Graph Homomorphism Tutorial

Field's Institute Covering Arrays Workshop 2006

Rick Brewster

Thompson Rivers University

Preparing this talk

What should I say?

Preparing this talk

What should I say?

What do you want to know?

Talk Outline

Talk Outline

- Basic Definitions;

Talk Outline

- Basic Definitions;
- Homomorphisms Generalize Colourings;

Talk Outline

- Basic Definitions;
- Homomorphisms Generalize Colourings;
- Graph Covering Arrays;

Talk Outline

- Basic Definitions;
- Homomorphisms Generalize Colourings;
- Graph Covering Arrays;
- Categorical Aspects;

Talk Outline

- Basic Definitions;
- Homomorphisms Generalize Colourings;
- Graph Covering Arrays;
- Categorical Aspects;
- Computational Aspects.

References

- P. Hell and J. Nešetřil, *Graphs and Homomorphisms*, Oxford University Press, 2004.
- C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer-Verlag, 2001.
- A. Pultr and V. Trnková, *Combinatorial, Algebraic, and Topological Representations of Groups, Semigroups and Categories*, North-Holland, 1980.

References

- G. Hahn and C. Tardif, Graph homomorphisms: structure and symmetry, in *Graph Symmetry, Algebraic Methods and Applications* (G. Hahn and G. Sabidussi eds.) NATO ASI Series C 497, Kluwer 1997.
- G. Hahn and G. MacGillivray, Graph homomorphisms: computational aspects and infinite graphs, manuscript, 2002.
- P. Hell, Algorithmic aspects of graph homomorphisms, in *Surveys in Combinatorics 2003* (C. D. Wensley ed.) *London Math. Soc. Lecture Notes Series* **307** Cambridge University Press.

Graph Homomorphisms

Definition 1 *Let G and H be graphs. A homomorphism of G to H is a*

Graph Homomorphisms

Definition 1 *Let G and H be graphs. A homomorphism of G to H is a function $f : V(G) \to V(H)$ such that*

$$xy \in E(G) \Rightarrow f(x)f(y) \in E(H).$$

Graph Homomorphisms

Definition 1 *Let G and H be graphs. A homomorphism of G to H is a function $f : V(G) \to V(H)$ such that*

$$xy \in E(G) \Rightarrow f(x)f(y) \in E(H).$$

Adjacent vertices receive adjacent images.

Graph Homomorphisms

Definition 1 *Let G and H be graphs. A homomorphism of G to H is a function $f : V(G) \to V(H)$ such that*

$$xy \in E(G) \Rightarrow f(x)f(y) \in E(H).$$

We write $G \to H$ ($G \not\to H$) if there is a homomorphism (no homomorphism) of G to H.

Beyond graphs

Definition of a homomorphism naturally extends to:

- digraphs;
- edge-coloured graphs;
- relational systems.

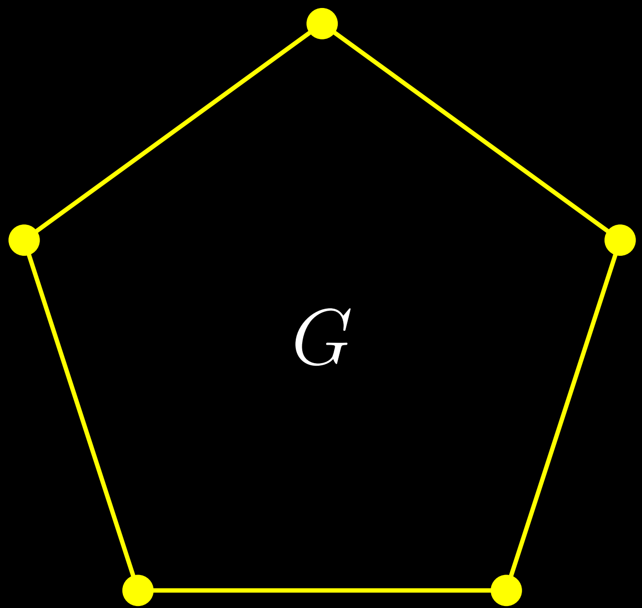
Beyond graphs

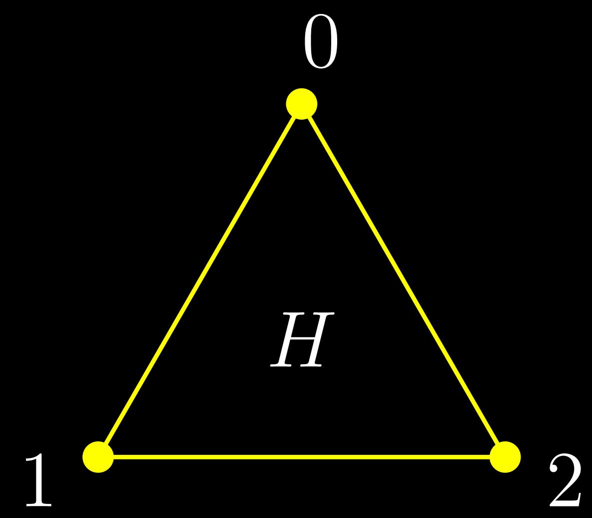
Definition of a homomorphism naturally extends to:

- digraphs;
- edge-coloured graphs;
- relational systems.

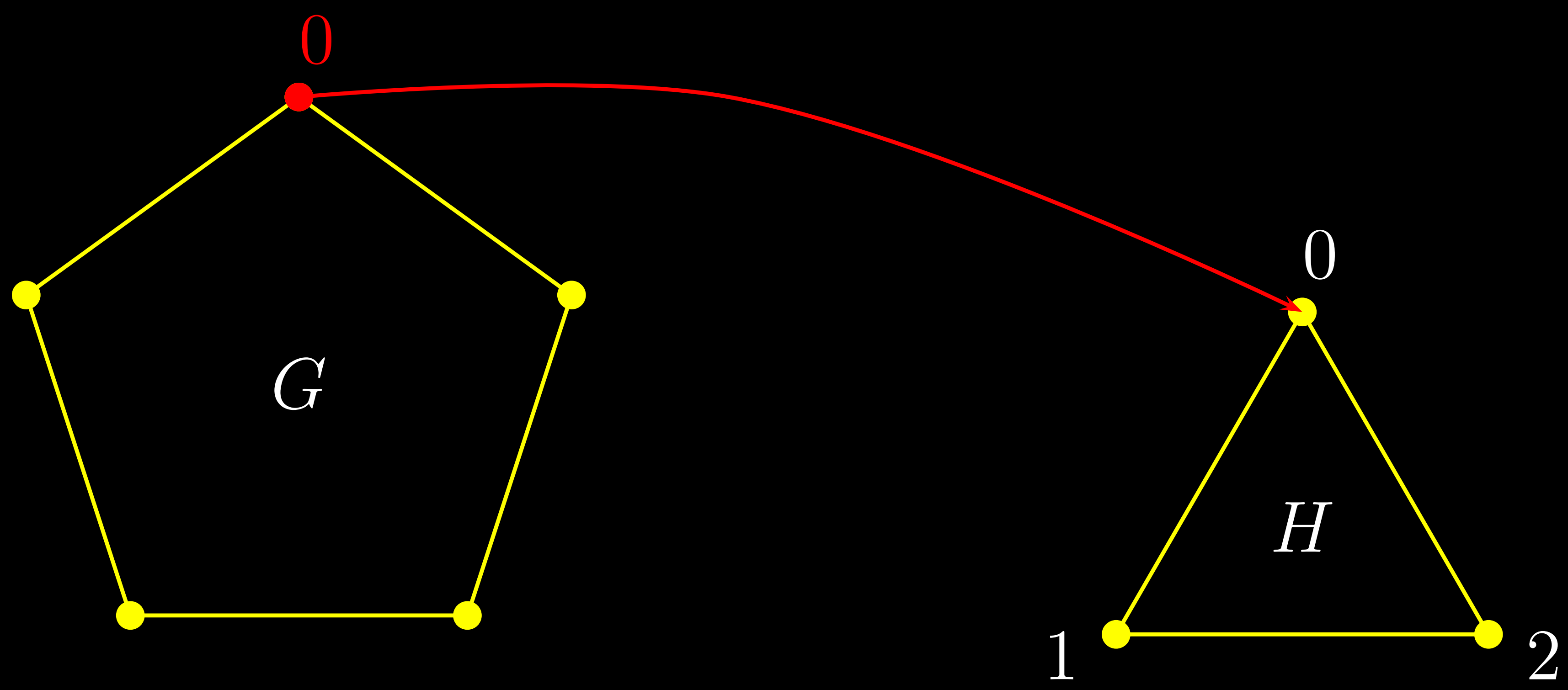
Hot idea: *Constraint Satisfaction Problems* encoded as homomorphisms.

An example

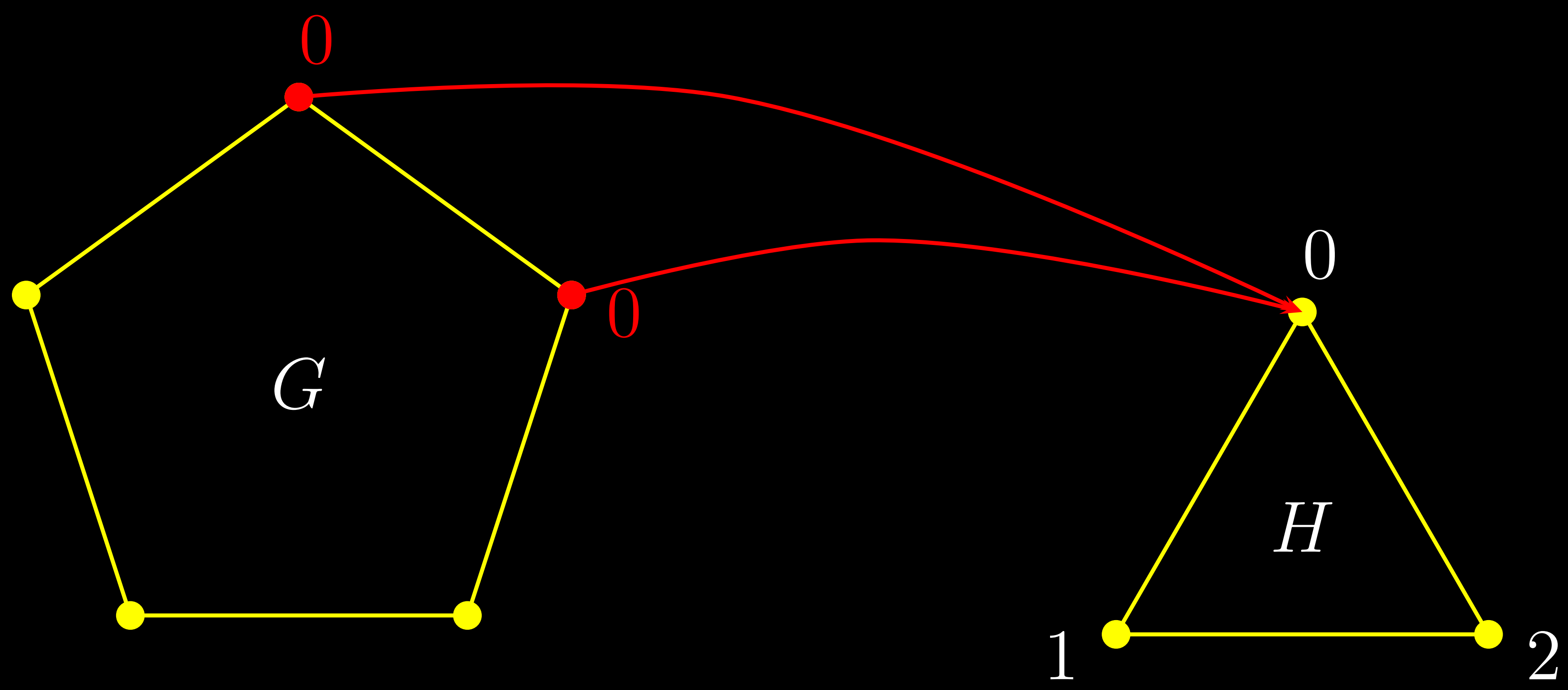




An example

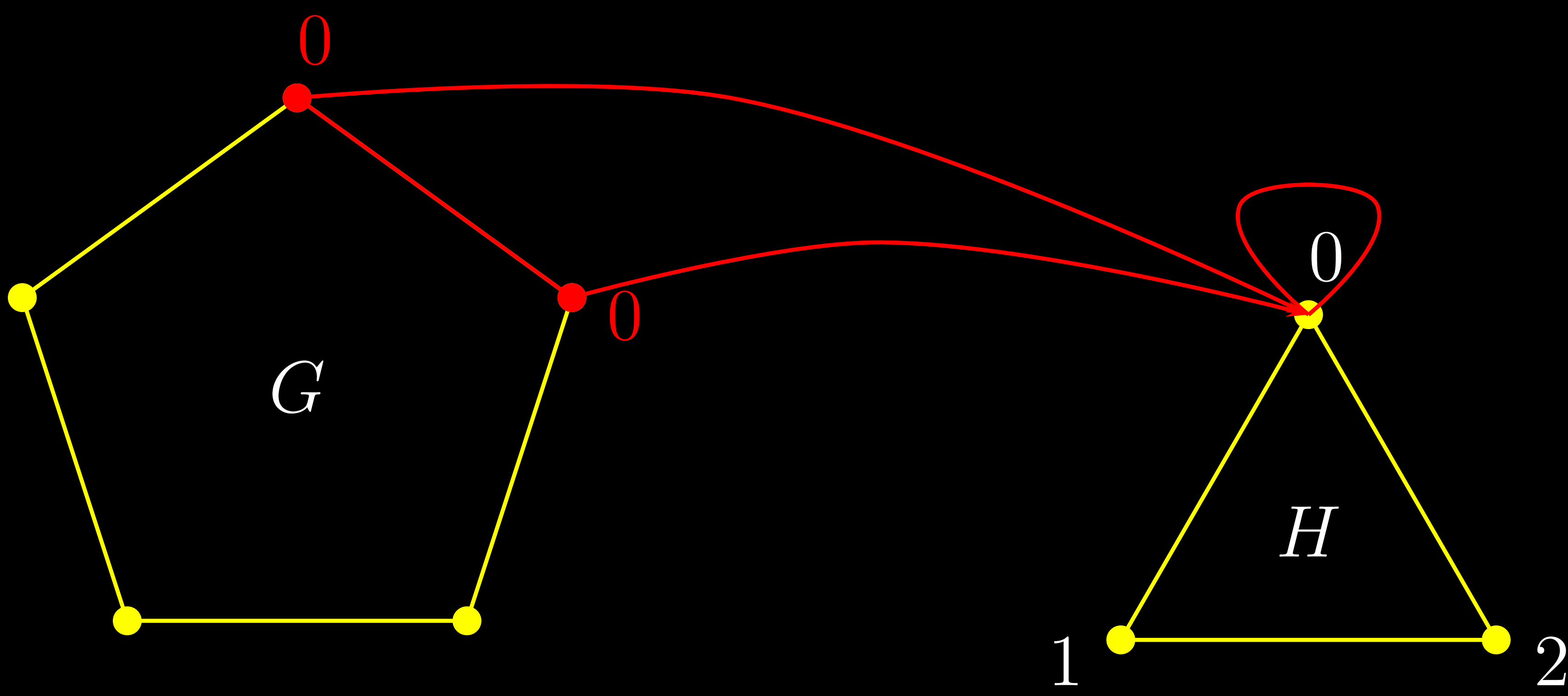


An example



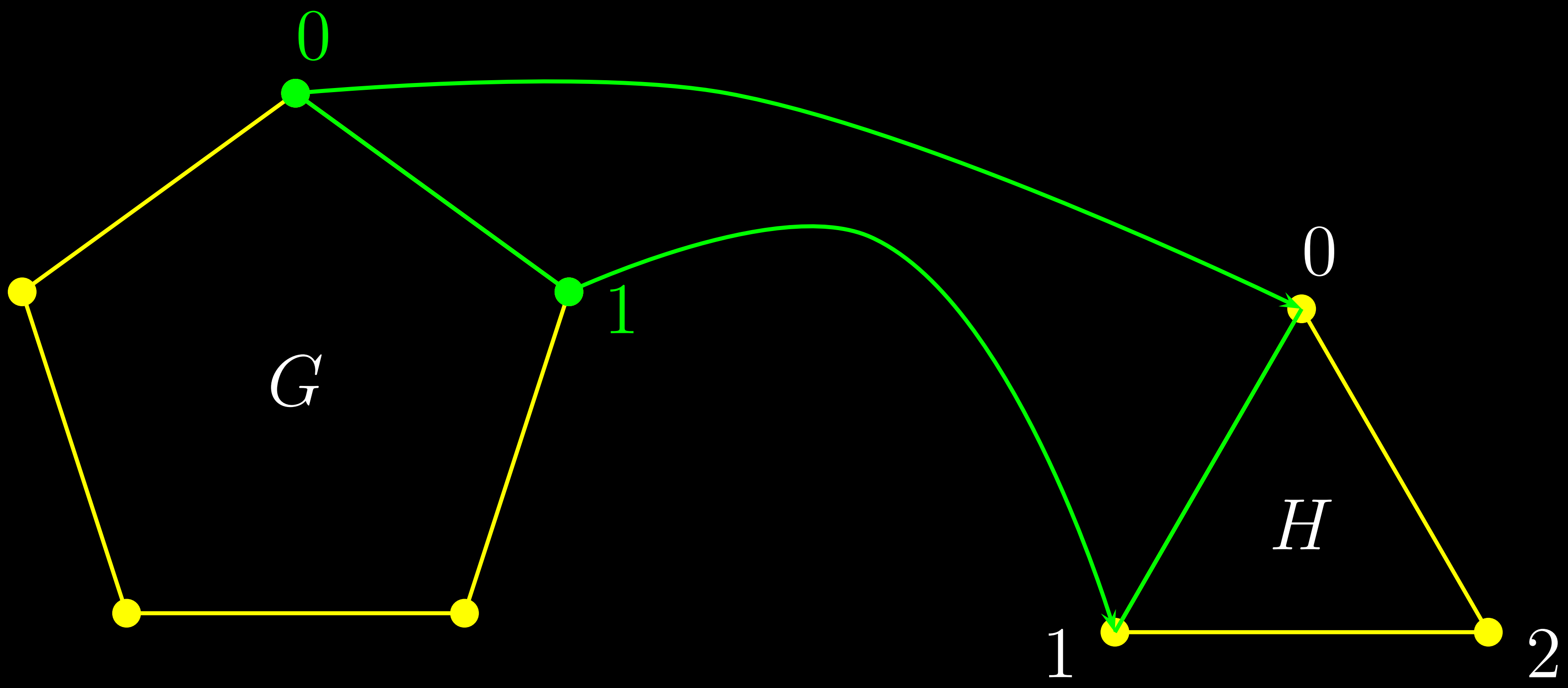
Why is this assignment not allowed?

