2) = \binom{n-1}{\lfloor n/2 \rfloor - 1}$. *Moreover, this bound is attained by a* $\lfloor n/2 \rfloor$*-uniform trivially* 1*-intersecting set system.*

Proof: Let $\mathcal{A}$ be the set system corresponding to a CA.

- **(Case 1)** n **even.**
- $\mathcal{A}^* = \{A, \overline{A} : A \in \mathcal{A}\}$ is Sperner.
- Sperner's theorem implies $|\mathcal{A}^*| \leq \binom{n}{n/2}$.
- $|\mathcal{A}| \leq \frac{1}{2}|\mathcal{A}^*| \leq \frac{1}{2}\binom{n}{n/2} = \binom{n-1}{n/2-1}$. $\square$

The binary covering array theorem

Theorem (Katona 1973, Kleitman and Spencer 1973)

$CAK(n, t = 2, g = 2) = \binom{n-1}{\lfloor n/2 \rfloor - 1}$. *Moreover, this bound is attained by a* $\lfloor n/2 \rfloor$*-uniform trivially* 1*-intersecting set system.*

Proof: Let $\mathcal{A}$ be the set system corresponding to a CA.

- **(Case 1)** n **even.**
- $\mathcal{A}^* = \{A, \overline{A} : A \in \mathcal{A}\}$ is Sperner.
- Sperner's theorem implies $|\mathcal{A}^*| \leq \binom{n}{n/2}$.
- $|\mathcal{A}| \leq \frac{1}{2}|\mathcal{A}^*| \leq \frac{1}{2}\binom{n}{n/2} = \binom{n-1}{n/2-1}$. $\square$
- **(Case 2)** n **odd.**

The binary covering array theorem

Theorem (Katona 1973, Kleitman and Spencer 1973)

$CAK(n, t = 2, g = 2) = \binom{n-1}{\lfloor n/2 \rfloor - 1}$. *Moreover, this bound is attained by a* $\lfloor n/2 \rfloor$*-uniform trivially* 1*-intersecting set system.*

Proof: Let $\mathcal{A}$ be the set system corresponding to a CA.

- **(Case 1)** n **even.**
- $\mathcal{A}^* = \{A, \overline{A} : A \in \mathcal{A}\}$ is Sperner.
- Sperner's theorem implies $|\mathcal{A}^*| \leq \binom{n}{n/2}$.
- $|\mathcal{A}| \leq \frac{1}{2}|\mathcal{A}^*| \leq \frac{1}{2}\binom{n}{n/2} = \binom{n-1}{n/2-1}$. $\square$
- **(Case 2)** n **odd.**
- Wlog assume $|A| \leq \lfloor n/2 \rfloor$, for all $A \in \mathcal{A}$.

The binary covering array theorem

Theorem (Katona 1973, Kleitman and Spencer 1973)

$CAK(n, t=2, g=2) = \binom{n-1}{\lfloor n/2 \rfloor - 1}$. *Moreover, this bound is attained by a* $\lfloor n/2 \rfloor$*-uniform trivially* 1*-intersecting set system.*

Proof: Let $\mathcal{A}$ be the set system corresponding to a CA.

- **(Case 1)** n **even.**
- $\mathcal{A}^* = \{A, \overline{A} : A \in \mathcal{A}\}$ is Sperner.
- Sperner's theorem implies $|\mathcal{A}^*| \leq \binom{n}{n/2}$.
- $|\mathcal{A}| \leq \frac{1}{2}|\mathcal{A}^*| \leq \frac{1}{2}\binom{n}{n/2} = \binom{n-1}{n/2-1}$. $\square$
- **(Case 2)** n **odd.**
- Wlog assume $|A| \leq \lfloor n/2 \rfloor$, for all $A \in \mathcal{A}$.
- $\mathcal{A}$ is 1-intersecting, so by the EKR theorem, $|\mathcal{A}| \leq \binom{n-1}{\lfloor n/2 \rfloor - 1}$.

