CSI 2101 Discrete Structures — Winter 2012
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Quiz #7

1. Use a loop invariant to prove that the following program segment for computing the nth power (where n is a positive integer) of a real number x is correct.

   power $= 1$
   $i = 1$
   while $i \leq n$ **do**
     power $=$ power $\times x$
     $i = i + 1$
   end while

   Let p be the assertion "$i \leq n+1$ and power $= x^{i-1}$". We must first show that p is a loop invariant: to do so, we have to show that if p is true at the beginning of an execution of the loop, then p is still true after the execution of the loop.

   Suppose that, at the beginning of one execution of the while loop, p is true and the condition of the while loop holds: in other words, we assume that power $= i^{n-1}$ and that $i \leq n$. The new values i_{new} and power$_{\text{new}}$ are:

   $$i_{\text{new}} = i + 1$$

   $$\text{power}_{\text{new}} = \text{power} \times x = x^{i-1} \times x = x^{i} = x^{i_{\text{new}}-1}$$

   Because $i \leq n$, we also have that $i_{\text{new}} \leq n+1$. Thus, p is still true at the end of the execution of the loop.

   We now must show that p was true before the loop executed. In the program header, we set $i = 1$. Since, by assumption, n is a positive integer, then $n \geq 1$, so we have that $i \leq n+1$. We also have power $= 1 = x^0 = x^{i-1}$. Thus, p is true before the loop is executed.

   Because p is a loop invariant, we have that when the loop terminates, it terminates with p true and the condition of the loop, $i \leq n$ false. Thus, $i = n+1$, and so power $= x^{(n+1)-i} = x^n$, which is the result that we want, so the progam calculates correctly.

   Finally, we need to check that the loop actually does terminate. The value of i is initialized to 1, and with every execution of the loop, is incremented by 1. Thus, the loop will terminate after n iterations.

   Thus, the program is correct.