An example



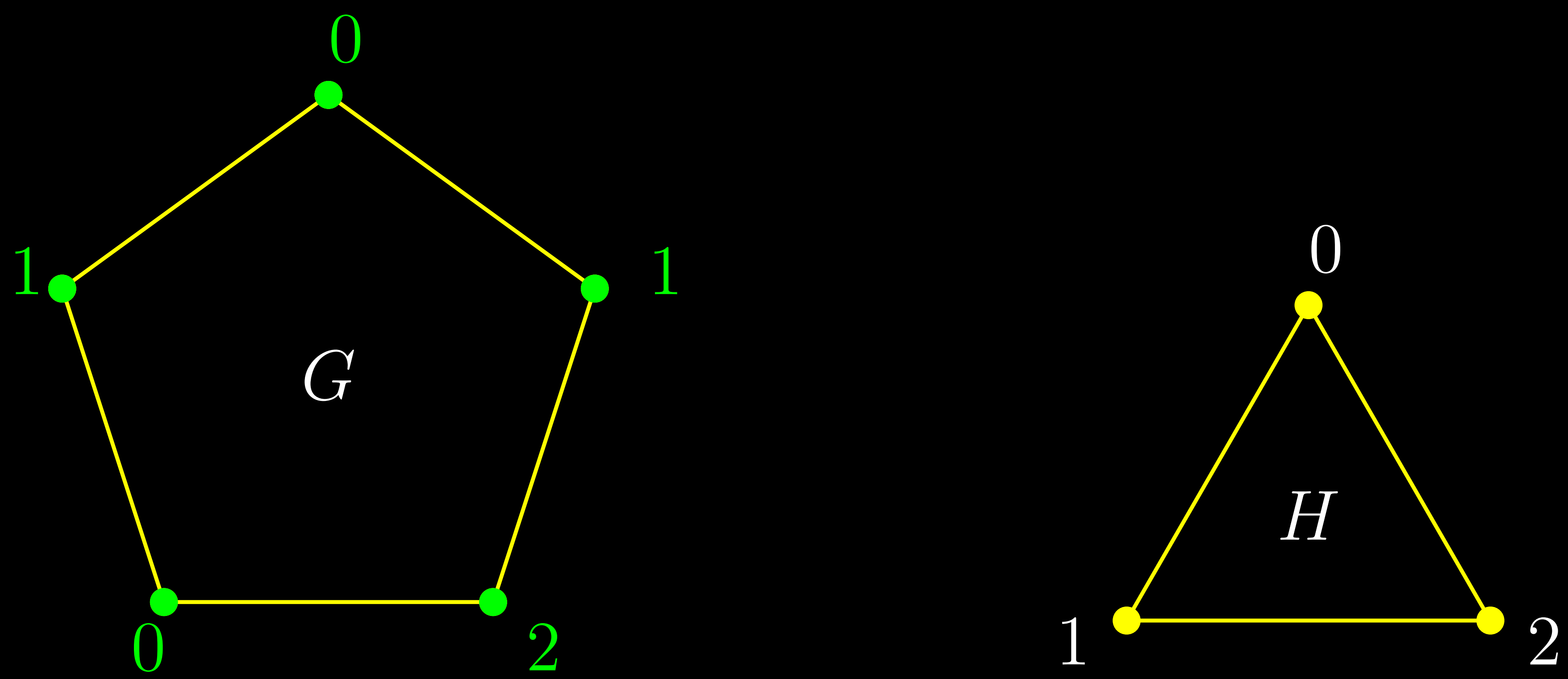
This assignment requires a loop on vertex 0 (in H)

An example



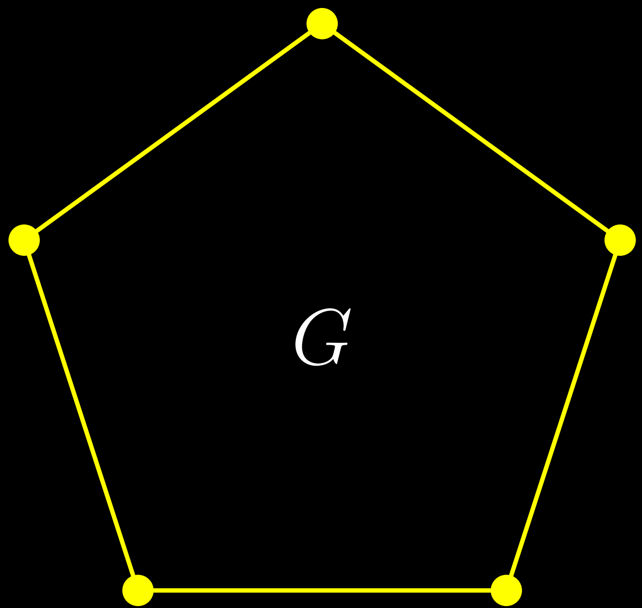
This assignment is allowed.

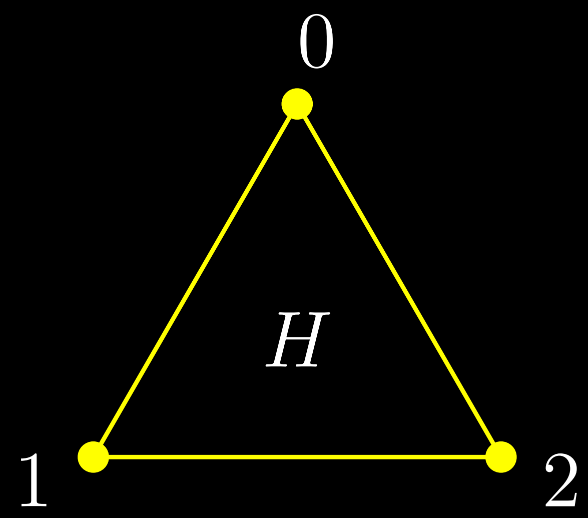
An example



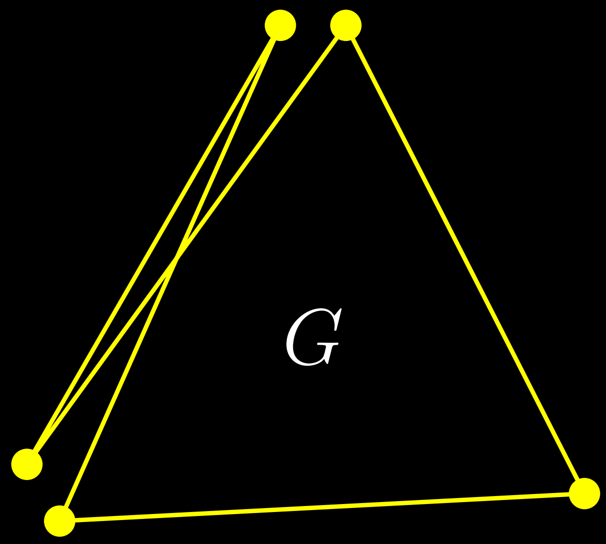
This labeling is a homomorphism $G \to H$.

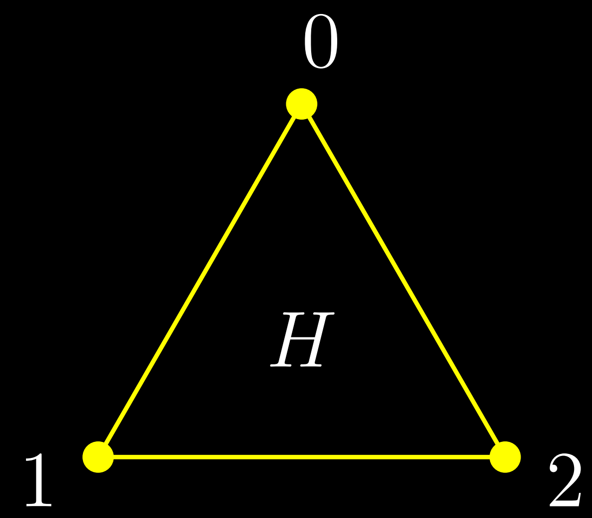
A partitioning problem



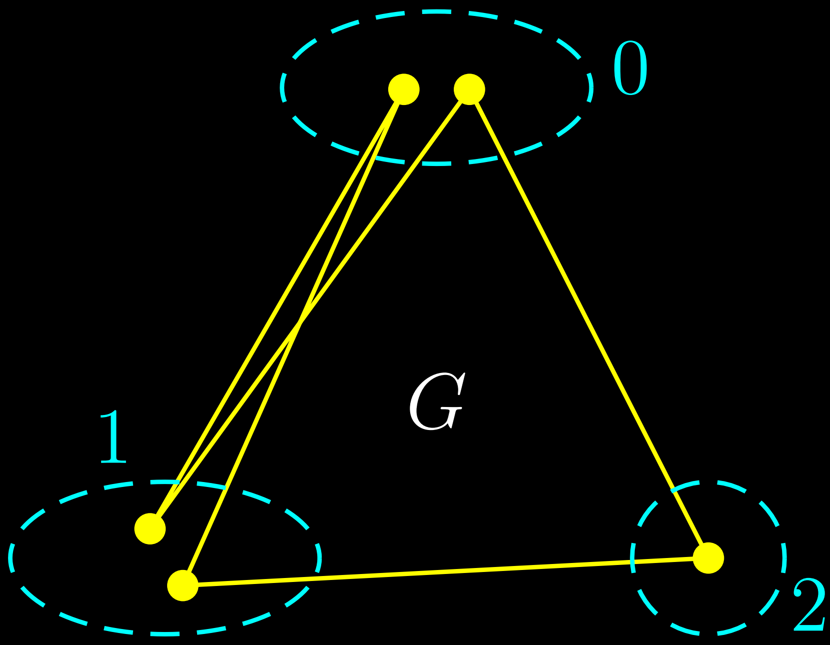


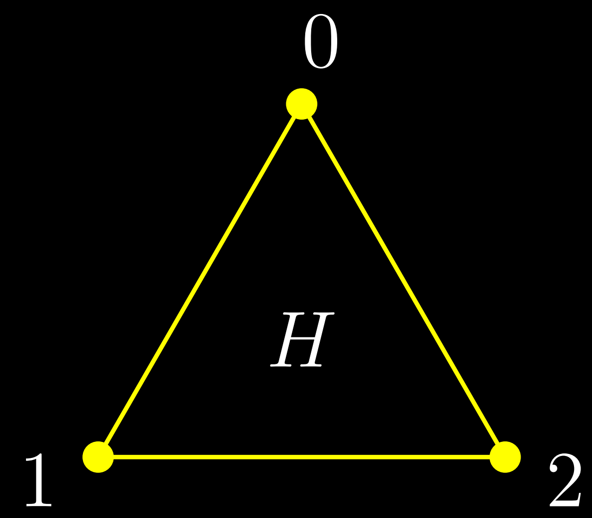
A partitioning problem





A partitioning problem





The *quotient* of the partition is a subgraph of H

The partition is the *kernel* of the map.

Some observations

Many keys ideas appear in our example:

Some observations

Many keys ideas appear in our example:

- $G \longrightarrow K_3$ iff G is 3-colourable.
- $G \longrightarrow K_n$ iff G is n-colourable.

Some observations

Many keys ideas appear in our example:

- $G \longrightarrow K_3$ iff G is 3-colourable.
- $G \longrightarrow K_n$ iff G is n-colourable.
- Homomorphisms generalize colourings.
- Testing the existence of a homomorphism is a hard problem.

Some observations

Many keys ideas appear in our example:

- $G \longrightarrow K_3$ iff G is 3-colourable.
- $G \longrightarrow K_n$ iff G is n-colourable.
- Homomorphisms generalize colourings.
- Testing the existence of a homomorphism is a hard problem.

Notation: $G \rightarrow H$ is an H-colouring of G.

The complexity of H-colouring

Let H be a fixed graph.

H-colouring
Instance: A graph G.
Question: Does G admit an H-colouring.

Theorem 1 (Hell and Nešetřil, 1990) *If H is bipartite or contains a loop, then H-colouring is polynomial time solvable; otherwise, H is NP-complete.*

About loops

- If H contains a loop, then the testing $G \overset{?}{\longrightarrow} H$ is trivial.
- Variants of H-colouring remain difficult when loops are allowed.

We will assume graphs are loop-free unless stated otherwise.

In the language of homomorphisms

- Chromatic number

$$\chi(G) = \min_n \{n | G \to K_n\}$$

- Clique number

$$\omega(G) = \max_n \{n | K_n \to G\}$$

- Odd girth

$$og(G) = \min_\ell \{2\ell + 1 | C_{2\ell+1} \to G\}$$

Homomorphism language con't

An H-colouring of G is a partition of $V(G)$ subject to the edge structure in H.

- Independence number

$$\alpha(G) = \max_f \{ |f^{-1}(1)| \mid f : G \to H \}$$

General partitioning problems

- Split graphs

- G is a *split-graph* iff $\exists g, g : G \rightarrow H$ such that $g^{-1}(0)$ is complete.

General partitioning problems

- Split graphs 

- G is a *split-graph* iff $\exists g, g : G \longrightarrow H$ such that $g^{-1}(0)$ is complete.
- M-partitions – Feder, Hell, Klein, and Motwani.
- Trigraphs in Hell and Nešetřil book.

General partitioning problems

- Split graphs 

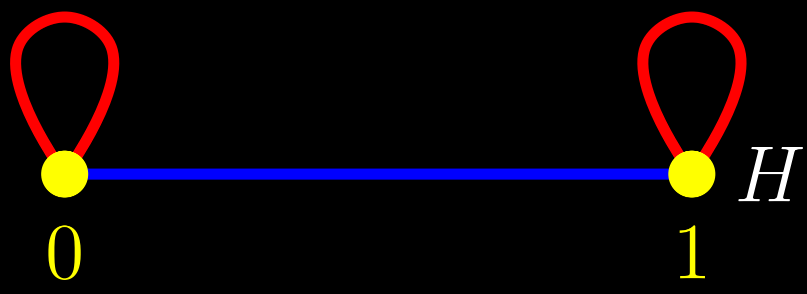
- G is a *split-graph* iff $\exists g, g : G \rightarrow H$ such that $g^{-1}(0)$ is complete.
- M-partitions – Feder, Hell, Klein, and Motwani.
- Trigraphs in Hell and Nešetřil book.
- clique-cut set, skew partition, homogenous set, ...

CSP encodings via Edge-coloured graphs

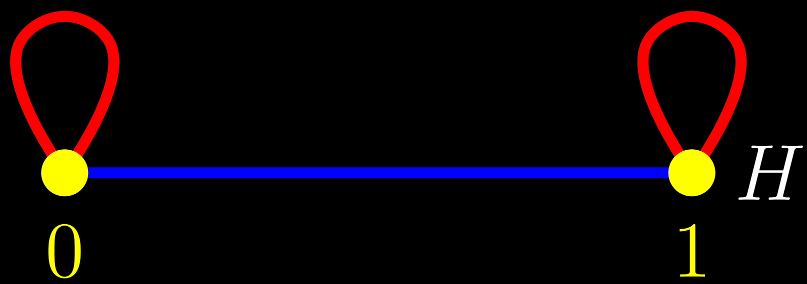
- Graphs have coloured edges.
- Homomorphisms preserve edges and their colours.