Covering arrays are systems of set partitions

- A covering array (strength 2) is a system of set-partitions:

	1	2	3	4	5	6	7	8	9	10			
column 1	0	0	0	1	1	1	2	2	2	0	{1,2,3,10}	{4,5,6}	{7,8,9}
column 2	0	1	2	0	1	2	0	1	2	0	{1,4,7,10}	{2,5,8}	{3,6,9}
column 3	0	1	2	2	0	1	1	2	0	1	{1,5,9}	{2,6,7,10}	{3,4,8}
column 4	0	0	2	1	2	0	2	0	1	1	{1,2,6,8}	{4,9,10}	{3,5,7}

Covering arrays are systems of set partitions

- A covering array (strength 2) is a system of set-partitions:

	1	2	3	4	5	6	7	8	9	10			
column 1	0	0	0	1	1	1	2	2	2	0	{1,2,3,10}	{4,5,6}	{7,8,9}
column 2	0	1	2	0	1	2	0	1	2	0	{1,4,7,10}	{2,5,8}	{3,6,9}
column 3	0	1	2	2	0	1	1	2	0	1	{1,5,9}	{2,6,7,10}	{3,4,8}
column 4	0	0	2	1	2	0	2	0	1	1	{1,2,6,8}	{4,9,10}	{3,5,7}

- Maximization problem:
  Given N, find a set partition system $\mathcal{P}$ with maximum $|\mathcal{P}|$ that is **(pairwise) strongly intersecting**:
  For all $P, Q \in \mathcal{P}$ we have

$$\text{for all } P_i \in P, Q_j \in Q, \quad P_i \cap Q_j \neq \emptyset.$$

Strongly intersecting condition: upper bound via 2-parts

Theorem (Stevens, Moura and Mendelsohn 1998)

$CAK(n,2,g) \leq \frac{1}{2}\binom{\lfloor \frac{2n}{g} \rfloor}{\lfloor \frac{n}{g} \rfloor}$.

This theorem only uses the two smallest parts of each partition, and the following fact:
Consider a pair of set systems, $A_1, A_2, \ldots, A_k$ and $B_1, B_2, \ldots, B_k$, with $|A_i| + |B_i| \leq c$ and $|A_i| \leq a \leq c/2$, and such that $A_i \cap B_i = \emptyset$, and all other sets intersect. Then, $k \leq \frac{1}{2}\binom{c}{a}$.
It is possible to relabel symbols of the covering array so that $|P_{1j}| \leq \lfloor \frac{n}{g} \rfloor$ and $|P_{1j}| + |P_{2j}| \leq \lfloor \frac{2n}{g} \rfloor$

Stronly intersecting versus Sperner formulation

Strongly intersecting formulation:
Partitions P and Q corresponding to two columns of a covering array must satisfy:

$$\text{for all } P_i \in P, Q_j \in Q, \quad P_i \cap Q_j \neq \emptyset.$$

Strongly Sperner formulation:
Partitions P and Q corresponding to two columns of a covering array must satisfy:

$$\text{for all } P_i \in P, Q_j \in Q, \quad P_i \not\subseteq \overline{Q}_j \text{ and } P_i \not\subseteq Q_j$$

Sperner's theorem for set-partition systems

largest cardinality k of a system $\mathcal{P}$ of g-partitions of $[1, n]$ such that for all $\mathcal{P}_i, \mathcal{P}_j \in \mathcal{P}$:

$$\forall P \in \mathcal{P}_i, \forall P' \in \mathcal{P}_j (P \not\subseteq P' \text{and} P' \not\subseteq P).$$ (Weakly) Sperner

Theorem (Meagher, Moura and Stevens 2005)

Let g, n such that $n = cg + r$ and $0 \leq r < g$. Then,

$$N_n(\forall, \forall) \leq \frac{1}{(g-r) + \frac{r(c+1)}{n-1}} \binom{n}{c}.$$

Theorem (Meagher, Moura and Stevens 2005)

Let g, n such that $g|n$. Then, $N_n(\forall, \forall) = \binom{n-1}{\frac{n}{g}-1}$. Moreover, this bound is met if and only if the g-partitions are uniform (all parts with cardinality $\frac{n}{g}$).

Example: weakly Sperner property

n=2g	***n=3g***
{1,2,3},{4,5,6}	***{1,2,3},{4,5,6},{7,8,9}***
{1,2,4},{3,5,6}	***{1,2,4},....***
{1,2,5},{3,4,6}	***{1,2,5},...***
{1,2,6},{3,4,5}	***{1,2,6},...***
{1,3,4},{2,5,6}	.
{1,3,5},{2,4,6}	.
{1,3,6},{2,4,5}	.
{1,4,5},{2,3,6}	.
{1,4,6},{2,3,5}	***{1,7,8},...***
{1,5,6},{2,3,4}	***{1,8,9},{2,3,4}, {5,6,7}***

Comparison of two bounds obtained

Theorem (Stevens, Moura and Mendelsohn 1998)

$CAK(n,2,g) \leq \frac{1}{2}\binom{\lfloor \frac{2n}{g} \rfloor}{\lfloor \frac{n}{g} \rfloor}$.

Theorem (Meagher, Moura and Stevens 2005)

If $g|n$, *then* $CAK(n,2,g) \leq \binom{n-1}{\frac{n}{g}-1}$.

if $g > 2$, $g|n$, then

$$\frac{1}{2}\binom{\frac{2n}{g}}{\frac{n}{g}} < \binom{n-1}{\frac{n}{g}-1}$$

Erdos-Ko-Rado theorem for set-partition systems

We are interested on a maximal partition system $\mathcal{P}$ such that:

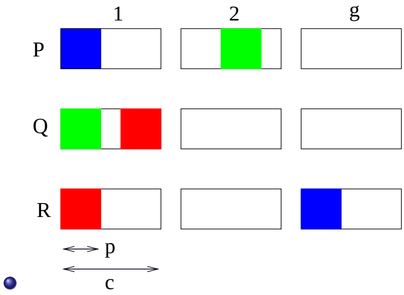
- each partition of $[1, n]$ have g parts of size $\frac{n}{g}$;
- two partitions $P, Q \in \mathcal{P}$ are such that there exists $P_i \in P$ and $Q_j \in Q$ such that $|P_i \cap Q_j| \leq p$.

Useful for bounds on "anti-covering-arrays" for certain uniform cases. Ex: $n = g^2$, $p = 2$

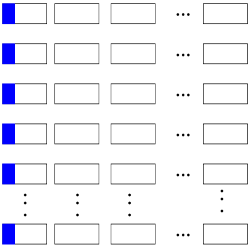
Conjecture

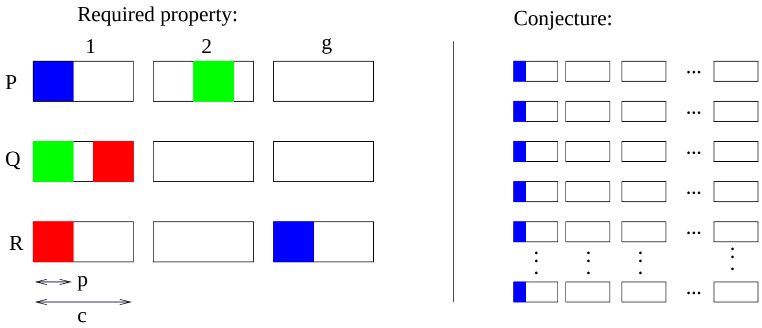
Suppose $g|n$, and let $c = n/g$ be the size of each part of the (uniform) partition system. $|\mathcal{P}| = \binom{n-p}{c-p} U(n - c, g - 1)$.