CSP encodings via Edge-coloured graphs

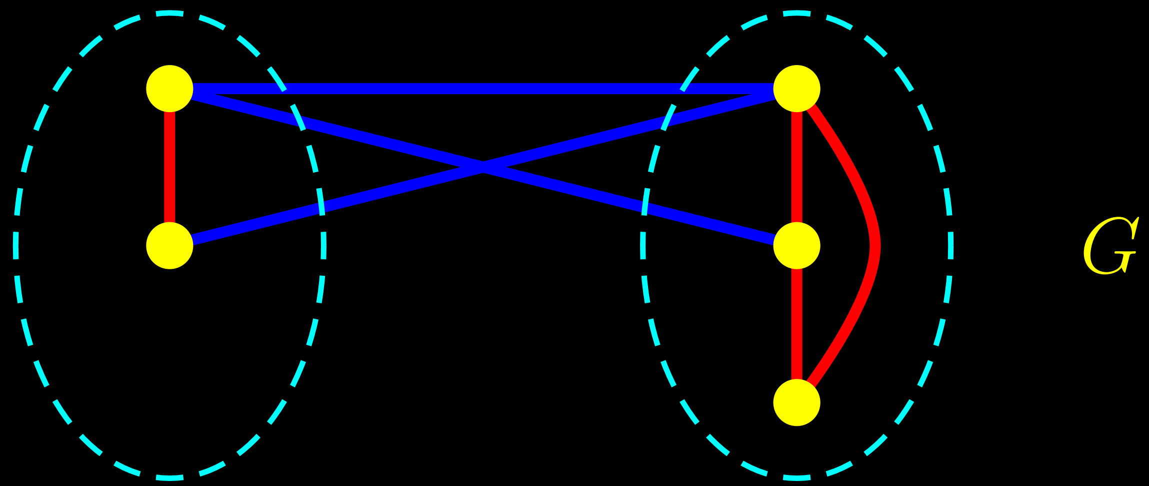
- Graphs have coloured edges.
- Homomorphisms preserve edges and their colours.
- Red edges encode *same*; and
- blue edges encode *different*.

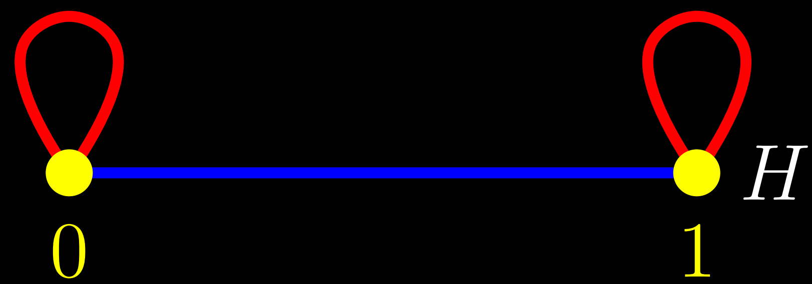


CSP encodings via Edge-coloured graphs



CSP encodings via Edge-coloured graphs





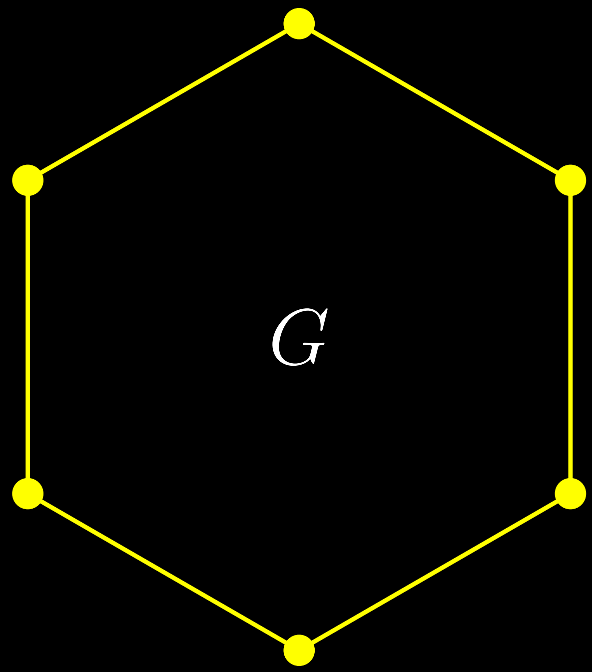
Colouring interpolation theorem

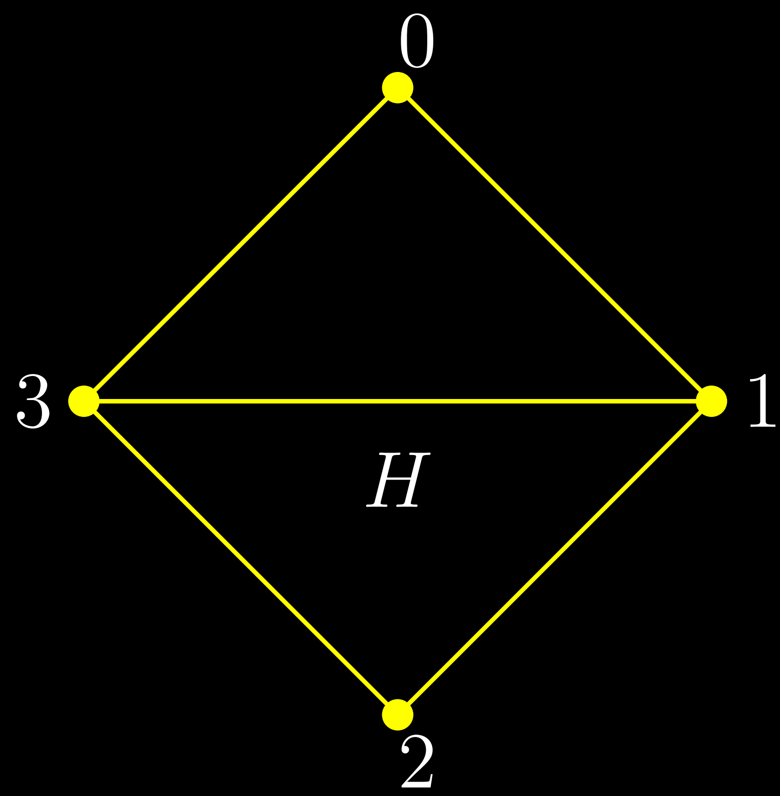
- *Achromatic number*

$$\psi(G) = \max_k \{k | G \xrightarrow{sur} K_k\}$$

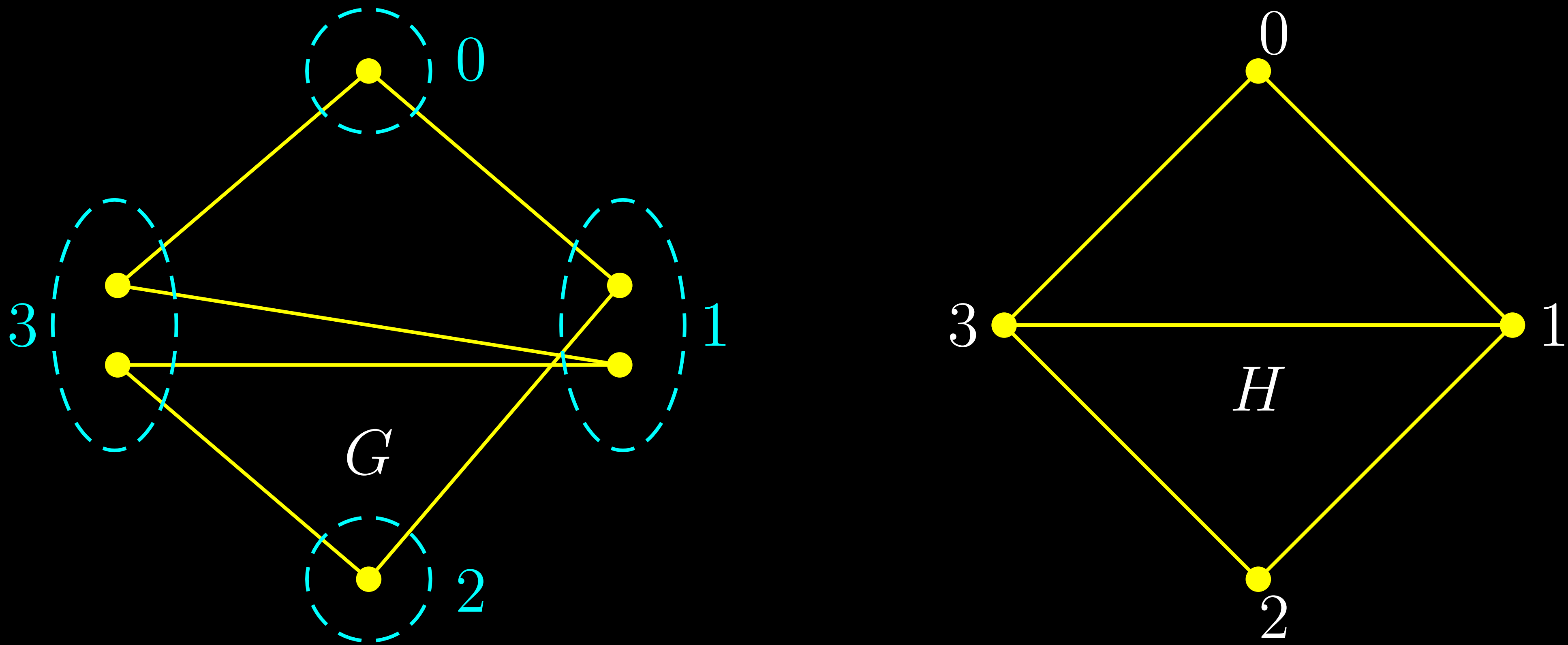
- *Complete* k-colourings
- **Theorem 1** *Let G be a graph. For each i, $\chi(G) \leq i \leq \psi(G)$, G admits a complete i-colouring.*

Another partitioning example



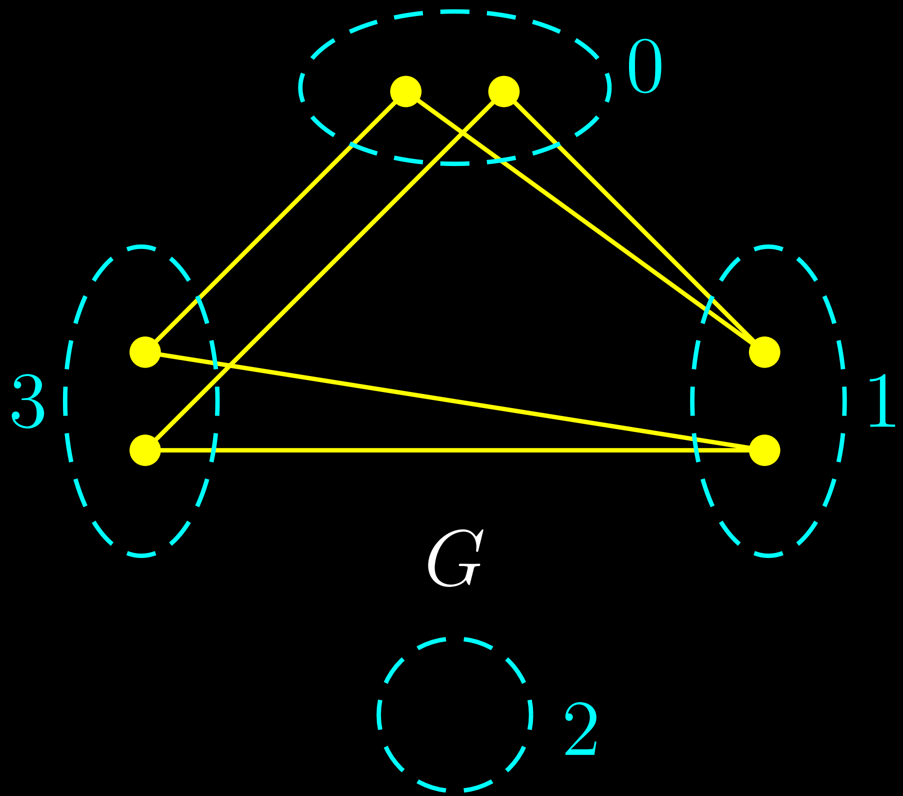


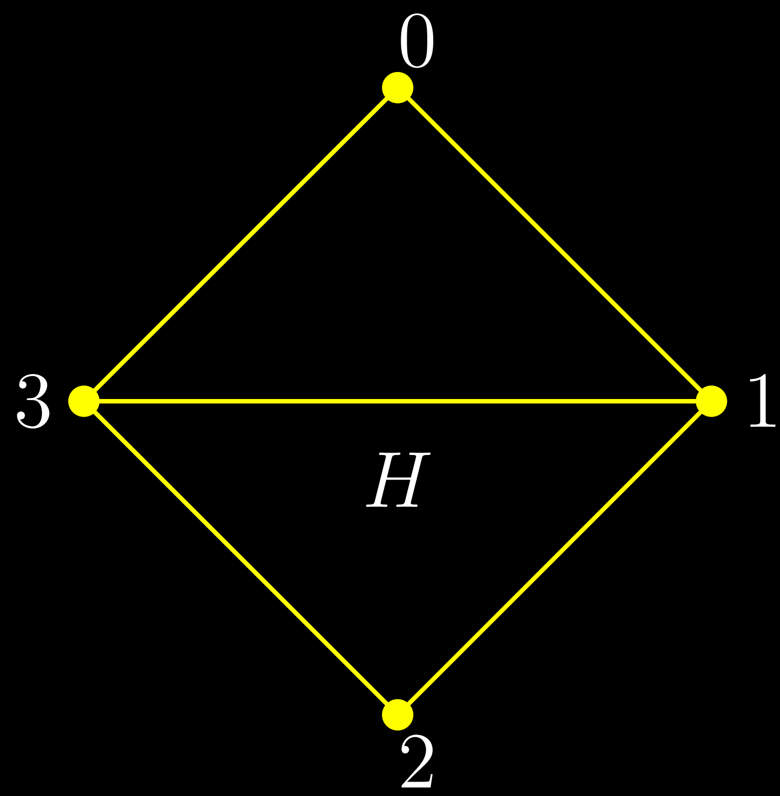
Another partitioning example



The quotient of the partition
is the *homomorphic image*, in this case H

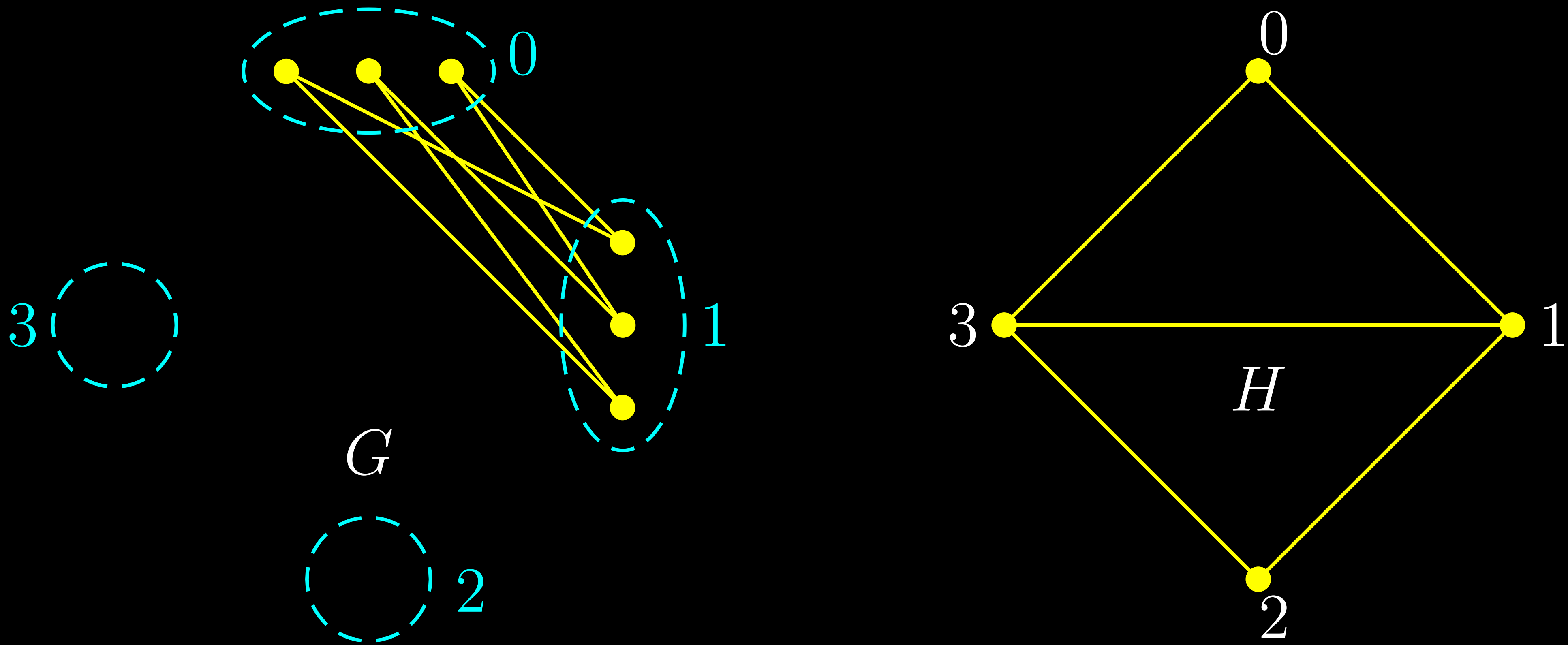
Another partitioning example





The homomorphic image is the upper triangle of H

Another partitioning example



The homomorphic image is the edge on $\{0, 1\}$
Note any bipartite graph will map to $\{0, 1\}$

A few natural questions about the hom-image

Given $f : G \to H$:

A few natural questions about the hom-image

Given $f : G \to H$:

- Is f vertex (edge) injective?