Required property:

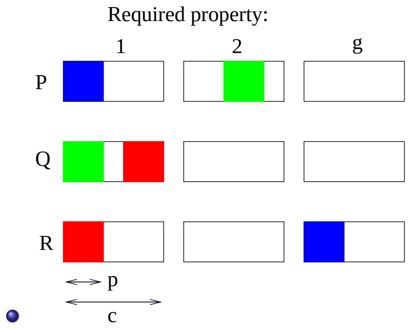


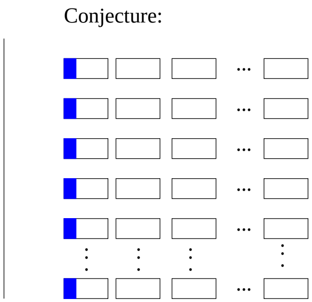
Conjecture:





- Conjecture has been proven for $p = c$:





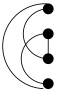
- Conjecture has been proven for $p = c$:

Theorem (Meagher and Moura 2005)

Let $n \geq g \geq 1$ and let $\mathcal{P} \subseteq U_g^n$ be a maximal partition system in which every two partitions share at least one class. Let $c = n/g$. Then, $|\mathcal{P}| = U(n-c, g-1)$

Covering array on graphs

factors influencing publication sucess with knwon SAFE INTERACTIONS:



Topic (superconnectivity / Cayley graphs)
Professor (Anna Llado' / Oriol Serra)
Student (Jordi Moragas /Amanda Montejano)
Meeting place (UPC / tapas bar)

Factors:	Topic	Professor	Student	Meeting place
paper1	1	1	1	1
paper2	1	1	0	0
paper3	0	0	0	0
paper4	0	0	1	1

for complete graph
min N = 5

For this graph
min N = 4

Covering array on graphs: definition

Definition: Covering Array

A *covering array* on a graph G with alphabet size g and size n, denoted by $CA(n; G, g)$, is an $n \times k = |V(G)|$ array with symbols from $[0, g-1]$ such that for every pair of columns corresponding to an edge of G, every ordered pair in $[0, g-1]^2$ is covered at least once.

Objective: given G and g find a covering array with mininum size n.

$$CAN(G, g) = \min\{n : \textit{ there exists a } CA(n; G, g)\}.$$

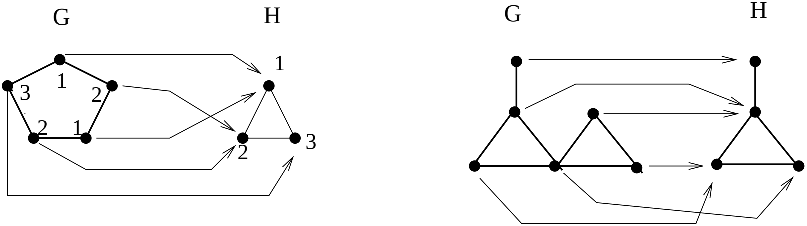
Determining $CAN(G, 2)$ is NP-complete (Serousi and Bshouty) reduction to 3-COLOUR.

Graph homomorphisms

Definition: graph homomorphism

A graph homomorphism from graph G to graph H, denoted $G \to H$ is a mapping from $V(G)$ to $V(H)$ that takes edges to edges.

Vertex colouring = homomorphism to the complete graph with `numberOfColours` vertices.

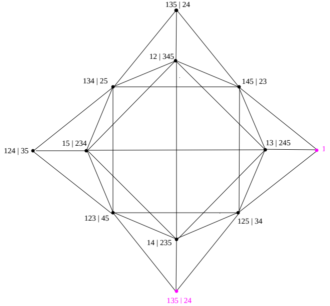


See book "Graphs and homomorphisms" by Hell and Nesetril, 2004.

Qualitative independence graph

The qualitative independence graph $QI(n, g)$ has:

- Vertex Set: all g-partitions of $[1, n]$ that have every class of size at least g.
- Edges: two vertices are connected if their partitions are qualitatively independent (P, Q are qualitatively independent if $P_i \cap Q_j \neq \emptyset$ for all i, j.)



Why $QI(n, g)$ are interesting?

Results by Meagher and Stevens (2005):

- A k-clique in $QI(n, g)$ corresponds to a $CA(n, k, g)$;
- A $CA(n, G, g)$ exists if and only if there exists a graph homomorphism $G \to QI(n, g)$;
- $CAN(G, g) = \min\{n : G \to QI(n, g)\}$.

Clique and chromatic bounds

Corollary (Meagher and Stevens 2005): If there exists a homomorphism $G \to H$, then $CAN(G, g) \leq CAN(H, g)$.

It is well-known that there exists homomorphisms:
$K_{\omega(G)} \to G \to K_{\chi(G)}$.

Therefore: $CAN(\omega(G), g) \leq CAN(G, g) \leq CAN(\chi(G), g)$.

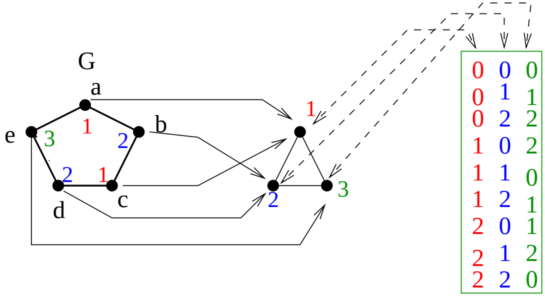
This gives the "colouring construction":

- k-colour the vertices of G.
- build a $CA(n, k, g)$.
- pull back a $CA(n, G, g)$.

Colouring construction

- k-colour the vertices of G.
- build a $CA(n, k, g)$.
- pull back a $CA(n, G, g)$.

a	b	c	d	e
0	0	0	0	0
0	1	0	1	1
0	2	0	2	2
1	0	1	0	2
1	1	1	1	0
1	2	1	2	1
2	0	2	0	1
2	1	2	1	2
2	2	2	2	0



Binary alphabet results

- $\omega(QI(n,2)) = \binom{n-1}{\lfloor \frac{n}{2} \rfloor}$ (Kleitman and Spencer, Katona 1973).

Binary alphabet results

- $\omega(QI(n,2)) = \binom{n-1}{\lfloor \frac{n}{2} \rfloor}$ (Kleitman and Spencer, Katona 1973).
- $\chi(QI(n,2)) = \lceil \frac{1}{2} \binom{n}{\lceil \frac{n}{2} \rceil} \rceil$ (Meagher and Stevens 2005)

Binary alphabet results

- $\omega(QI(n,2)) = \binom{n-1}{\lfloor \frac{n}{2} \rfloor}$ (Kleitman and Spencer, Katona 1973).
- $\chi(QI(n,2)) = \lceil \frac{1}{2} \binom{n}{\lceil \frac{n}{2} \rceil} \rceil$ (Meagher and Stevens 2005)
- If $CAN(G,2) \leq n$, then there exists a $CAN(n,G,2)$ with $\lceil \frac{n}{2} \rceil$ 0's per row (nearly balanced). (Meagher and Stevens 2005)

Binary alphabet results

- $\omega(QI(n,2)) = \binom{n-1}{\lfloor \frac{n}{2} \rfloor}$ (Kleitman and Spencer, Katona 1973).
- $\chi(QI(n,2)) = \lceil \frac{1}{2}\binom{n}{\lceil \frac{n}{2} \rceil} \rceil$ (Meagher and Stevens 2005)
- If $CAN(G,2) \leq n$, then there exists a $CAN(n,G,2)$ with $\lceil \frac{n}{2} \rceil$ 0's per row (nearly balanced). (Meagher and Stevens 2005)