A few natural questions about the hom-image

Given $f : G \to H$:

- Is f vertex (edge) injective?
- Is f vertex (edge) surjective?

A few natural questions about the hom-image

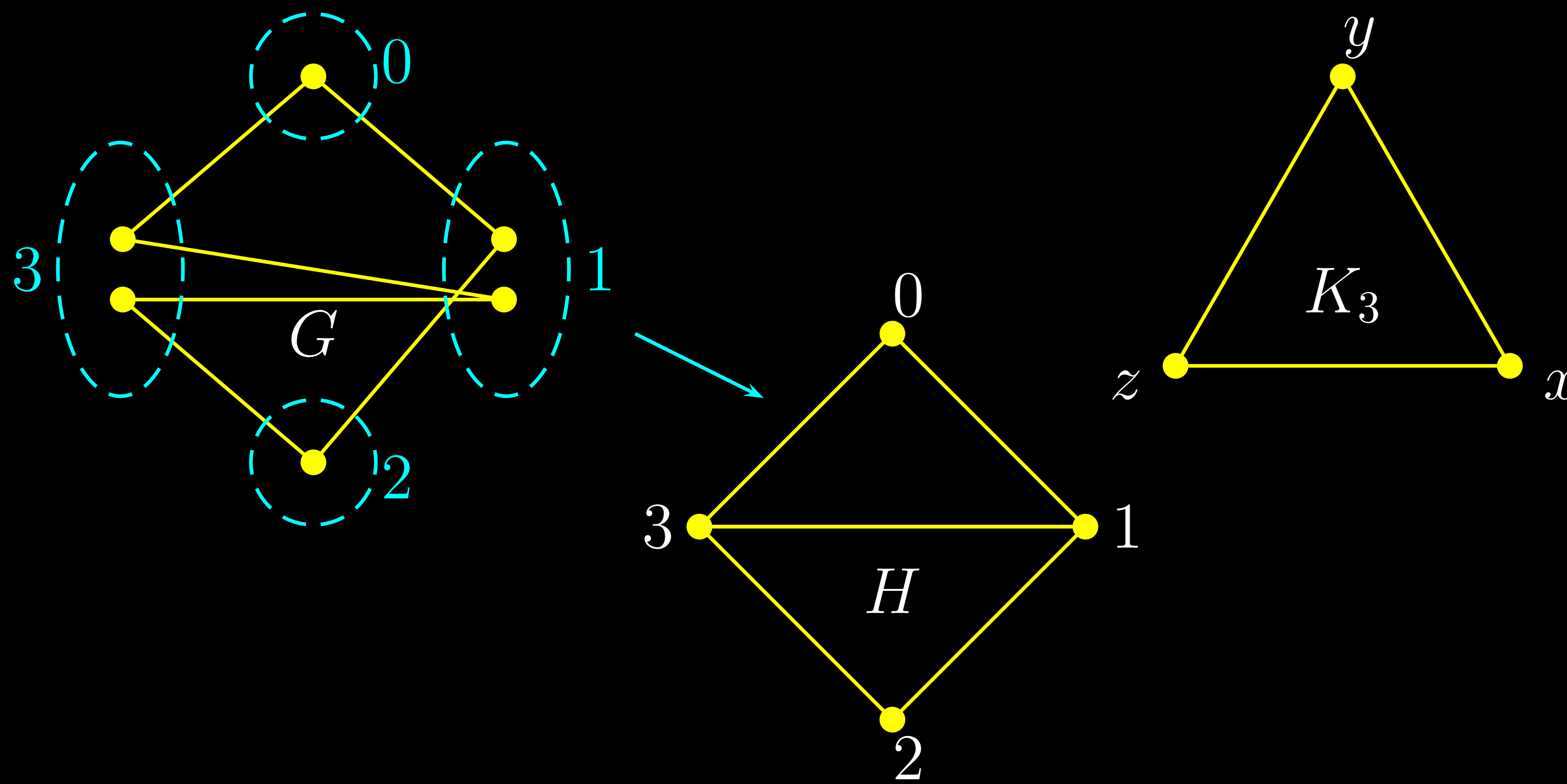
Given $f : G \to H$:

- Is f vertex (edge) injective?
- Is f vertex (edge) surjective?

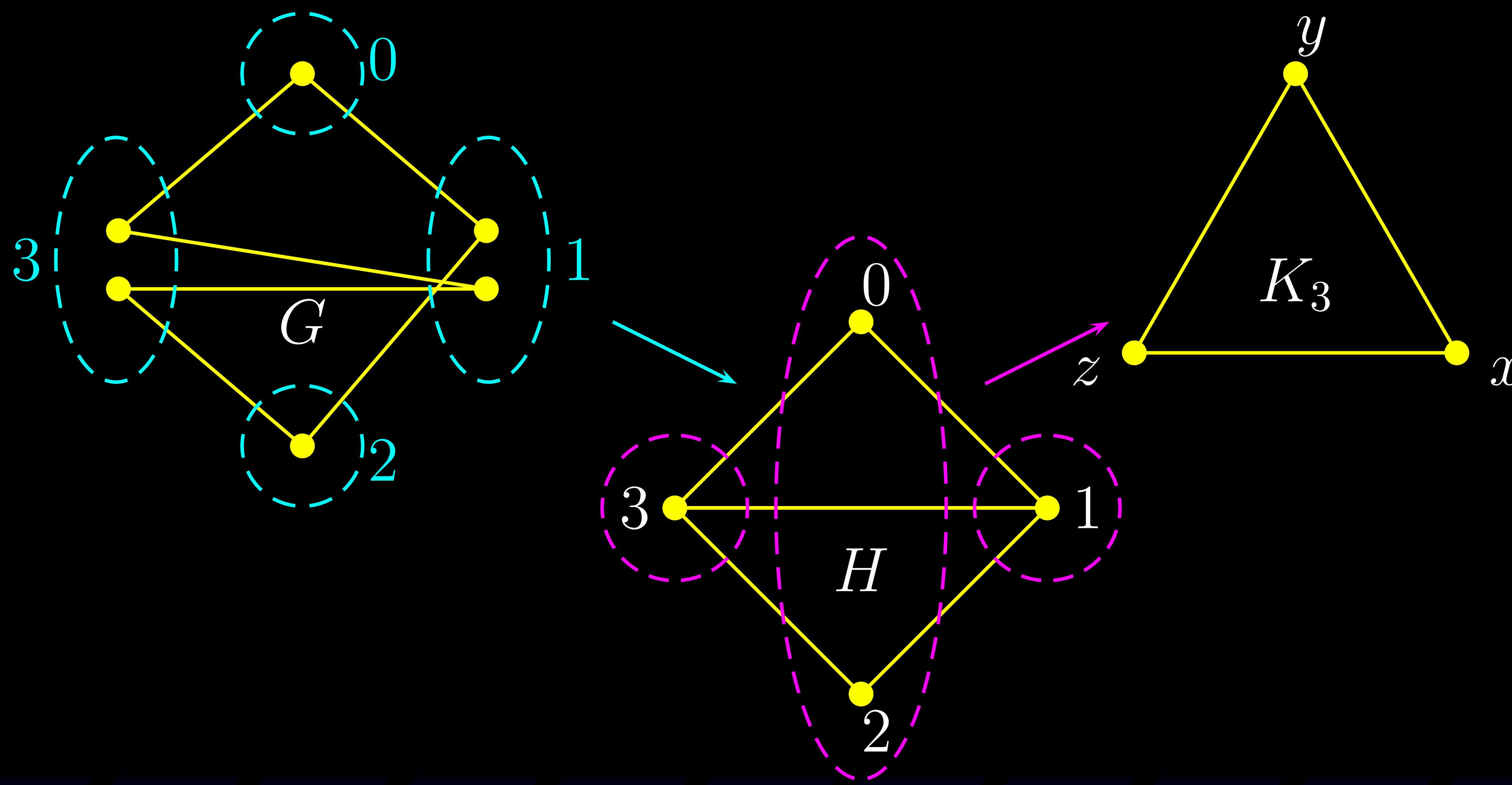
Homomorphisms generalize isomorphisms.

NP-complete versus Graph-Isomorphism complete.

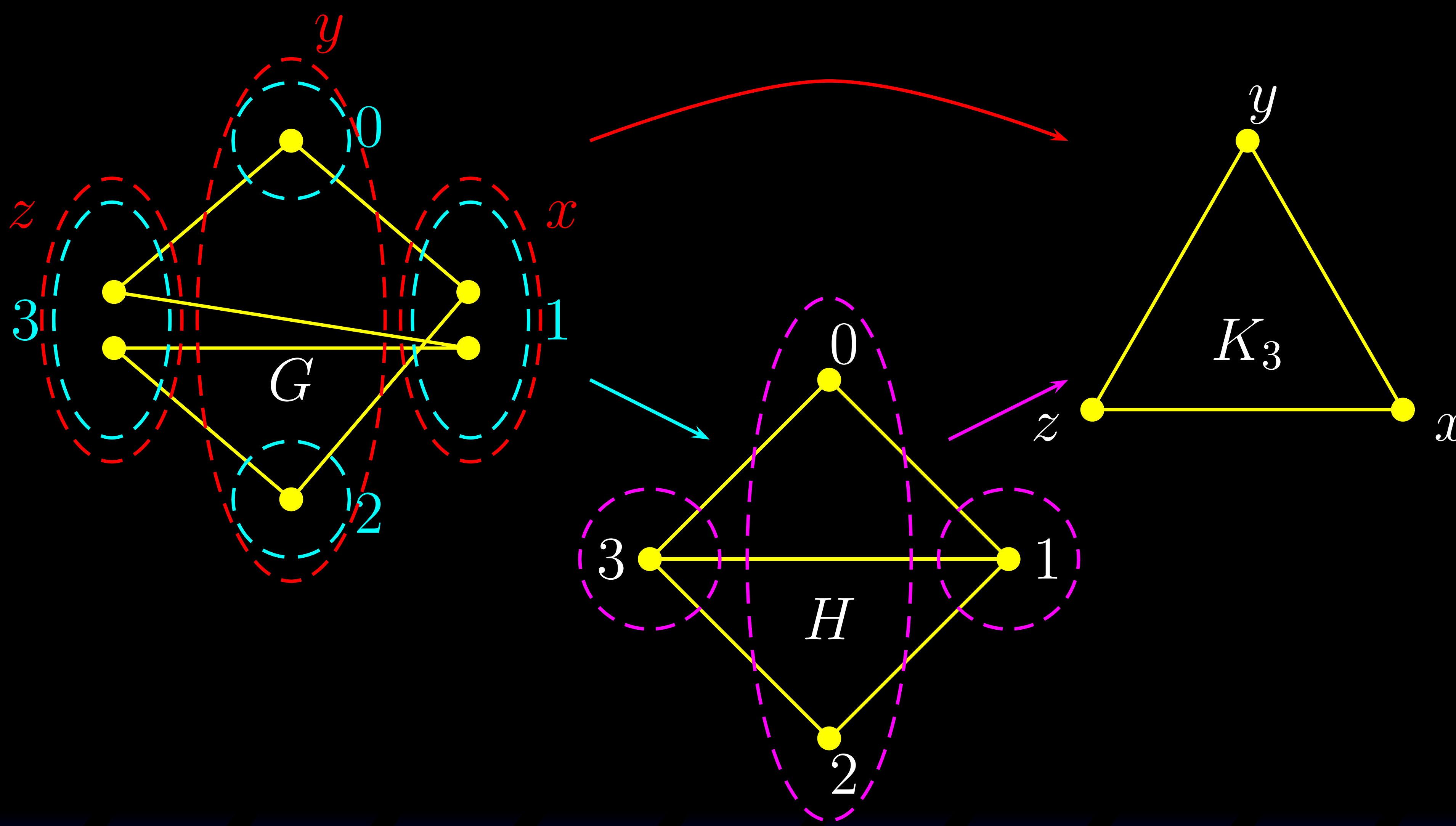
Homomorphisms compose



Homomorphisms compose



Homomorphisms compose



Learning to say no

Let G and H be graphs.

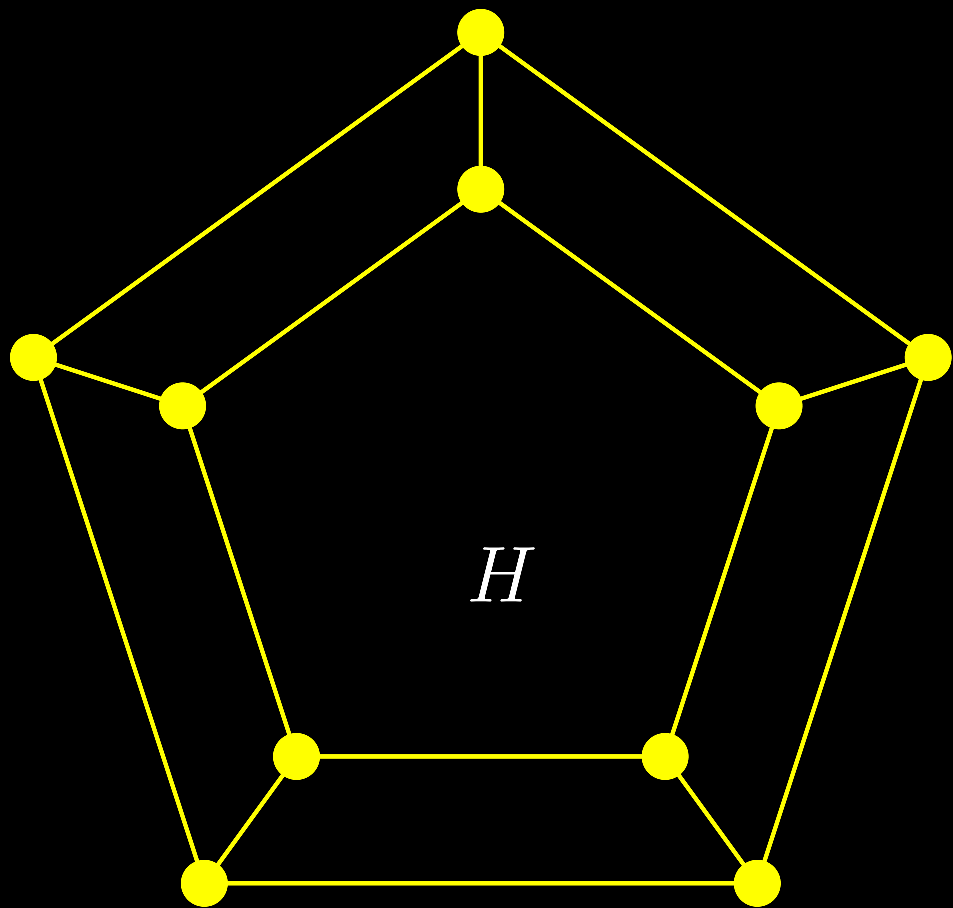
- If $\chi(G) > \chi(H)$, then $G \not\to H$.
- If $og(G) < og(H)$, then $G \not\to H$.
- If $F \to G$ and $F \not\to H$, then $G \not\to H$.

The core of a graph

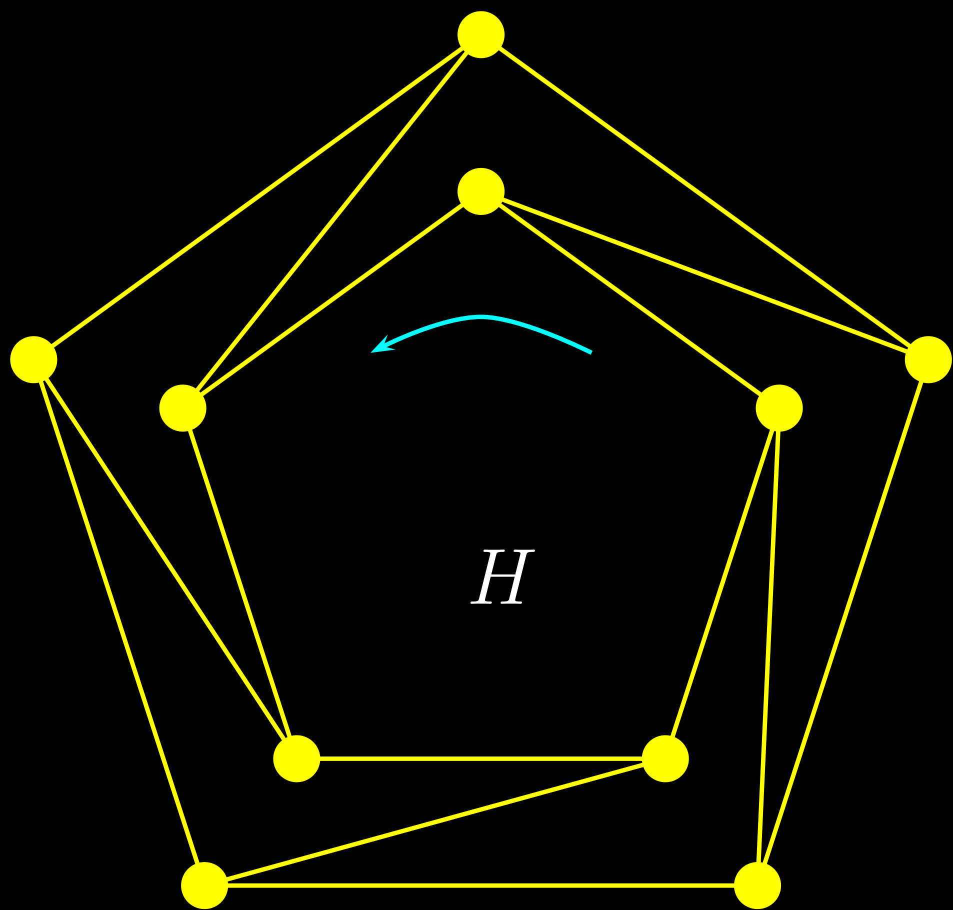
In our example,

- $H \longrightarrow K_3$ and $K_3 \hookrightarrow H$.
- H and K_3 are *homomorphism equivalent*.
- Every graph has a unique (up to iso) inclusion minimal subgraph to which it is hom-equivalent called the *core* of the graph.

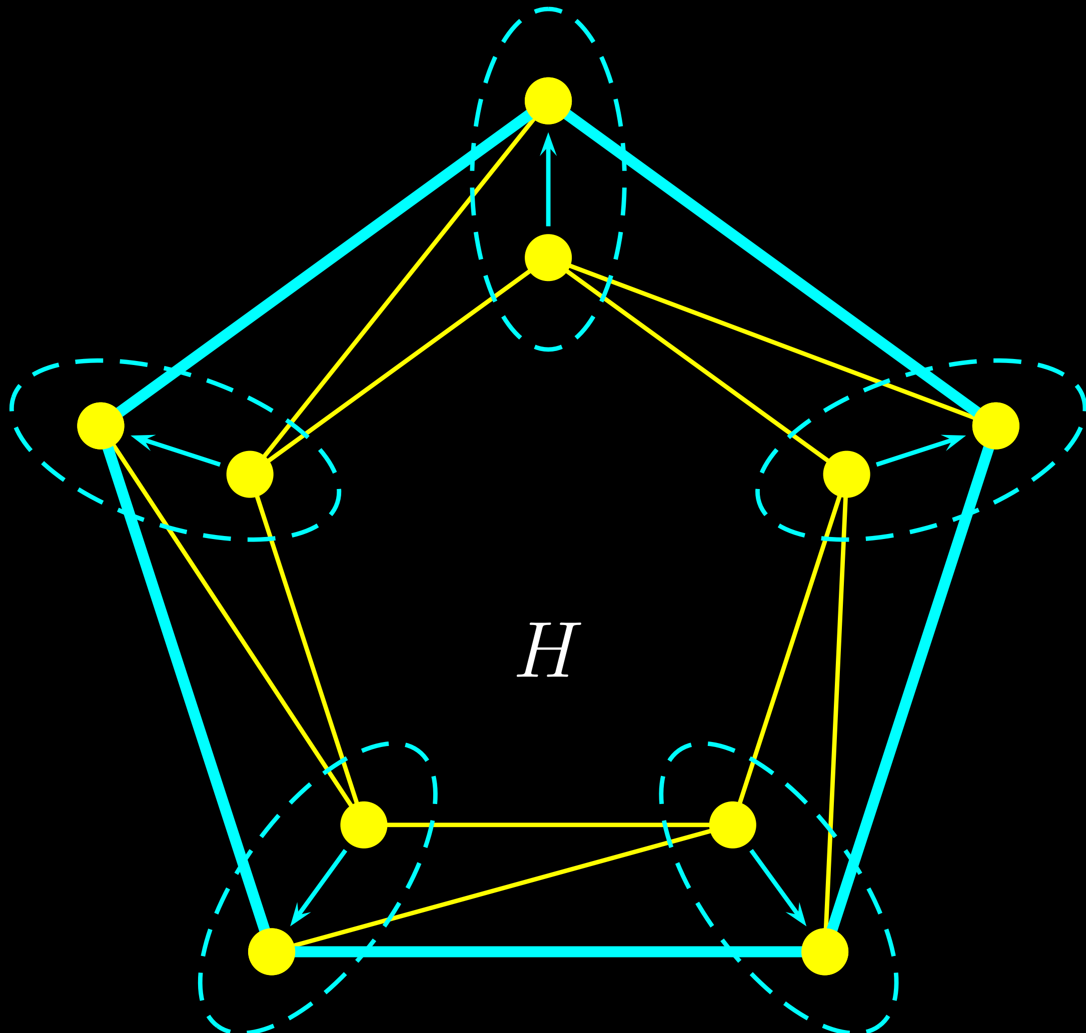
Core examples



Core examples



Core examples



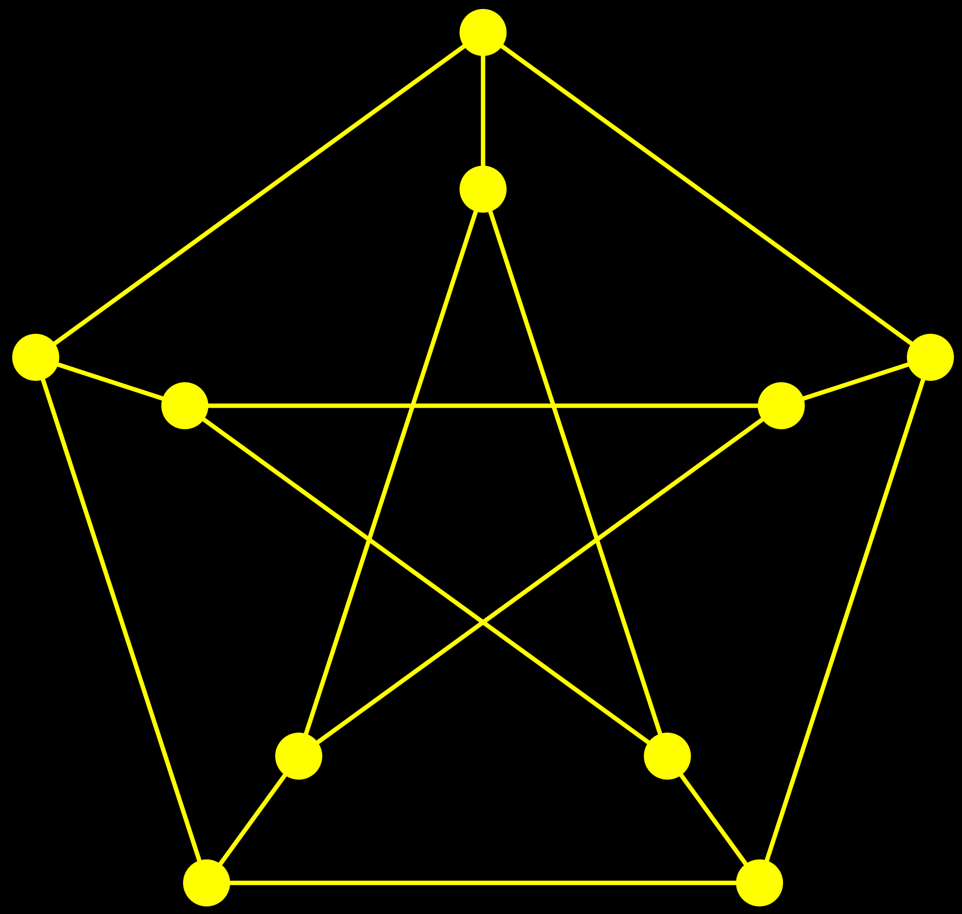
The mapping to the core

- C_5 is a subgraph of H.
- H maps to C_5.
- $C_5 \xrightarrow{g} H \xrightarrow{h} C_5$
- $h \circ g = \mathrm{id}_{C_5}$
- The map h is a *retraction*.

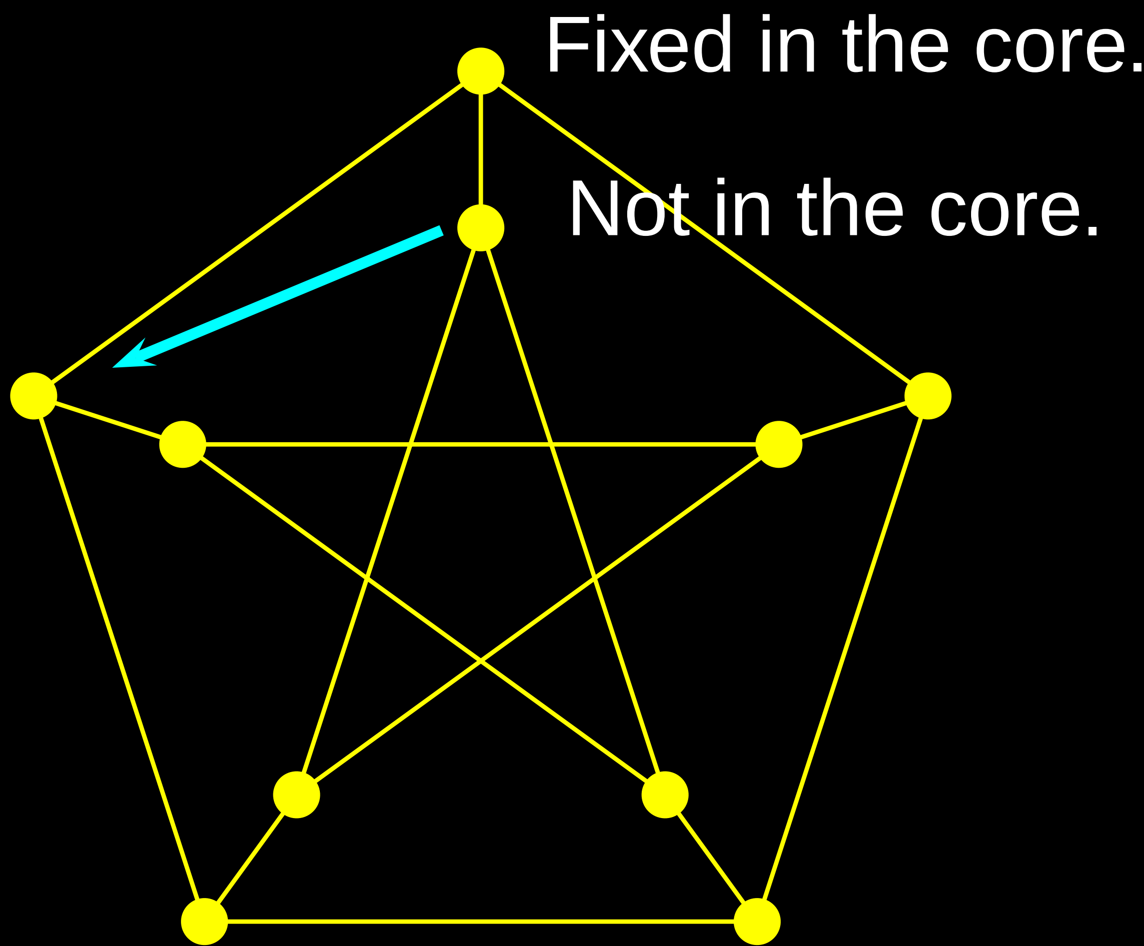
The mapping to the core

- C_5 is a subgraph of H.
- H maps to C_5.
- $C_5 \xrightarrow{g} H \xrightarrow{h} C_5$
- $h \circ g = \mathrm{id}_{C_5}$
- The map h is a *retraction*.
- Let $H' \subseteq H$. A retraction $f : H \to H'$ is a hom that is the identity on H'.

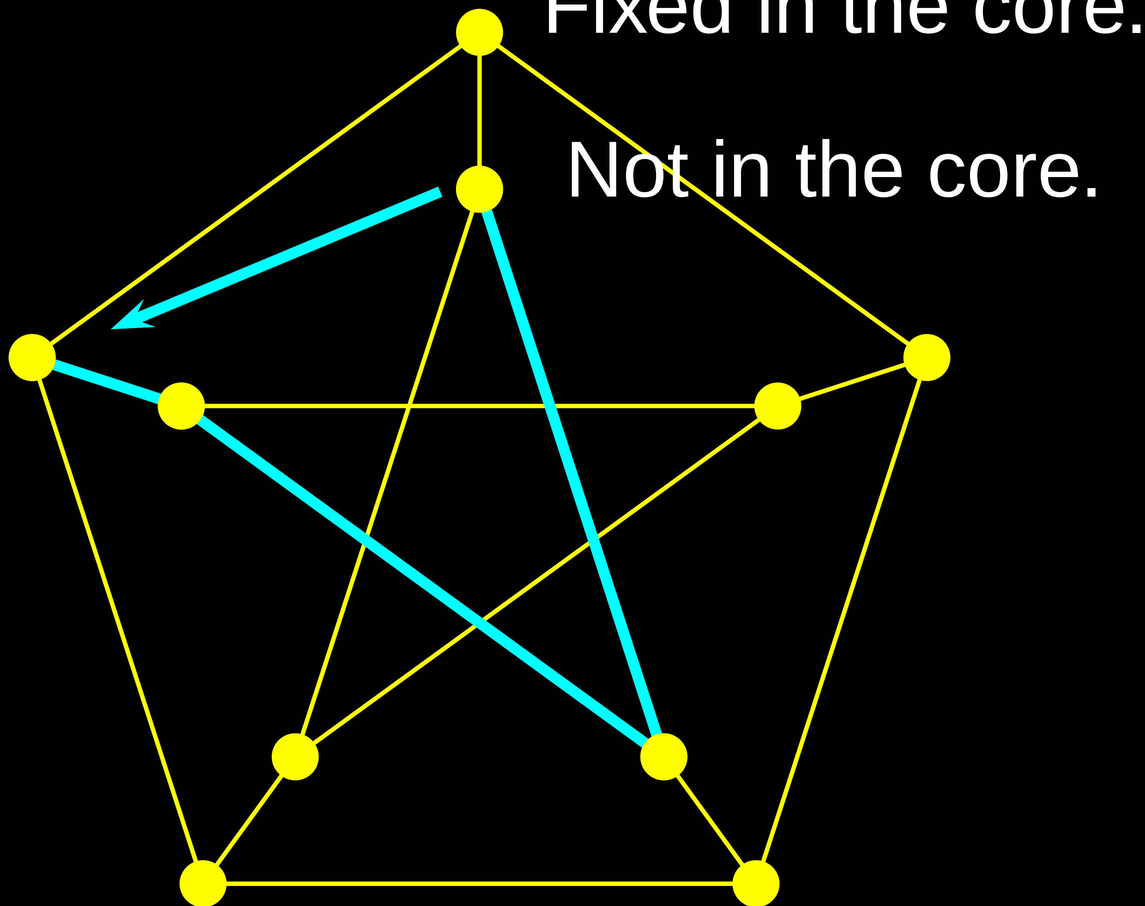
Core of an old friend



Core of an old friend



Core of an old friend



The hom-image contains $K_3 \not\subseteq P_{10}$.
Petersen graph is a core.

Cores

- Every graph H contains a core, denoted $H^\bullet$.
- The core is a subgraph.
- There is a retraction $r : H \longrightarrow H^\bullet$ (which fixes $H^\bullet$).
- For all G,

$$G \longrightarrow H \Leftrightarrow G \longrightarrow H^\bullet$$

- If $H = H^\bullet$, then H is a core.

Some popular cores

The following graphs are cores:

- complete graphs K_n;
- odd cycles C_{2n+1};
- directed cycles $\vec{C}_k$.

Resumé

- Homomorphisms generalize colourings.
- Homomorphisms generalize isomorphism.
- Each graph contains a unique core.
- Let $H' \subseteq H$. A retraction $f : H \to H'$ is a hom that the identity on H'.

Colouring Problems

Key idea: Many colouring problems can be formulated as homomorphism problems by defining a suitable collection of target graphs.

Circular colourings

A (p/q)*-colouring* of a graph G is:

- a function $c : V(G) \to \{0, 1, 2, \ldots, p-1\}$;
- where $uv \in E(G)$ implies $q \leq |c(u) - c(v)| \leq p - q$.

In other words, adjacent vertices receive colours that differ by a least q modulo p.

Circular colourings

A (p/q)-*colouring* of a graph G is:

- a function $c : V(G) \to \{0, 1, 2, \ldots, p-1\}$;
- where $uv \in E(G)$ implies
  $q \leq |c(u) - c(v)| \leq p - q$.

In other words, adjacent vertices receive colours that differ by a least q modulo p.

- Introduced by Vince (1988).
- Combinatorial setting Bondy and Hell (1990).
- Survey Zhu (2001).