Binary alphabet results

- $\omega(QI(n,2)) = \binom{n-1}{\lfloor \frac{n}{2} \rfloor}$ (Kleitman and Spencer, Katona 1973).
- $\chi(QI(n,2)) = \lceil \frac{1}{2} \binom{n}{\lceil \frac{n}{2} \rceil} \rceil$ (Meagher and Stevens 2005)
- If $CAN(G,2) \leq n$, then there exists a $CAN(n,G,2)$ with $\lceil \frac{n}{2} \rceil$ 0's per row (nearly balanced). (Meagher and Stevens 2005)

Conjecture for general g (Meagher)

If $CAN(G,g) \leq n$, then there exists a $CAN(n,G,g)$ that is nearly balanced (each symbol appears either $\lceil \frac{n}{g} \rceil$ or $\lfloor \frac{n}{g} \rfloor$ times).

Binary alphabet results

- $\omega(QI(n,2)) = \binom{n-1}{\lfloor \frac{n}{2} \rfloor}$ (Kleitman and Spencer, Katona 1973).
- $\chi(QI(n,2)) = \lceil \frac{1}{2} \binom{n}{\lceil \frac{n}{2} \rceil} \rceil$ (Meagher and Stevens 2005)
- If $CAN(G,2) \leq n$, then there exists a $CAN(n,G,2)$ with $\lceil \frac{n}{2} \rceil$ 0's per row (nearly balanced). (Meagher and Stevens 2005)

Conjecture for general g (Meagher)

If $CAN(G,g) \leq n$, then there exists a $CAN(n,G,g)$ that is nearly balanced (each symbol appears either $\lceil \frac{n}{g} \rceil$ or $\lfloor \frac{n}{g} \rfloor$ times).

- If true, we can concentrate on $AUQI(n,g)$ (almost uniform qualitative independence graphs)!

Mixed covering arrays

Different factors/parameters can have different alphabet sizes.

gi =	2	2	2	2	3
	0	*0*	*0*	*0*	*0*
	1	*1*	*1*	*1*	*0*
	1	*0*	*1*	*0*	*1*
	0	*1*	*0*	*1*	*1*
	1	*0*	*0*	*1*	*2*
	0	*1*	*1*	*0*	*2*

References:

- Moura, Stardom, Stevens and Williams, "Mixed covering arrays" (2003)
- Colbourn, Martirosian, Mullen, Shasha, Sherwood and Yucas, "Pruducts of mixed covering arrays of strength two" (2006)
- Sherwood, "A column expansion construction for optimal and near-optimal mixed covering arrays" (preprint).

Mixed covering arrays on graphs

Combine covering array on graphs with mixed alphabet sizes.
Reference:

- Meagher, Moura, Zekaoui, 'Mixed covering arrays on graphs",(to appear):
  generalize graph homomorphism results; give optimal constructions for special classes of graphs.
- Cheng, "The Test Suite Generation Problem: Optimal Instances and Their Implications", preprint.
  Give optimal constructions for special classes of graphs and for hypertrees.

Higher strength ($t \geq 3$)

t=3
k=5
g=2

1	1	1	1	1
1	0	0	1	0
0	1	0	1	1
0	0	1	0	1
1	1	0	0	1
0	0	0	0	0
0	1	1	0	0
1	0	1	0	0
1	1	0	1	0
0	0	1	1	0

References:

- Chateauneuf and Kreher, "On the state of covering arrays of strength three ", 2002.
- Colbourn, Martirosyan, Trung, and Walker, "Roux-type Constructions for Covering Arrays of Strengths Three and Four" (2006).
- etc.