Circular chromatic number

The *circular chormatic number* of a graph G is

$$\chi_c(G) = \inf\left\{\frac{p}{q} \mid G \text{ is } (p/q) - \text{colourable}\right\}.$$

Circular chromatic number

The *circular chormatic number* of a graph G is

$$\chi_c(G) = \inf\left\{ \frac{p}{q} \mid G \text{ is } (p/q) - \text{colourable} \right\}.$$

Prop 2

- *A $(p,1)$-colouring is simply a p-colouring. Hence, (p,q)-colourings generalize classical colourings.*
- $\chi_c(K_n) = \chi(K_n) = n$;
- $\chi_c(C_{2k+1}) = 2 + 1/k$.

Circular chromatic number in the language of homomorphisms

We require a suitable colleciton of *calibrating graphs*.
What is the correct target H for a (p/q)-colouring?

Circular chromatic number in the language of homomorphisms

We require a suitable colleciton of *calibrating graphs*.
What is the correct target H for a (p/q)-colouring?

- $V(H) = \{0, 1, 2, \ldots, p-1\}$;
- $E(H) = \{ij \mid q \leq |i-j| \leq p-q\}$.

Circular chromatic number in the language of homomorphisms

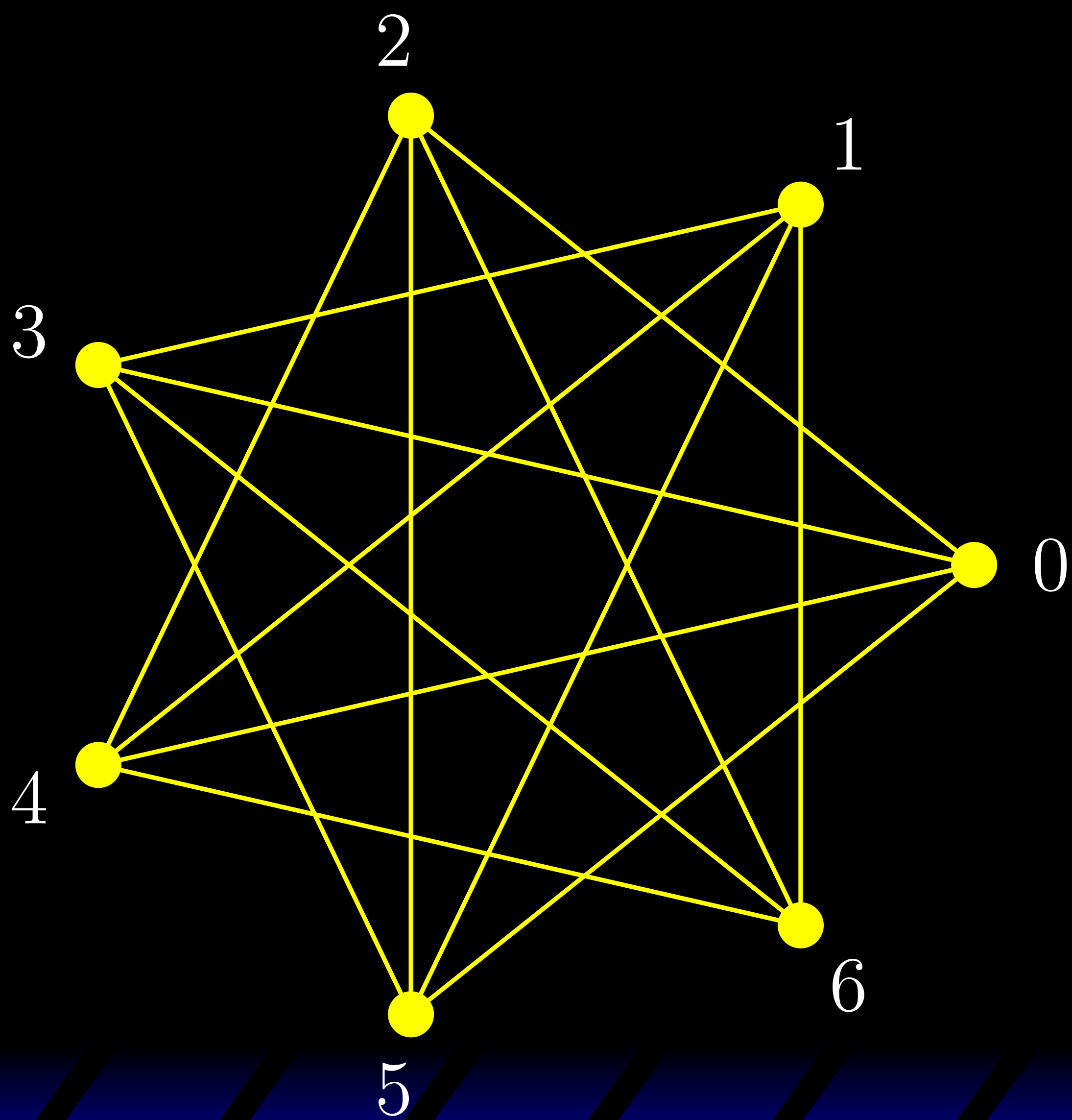
We require a suitable colleciton of *calibrating graphs*.
What is the correct target H for a (p/q)-colouring?

- $V(H) = \{0, 1, 2, \ldots, p-1\}$;
- $E(H) = \{ij \mid q \leq |i-j| \leq p-q\}$.

We call these graphs the $K_{p/q}$ *cliques*.

$$\chi_c(G) = \inf \left\{ \frac{p}{q} \mid G \to K_{p/q} \right\}$$

The circular clique $K_{7/2}$



Circular cliques behave

The circular cliques have many nice properties we recognize from classical cliques.

Circular cliques behave

The circular cliques have many nice properties we recognize from classical cliques.

- For rationals $r' < r$, $K_{r'} \to K_r$;

Circular cliques behave

The circular cliques have many nice properties we recognize from classical cliques.

- For rationals $r' < r$, $K_{r'} \to K_r$;
- for $(p,q) = 1$ and $p/q \geq 2$

$$(K_{p/q} - \{x\}) \to K_{p'/q'}$$

with $p'/q' < p/q$, $p' < p$ and $q' < q$;

Circular cliques behave

The circular cliques have many nice properties we recognize from classical cliques.

- For rationals $r' < r$, $K_{r'} \to K_r$;
- for $(p, q) = 1$ and $p/q \geq 2$

$$(K_{p/q} - \{x\}) \to K_{p'/q'}$$

  with $p'/q' < p/q$, $p' < p$ and $q' < q$;
- we can replace $\inf$ with $\min$.

Circular colouring comments

Circular colouring comments

- $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$

Circular colouring comments

- $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$
- An *orientation* $\vec{G}$ of G is obtained by assigning a direction to each edge in G.
- Given a cycle C in G, $C^+(C^-)$ is number of forward (backward) arcs.

Circular colouring comments

- $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$
- An *orientation* $\vec{G}$ of G is obtained by assigning a direction to each edge in G.
- Given a cycle C in G, $C^+(C^-)$ is number of forward (backward) arcs.
- Minty, 1962: $\chi(G) = \min_{\vec{G}} \max_{C} \left\lceil \frac{|C^+|}{|C^-|} + 1 \right\rceil$
- Goddyn, Tarsi, Zhang, 1998:

$$\chi_c(G) = \min_{\vec{G}} \max_{C} \frac{|C^+|}{|C^-|} + 1$$

Fractional Colourings

A *k-tuple, n-colouring* of a graph G is:

Fractional Colourings

A *k-tuple, n-colouring* of a graph G is:

- an assignment to each vertex v, a k-set of colours from an n-set;
- adjacent vertices receive disjoint sets.

Fractional Colourings

A *k-tuple, n-colouring* of a graph G is:

- an assignment to each vertex v, a k-set of colours from an n-set;
- adjacent vertices receive disjoint sets.

When $k = 1$ we have a classical vertex colouring.

Fractional Colouring Targets

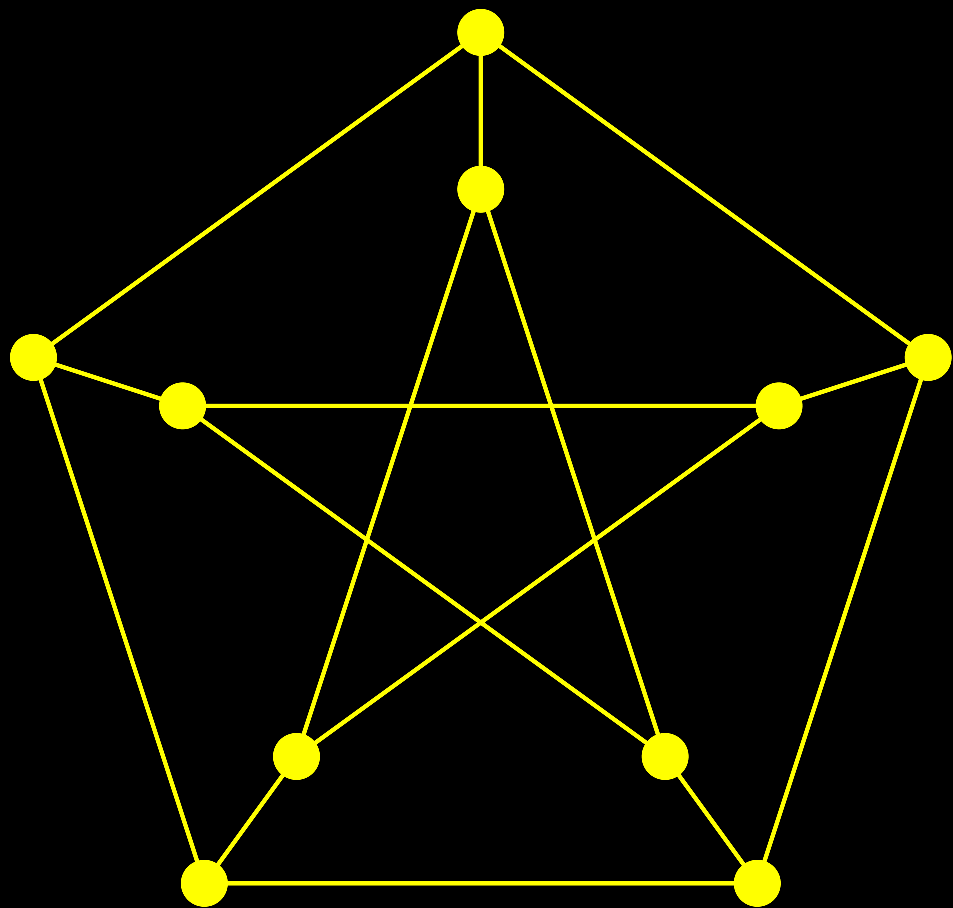
The *Kneser graph* $K(n, k)$ is defined as follows:

Fractional Colouring Targets

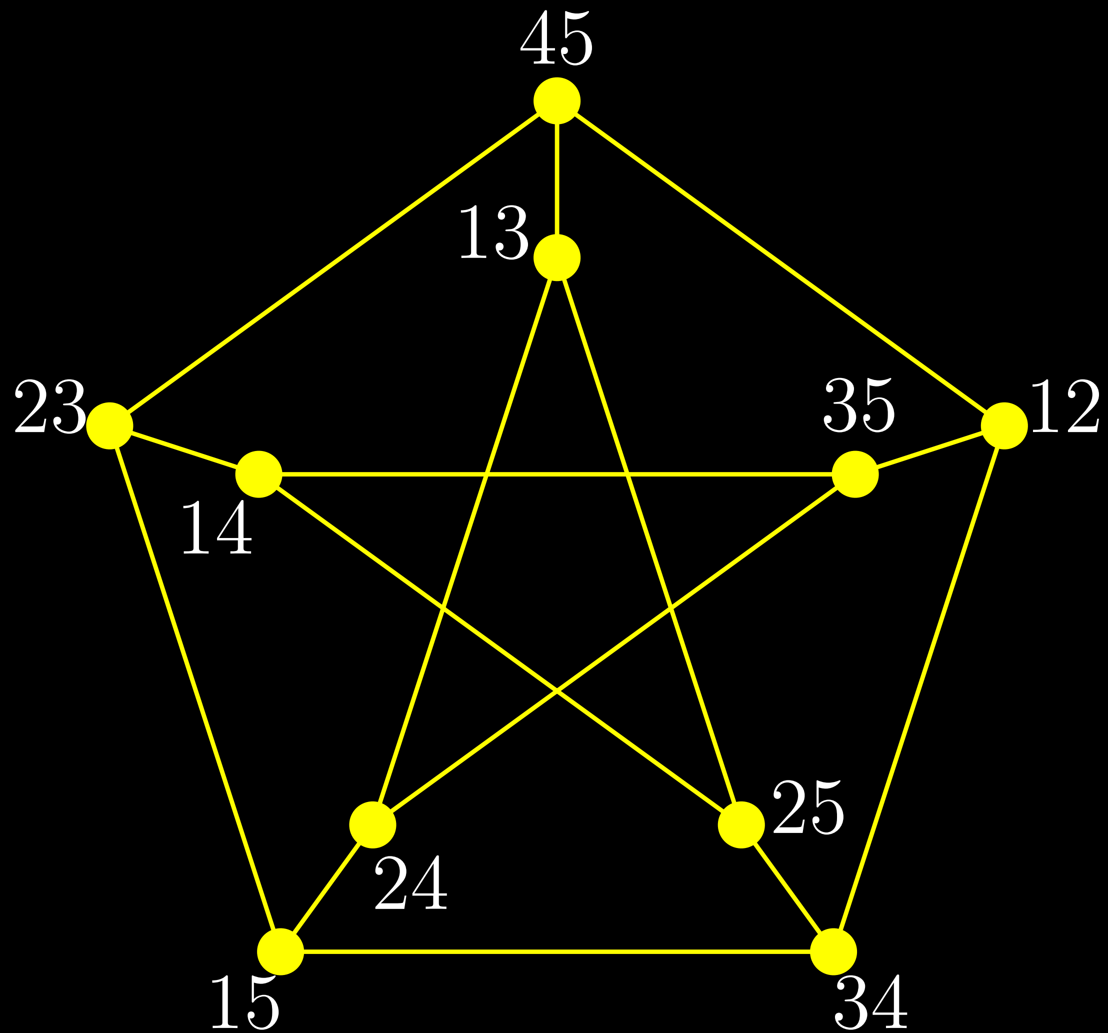
The *Kneser graph* $K(n, k)$ is defined as follows:

- vertices k-sets from an n-set;
- two vertices are adjacent if they are disjoint.

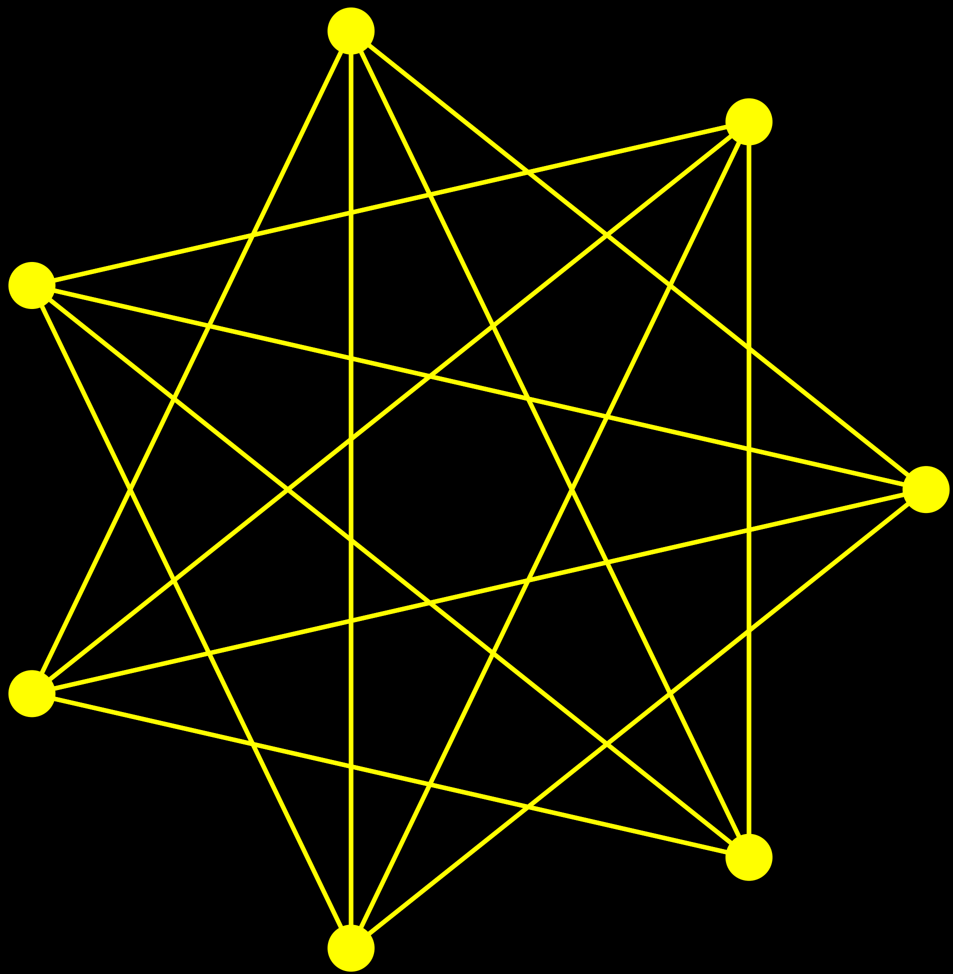
An old friend returns: $K(5,2)$



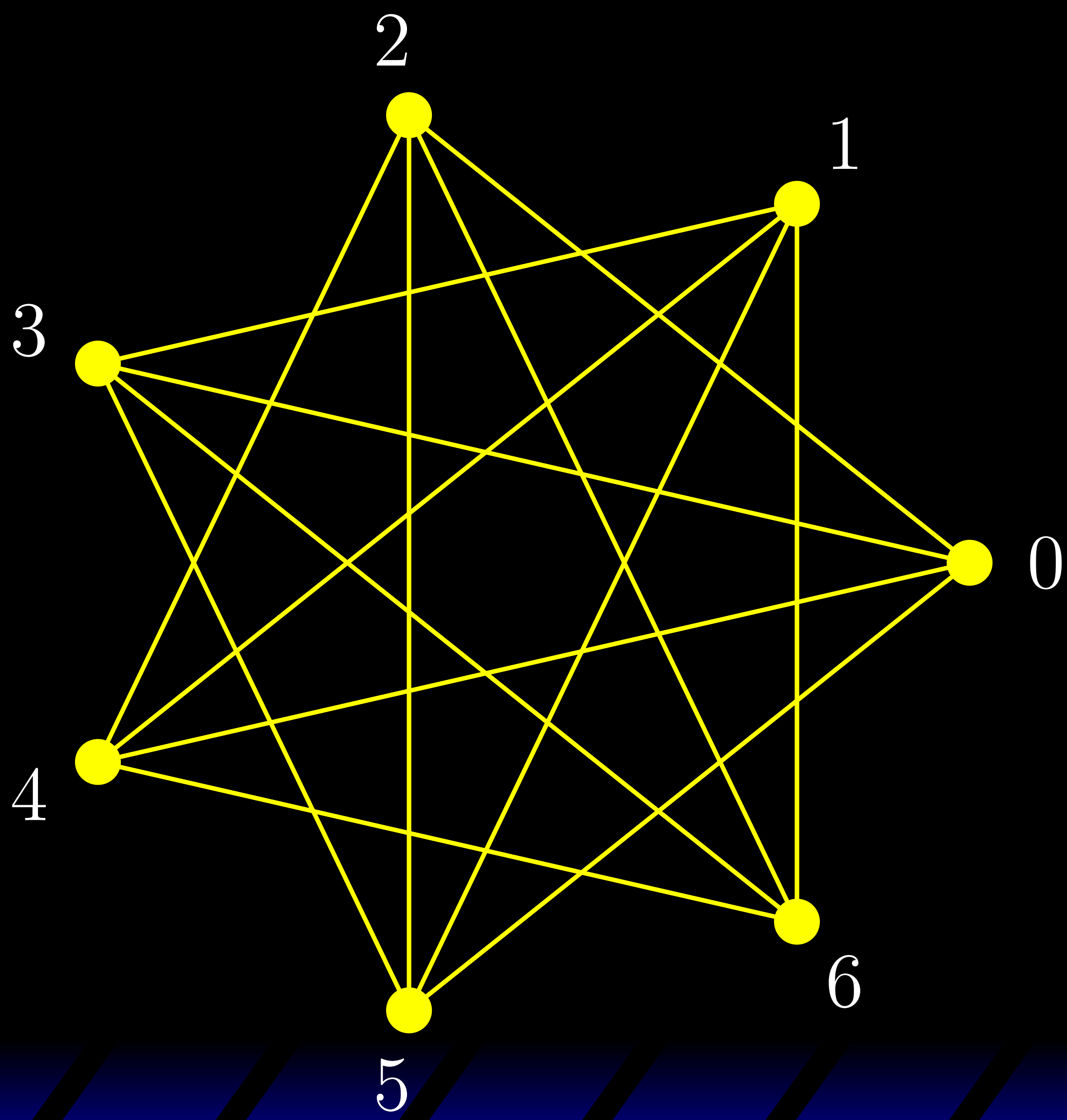
An old friend returns: $K(5, 2)$



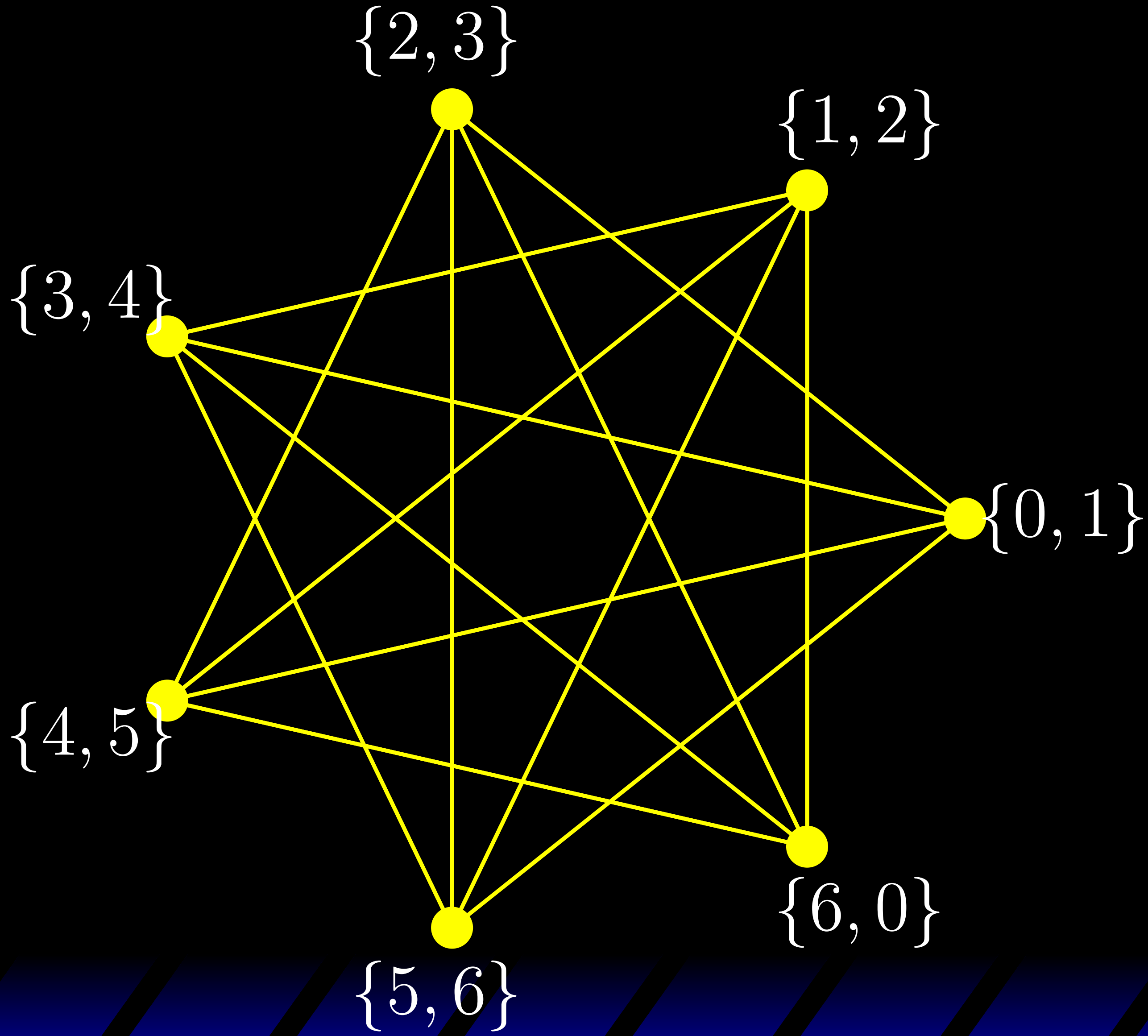
$K_{7/2} \to K(7,2)$



$K_{7/2} \to K(7,2)$



$K_{7/2} \to K(7,2)$



Integer Programming

We can formulate ordinary chromatic number as an integer program. Recall χ is the smallest number of independent sets into which we can partition $V(G)$.

Integer Programming

We can formulate ordinary chromatic number as an integer program. Recall χ is the smallest number of independent sets into which we can partition $V(G)$.

- For each independent set I, create a 01-variable x_I.

Integer Programming

We can formulate ordinary chromatic number as an integer program. Recall χ is the smallest number of independent sets into which we can partition $V(G)$.

- For each independent set I, create a 01-variable x_I.
- χ is the optimum value of:

$$\min \sum_I x_I$$

$$\sum_{v \in I} = 1, \text{ for all } v \in V(G)$$

Fractional Relaxation

It is easy to verify that χ_f is the optimal value of the fractional relaxation of the IP above:

$$\min \textstyle\sum_I x_I$$

$$\textstyle\sum_{v \in I} = 1, \text{ for all } v \in V(G)$$

$$x_I \geq 0$$

Fractional Relaxation (2)

The dual to this problem (in standard form) defines the *fractional clique.* Gives lower bounds on χ_f. For example,

$$\chi_f(G) \geq \frac{|V(G)|}{\alpha}$$

Fractional Relaxation (2)

The dual to this problem (in standard form) defines the *fractional clique.* Gives lower bounds on χ_f. For example,

$$\chi_f(G) \geq \frac{|V(G)|}{\alpha}$$

Using this we get $\chi_f(K(n,k)) = \frac{n}{k}$.

Kneser graphs

- Unlike the circular cliques, we do not have a full understanding of the homomorphism structure between $K(n,k)$.
- We do know for $n \geq 2k \geq 2$
  - $K(n,k) \to K(n+1,k)$
  - $K(n,k) \to K(tn,tk)$, for every positive integer t
  - $K(n,k) \to K(n-2,k-1)$, for $k > 1$

Chromatic number of Kneser graphs

Theorem 3 (Lovász, 1978) *For every* $n, k, n \geq 2k$*,*

$$\chi(K(n,k)) = n - 2k + 2.$$

Chromatic number of Kneser graphs

Theorem 3 (Lovász, 1978) *For every* $n, k, n \geq 2k$,

$$\chi(K(n,k)) = n - 2k + 2.$$

- Topological methods;
- Uses $\alpha(K(n,k)) = \binom{n-1}{k-2}$ from the Erdős-Ko-Rado Theorem.

Chromatic number of Kneser graphs

Theorem 3 (Lovász, 1978) *For every* $n, k, n \geq 2k$,

$$\chi(K(n,k)) = n - 2k + 2.$$

- Topological methods;
- Uses $\alpha(K(n,k)) = \binom{n-1}{k-2}$ from the Erdős-Ko-Rado Theorem.

Stahl (and others) conjecture

$K(n,k) \not\to K(tn-2k+1, tk-k+1)$.

Covering Arrays and Homomorphisms

Can we express covering array problems in the language of homomorphisms?

Natural problems? Interesting?

Covering Arrays and Homomorphisms

Can we express covering array problems in the language of homomorphisms?
Natural problems? Interesting?

- Karen Meagher and Brett Stevens
- Karen Meagher, Lucia Moura, and Latifa Zekaoui
- Chris Godsil, Karen Meagher, and Reza Naserasr

Covering Arrays Targets

The graph $QI(n,g)$ (with $n \geq g^2$)

- V strings of length n over $\{0, 1, \ldots, g-1\}$;
- E pairs of qualitatively independent strings.

Covering Arrays Targets

The graph $QI(n, g)$ (with $n \geq g^2$)

- V strings of length n over $\{0, 1, \ldots, g-1\}$;
- E pairs of qualitatively independent strings.

A k-clique in $QI(n, g)$ corresponds to a $n \times k$ covering array.

CA in the language of homomorphisms

There exists a $CA(n, k, g)$

CA in the language of homomorphisms

There exists a $CA(n,k,g)$
iff $k \leq \omega(QI(n,g))$

CA in the language of homomorphisms

There exists a $CA(n, k, g)$
iff $k \leq \omega(QI(n, g))$
iff $K_k \to \omega(QI(n, g))$

CA in the language of homomorphisms

There exists a $CA(n, k, g)$
iff $k \leq \omega(QI(n, g))$
iff $K_k \rightarrow \omega(QI(n, g))$

Again, we may restrict our attention to cores.

Observe $QI^\bullet(4, 2) = K_3$, and is induced by the *balance strings starting with* 0.

Continuing with homomorphisms

Let's ask the question, for which graph G

$$G \xrightarrow{?} QI(n,g)$$

Continuing with homomorphisms

Let's ask the question, for which graph G

$$G \xrightarrow{?} QI(n, g)$$

Covering array on a graph G is a homomorphism $G \rightarrow QI(n, g)$.

Continuing with homomorphisms

Let's ask the question, for which graph G

$$G \xrightarrow{?} QI(n, g)$$

Covering array on a graph G is a homomorphism $G \to QI(n, g)$.

$CA(G, g) = \min_{\ell \in \mathbb{N}}\{\ell : \exists CA(\ell, G, g)\}$

Note: $CAN(K_k, g) = CAN(k, g)$

Some Results

Some Results

Prop 4 *If* $G \to H$*, then* $CAN(G, g) \leq CAN(H, g)$*. In particular,*

$$CAN(K_{\omega(G)}, g) \leq CAN(G, g) \leq CAN(K_{\chi(G)}, g)$$

Some Results

Prop 4 *If* $G \to H$*, then* $CAN(G, g) \leq CAN(H, g)$*. In particular,*

$$CAN(K_{\omega(G)}, g) \leq CAN(G, g) \leq CAN(K_{\chi(G)}, g)$$

Meagher and Stevens examined the problem of finding graphs such that

$$CAN(G, 2) < CAN(K_{\chi(G)}, 2)$$

Some Results

Prop 4 *If $G \to H$, then $CAN(G,g) \leq CAN(H,g)$. In particular,*

$$CAN(K_{\omega(G)}, g) \leq CAN(G,g) \leq CAN(K_{\chi(G)}, g)$$

Meagher and Stevens examined the problem of finding graphs such that

$$CAN(G,2) < CAN(K_{\chi(G)}, 2)$$

$QI(5,2)$ is such a graph.

Do the target graphs behave?

Do the target graphs behave?

(The core of) $QI(5,2)$ is the complement of the Petersen graph.

Do the target graphs behave?

(The core of) $QI(5,2)$ is the complement of the Petersen graph.

Theorem 5 (MS) *$QI^\bullet(n,2)$ is the complement of a Kneser graph.*

- *for n even the core is $K_{\binom{n}{n/2}/2}$;*
- *for n odd the core is $F(n,2) =$ subgraph induced by vectors of weight $\lfloor n/2 \rfloor$.*

Karen’s Questions

Karen's Questions

- $QI(n,g) \to BQI(n,g)$?

Karen's Questions

- $QI(n,g) \rightarrow BQI(n,g)$?
- What is $Aut(QI(n,k))$ or $Aut(QI(ck,k))$?

Karen's Questions

- $QI(n,g) \to BQI(n,g)$?
- What is $Aut(QI(n,k))$ or $Aut(QI(ck,k))$?
- Is $BQI(k^2,k)$ a core?

Karen's Questions

- $QI(n,g) \to BQI(n,g)$?
- What is $Aut(QI(n,k))$ or $Aut(QI(ck,k))$?
- Is $BQI(k^2,k)$ a core?
- $\chi(BQI(k^2,k)) = \binom{k+1}{2}$?

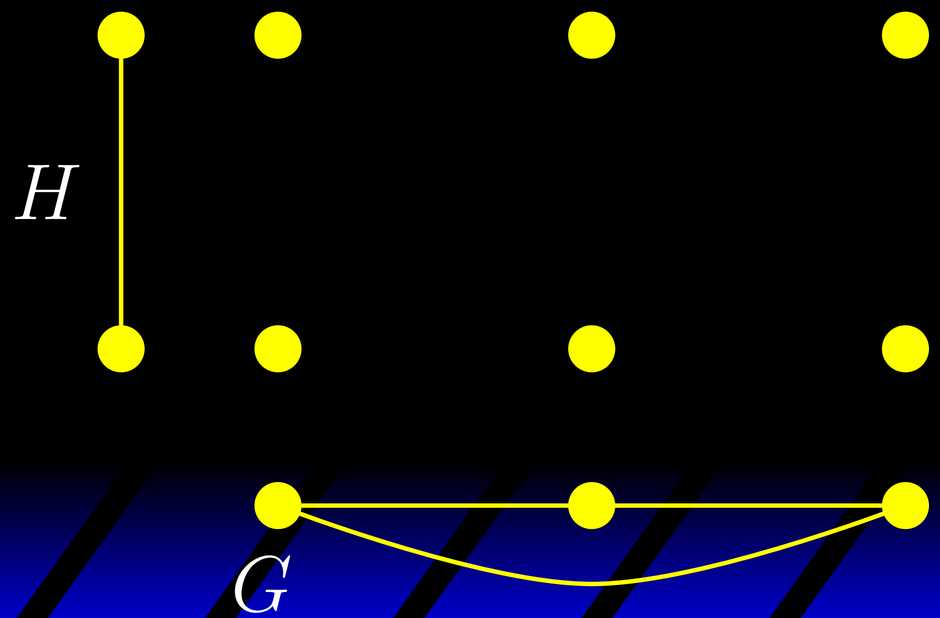
List Homomorphisms

Definition 5 *Let* G *and* H *be graphs. Let* $L(v)$ *be a subset of* $V(H)$ *for each vertex* $v \in V(G)$*. A* list homomorphism $f : G \to H$ *is a homomorphism such that* $f(v) \in L(v)$ *for all* v*.*

Products

The natural product with homomorphisms is the *categorical product* $G \times H$.