Locating arrays

We not only want to detect that an error exists, but we want to know which t-interaction caused the error.
Related to design of experiments and combinatorial group testing.

mixed covering array 2*3*3*3*3*3

d=1 t=2

C^T

0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0
0	1	2	2	1	0	2	1	1	2	1	2	0	0	2	1	0	2	2	0
1	0	1	2	2	2	0	2	1	1	2	1	2	0	0	0	1	0	2	2
2	1	0	1	2	1	2	0	2	1	0	2	1	2	0	2	0	1	0	2
2	2	1	0	1	1	1	2	0	2	0	0	2	1	2	2	2	0	1	0
1	2	2	1	0	2	1	1	2	0	2	0	0	2	1	0	2	2	0	1

Reference:

- Colbourn and McClary, "Locating and detecting arrays: existence and minimization", preprint.

Work in progress by myself with Martinez, Panario and Stevens on the adaptive problem.

References

1. R. Ahlswede, N. Cai and Z. Zhang. Higher level extremal problems. *J. Combin. Inform. System Sci.*, 21:185–210, 1996.
2. M. Chateauneuf and D. L. Kreher. On the state of strength-three covering arrays *J. Combin. Des.*, 10:217-238, 2002.
3. C. Cheng. The Test Suite Generation Problem: Optimal Instances and Their Implications, *preprint*.
4. C.J. Colbourn, M.B. Cohen, and R.C. Turban. A Deterministic Density Algorithm for Pairwise Interaction Coverage. *Proceedings of the International Conference on Software Engineering (SE 2004), Innsbruck, Austria*, pp. 245-252, February 2004.
5. D.M. Cohen, S.R. Dalal, M.L. Fredman and G.C.Patton. The AETG system: an approach to testing based on combinatorial design. *IEEE Trans. Soft. Engin.*, 23:437-444, 1997.
6. C.J. Colbourn, S.S. Martirosyan, G.L. Mullen, D.E. Shasha, G.B. Sherwood, and J.L. Yucas. Products of Mixed Covering Arrays of Strength Two. *J. Combin. Des.*, 14:124–138, 2006.
7. C.J. Colbourn, S.S. Martirosyan, Tran Van Trung, and R.A. Walker II. Roux-type Constructions for Covering Arrays of Strengths Three and Four. *Designs, Codes and Cryptography*, 41:33–57, 2006.
8. P. Erdos, C. Ko and R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser.* 12:313–320, 1961.
9. G. Katona. Two applications (for search theory and truth functions) of Sperner type theorems. *Period. Math. Hungar.*, 3:19–26, 1973.
10. D.J. Kleitman and J. Spencer. Families of k-independent sets. *Discrete Math.*, 6:255–262, 1973.
11. K. Meagher. Covering arrays on graphs: qualitative independence graphs and extremal set-partition theory.*Ph.D. thesis*, University of Ottawa, 177 pages, Sept. 2005.
12. K. Meagher and L. Moura. Erdos-Ko-Rado theorems for uniform set-partition systems. *Electr. J. Combin.* 12, R40, 12 pages, 2005.
13. K. Meagher, L. Moura and B. Stevens. A Sperner-type theorem for set-partition systems. *Electr. J. Combin.* 12, N20, 6 pages, 2005.
14. K. Meagher and B. Stevens. Covering arrays on graphs. *Journal of Combinatorial Theory B*, 95(1):134–151, 2005.

References continued

15. L. Moura, J. Stardom, B. Stevens and A. Williams. Covering arrays with mixed alphabet sizes. *Journal of Combinatorial Designs*, 11:413–432, 2003. (2003)
16. G. Sherwood,. A column expansion construction for optimal and near-optimal mixed covering arrays. *preprint*.
17. N. Sloane. Covering arrays and intersection codes. *Journal of Combinatorial Designs*, 1:51–63, 1993.
18. E. Sperner. Ein Satz uber Untermengen einer endlichen Menge. *Math. Z.*, 27:544-548, 1928.
19. B. Stevens, L. Moura and E. Mendelsohn. Lower bounds for transversal covers. *Des., Codes and Cryptogr.*, 15:279-299, 1998.