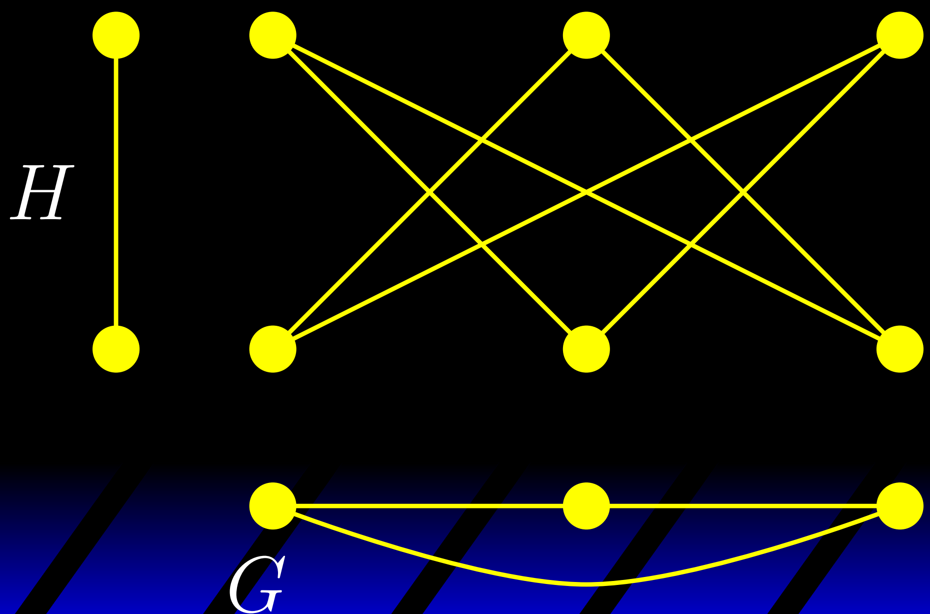
$$(g_1, h_1)(g_2, h_2) \in E(G \times H)$$
$$\Leftrightarrow \quad g_1 g_2 \in E(G) \text{ and } h_1 h_2 \in E(H)$$



Products

The natural product with homomorphisms is the *categorical product* $G \times H$.

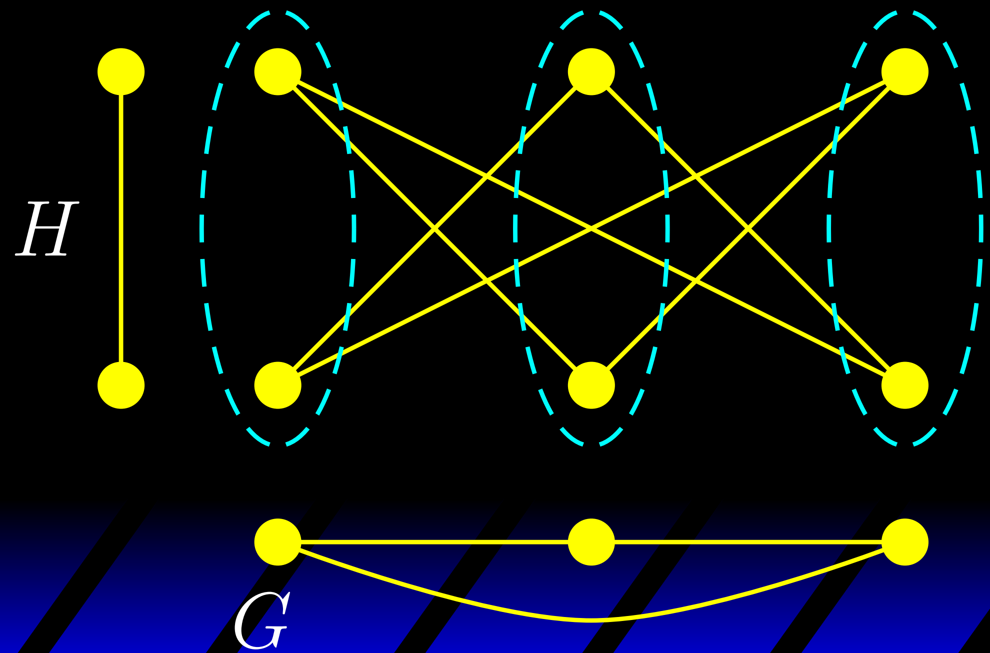
$$(g_1, h_1)(g_2, h_2) \in E(G \times H)$$
$$\Leftrightarrow \quad g_1 g_2 \in E(G) \text{ and } h_1 h_2 \in E(H)$$



Products

The natural product with homomorphisms is the *categorical product $G \times H$*.

$$
\begin{aligned}
&(g_1, h_1)(g_2, h_2) \in E(G \times H) \\
&\Leftrightarrow \quad g_1 g_2 \in E(G) \text{ and } h_1 h_2 \in E(H)
\end{aligned}
$$



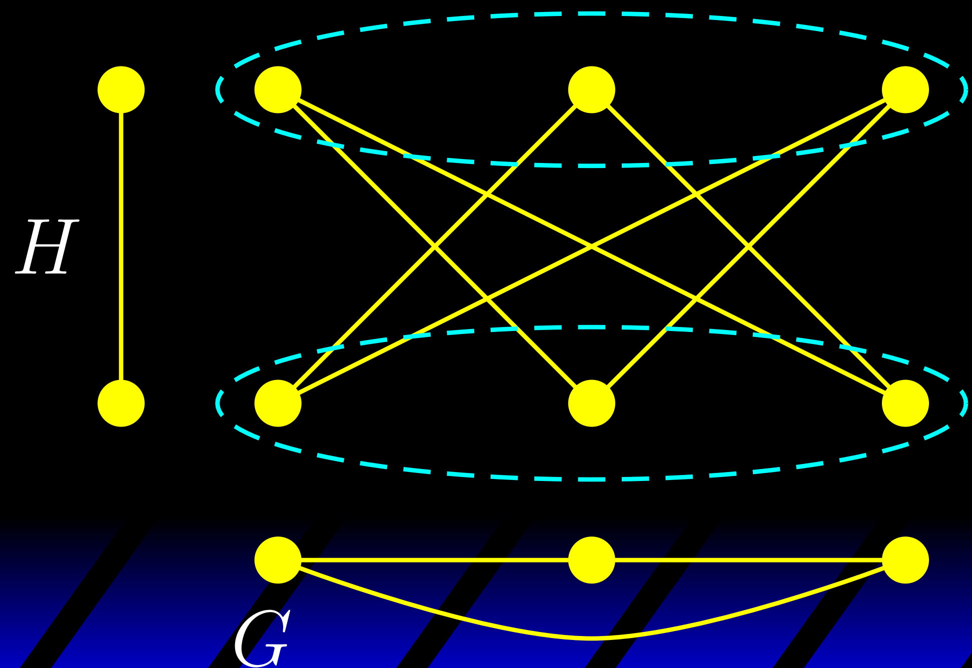
$\pi_1 : G \times H \to G$

Projections are homomorphisms

Products

The natural product with homomorphisms is the *categorical product* $G \times H$.

$$(g_1, h_1)(g_2, h_2) \in E(G \times H)$$
$$\Leftrightarrow \;\; g_1 g_2 \in E(G) \text{ and } h_1 h_2 \in E(H)$$



$\pi_2 : G \times H \to H$

Projections are homomorphisms

Products (2)

Prop 6 $X \to G \times H$ *iff* $X \to G$ *and* $X \to H$

Products (2)

Prop 6 $X \to G \times H$ *iff* $X \to G$ *and* $X \to H$

$$\begin{tikzcd} G & & H \\ & X \arrow[ul, bend right, "\phi_1"] \arrow[ur, bend left, "\phi_2"'] & \end{tikzcd}$$

Products (2)

Prop 6 $X \to G \times H$ *iff* $X \to G$ *and* $X \to H$

$$
\begin{tikzcd}
G & & H \\
& G \times H \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] & \\
& X \arrow[uul, bend left, "\phi_1"] \arrow[uur, bend right, "\phi_2"'] &
\end{tikzcd}
$$

Products (2)

Prop 6 $X \to G \times H$ *iff* $X \to G$ *and* $X \to H$

$$\begin{tikzcd}
G & & H \\
 & G \times H \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] & \\
 & X \arrow[u, "\alpha"'] \arrow[uul, bend left, "\phi_1"] \arrow[uur, bend right, "\phi_2"'] &
\end{tikzcd}$$

$$\alpha(x) := (\phi_1(x), \phi_2(x)) \qquad (= \phi_1 \times \phi_2(x))$$

$$\phi_1 = \pi_1 \circ \alpha \qquad \phi_2 = \pi_2 \circ \alpha$$

Varieties

- A *variety* is a set of graphs closed under retracts and products.
- Let C be a family of graphs. The *variety generated by* C is the smallest variety containing C. Denoted $\mathcal{V}(C)$.
- Example, the variety generated by finite, reflexive paths is important in the study of graph retraction problems. Well characterized.

Cops and Robbers

Consider reflexive graphs.

- Cop picks a vertex.
- Robber picks a vertex.
- Take turns moving to an adjacent vertex.

Cops and Robbers

Consider reflexive graphs.

- Cop picks a vertex.
- Robber picks a vertex.
- Take turns moving to an adjacent vertex.
- Cop wins by occupying the same vertex as the robber. A graph is *cop-win* if the cop has a winning strategy.
- Observation: Cop-win graphs form a variety.
- Nowakowski and Winkler, *Disc Math*, 1983.

Homomorphism Partial Order

- Let $\mathcal{G}$ be the set of all finite graphs.
- $G \preceq H$ if $G \to H$.
- Reflexive and Transitive: *quasi-order*.

Homomorphism Partial Order

- Let $\mathcal{G}$ be the set of all finite graphs.
- $G \preceq H$ if $G \longrightarrow H$.
- Reflexive and Transitive: *quasi-order*.
- Not-antisymmetric: $C_6 \longrightarrow K_2$ and $K_2 \longrightarrow C_6$.
- Usual operation of moding out by hom-equiv to obtain a partial order.
- Cores are the natural representation of the classes.

Homomorphism Partial Order (2)

- Rich structure.
- Distributive lattice.

Homomorphism Partial Order (2)

- Rich structure.
- Distributive lattice.
- meet: $G \wedge H = G \times H$;
- join: $G \vee H = G + H$, disjoint union or *co-product*.

Chains and Antichains

- $K_1 \longrightarrow K_2 \longrightarrow K_3 \longrightarrow \cdots$
- $\cdots \longrightarrow C_7 \longrightarrow C_5 \longrightarrow C_3$

Chains and Antichains

- $K_1 \to K_2 \to K_3 \to \cdots$
- $\cdots \to C_7 \to C_5 \to C_3$
- Recall:

Chains and Antichains

- $K_1 \rightarrow K_2 \rightarrow K_3 \rightarrow \cdots$
- $\cdots \rightarrow C_7 \rightarrow C_5 \rightarrow C_3$
- Recall:
  - $\chi(G) > \chi(H) \Rightarrow G \not\rightarrow H$.
  - $og(G) > og(H) \Rightarrow G \not\leftarrow H$.
  - Erdös: $\forall i \geq 3$, there exists a graph R_i such that $\chi(R_i) = i$ and $og(R_i) = 2i + 1$.

Chains and Antichains

- $K_1 \to K_2 \to K_3 \to \cdots$
- $\cdots \to C_7 \to C_5 \to C_3$
- Recall:
  - $\chi(G) > \chi(H) \Rightarrow G \not\to H$.
  - $og(G) > og(H) \Rightarrow G \not\leftarrow H$.
  - Erdös: $\forall i \geq 3$, there exists a graph R_i such that $\chi(R_i) = i$ and $og(R_i) = 2i + 1$.
- R_i, $i \geq 3$ form an antichain.

Density

Given $G \to H$ and $G \not\leftarrow H$:

Density

Given $G \to H$ and $G \not\leftarrow H$:

- find Z such that
  $G \to Z \to H$ and $G \not\leftarrow Z \not\leftarrow H$

Density

Given $G \to H$ and $G \not\leftarrow H$:

- find Z such that
  $G \to Z \to H$ and $G \not\leftarrow Z \not\leftarrow H$
- **Theorem 7 (Welzl, 1982)** *If $\{G, H\} \neq \{K_1, K_2\}$ with $G \to H$ and $G \not\leftarrow H$, then there exists Z such that*

$$G \to Z \to H \text{ and } G \not\leftarrow Z \not\leftarrow H$$

Density

Given $G \to H$ and $G \not\leftarrow H$:

- find Z such that
  $G \to Z \to H$ and $G \not\leftarrow Z \not\leftarrow H$
- **Theorem 7 (Welzl, 1982)** *If $\{G, H\} \neq \{K_1, K_2\}$ with $G \to H$ and $G \not\leftarrow H$, then there exists Z such that*

  $$G \to Z \to H \text{ and } G \not\leftarrow Z \not\leftarrow H$$

- Proof indep Nešetřil and Perles (1990).

Duality and Gaps

Define $\rightarrow H := \{G \mid G \rightarrow H\}$.
When can we nicely describe $\rightarrow H$?

Duality and Gaps

Define $\rightarrow H := \{G \mid G \rightarrow H\}$.
When can we nicely describe $\rightarrow H$?

- $G \rightarrow K_2$ iff $C \not\rightarrow G$ for all odd cycles C.
- $G \rightarrow K_1$ iff $K_2 \not\rightarrow G$.

Duality and Gaps

Define $\rightarrow H := \{G \mid G \rightarrow H\}$.
When can we nicely describe $\rightarrow H$?

- $G \rightarrow K_2$ iff $C \not\rightarrow G$ for all odd cycles C.
- $G \rightarrow K_1$ iff $K_2 \not\rightarrow G$.
- *duality pair*: (F, H)

$$\forall G, G \rightarrow H \Leftrightarrow F \not\rightarrow G$$

Duality and Gaps

Define $\rightarrow H := \{G \mid G \rightarrow H\}$.
When can we nicely describe $\rightarrow H$?

- $G \rightarrow K_2$ iff $C \not\rightarrow G$ for all odd cycles C.
- $G \rightarrow K_1$ iff $K_2 \not\rightarrow G$.
- *duality pair*: (F, H)

$$\forall G, G \rightarrow H \Leftrightarrow F \not\rightarrow G$$

- *finite duality*: $(\{F_1, \ldots, F_t\}, H)$

$$\forall G, G \rightarrow H \Leftrightarrow \forall i, F_i \not\rightarrow G$$

Gaps

A pair $[G, H]$ with $G < H$ is a *gap* if no X satisfies $G < X < H$.

Gaps

A pair $[G, H]$ with $G < H$ is a *gap* if no X satisfies $G < X < H$.
The result of Welzl tell us that $[K_1, K_2]$ is the only gap in $\mathcal{G}$.

Gaps

A pair $[G, H]$ with $G < H$ is a *gap* if no X satisfies $G < X < H$.

The result of Welzl tell us that $[K_1, K_2]$ is the only gap in $\mathcal{G}$.

Theorem 8 (Nešetřil and Tardif, 2000)

- *If cores (F, H) form a duality pair, then $[F \times H, F]$ is a gap.*
- *If cores $[A, B]$ form a gap and B is connected, then (B, A^B) is a duality pair.*

Gaps

A pair $[G, H]$ with $G < H$ is a *gap* if no X satisfies $G < X < H$.

The result of Welzl tell us that $[K_1, K_2]$ is the only gap in $\mathcal{G}$.

Theorem 8 (Nešetřil and Tardif, 2000)

- *If cores (F, H) form a duality pair, then $[F \times H, F]$ is a gap.*
- *If cores $[A, B]$ form a gap and B is connected, then (B, A^B) is a duality pair.*

Finite duality implies H-colouring is polynomial.

Representation

Frucht, 1938 Every group is isomorphic to the automorphism group of a graph.

Representation

Frucht, 1938 Every group is isomorphic to the automorphism group of a graph.

Hedrlín and Pultr, 1965 Every monoid is isomorphic to the endomorphism monoid of a suitable digraph G.

Representation

Frucht, 1938 Every group is isomorphic to the automorphism group of a graph.

Hedrlín and Pultr, 1965 Every monoid is isomorphic to the endomorphism monoid of a suitable digraph G.

Pultr and Trnková, 1980 Any countable partial order is isomorphic to a suborder of the digraph poset.

Representation

Frucht, 1938 Every group is isomorphic to the automorphism group of a graph.

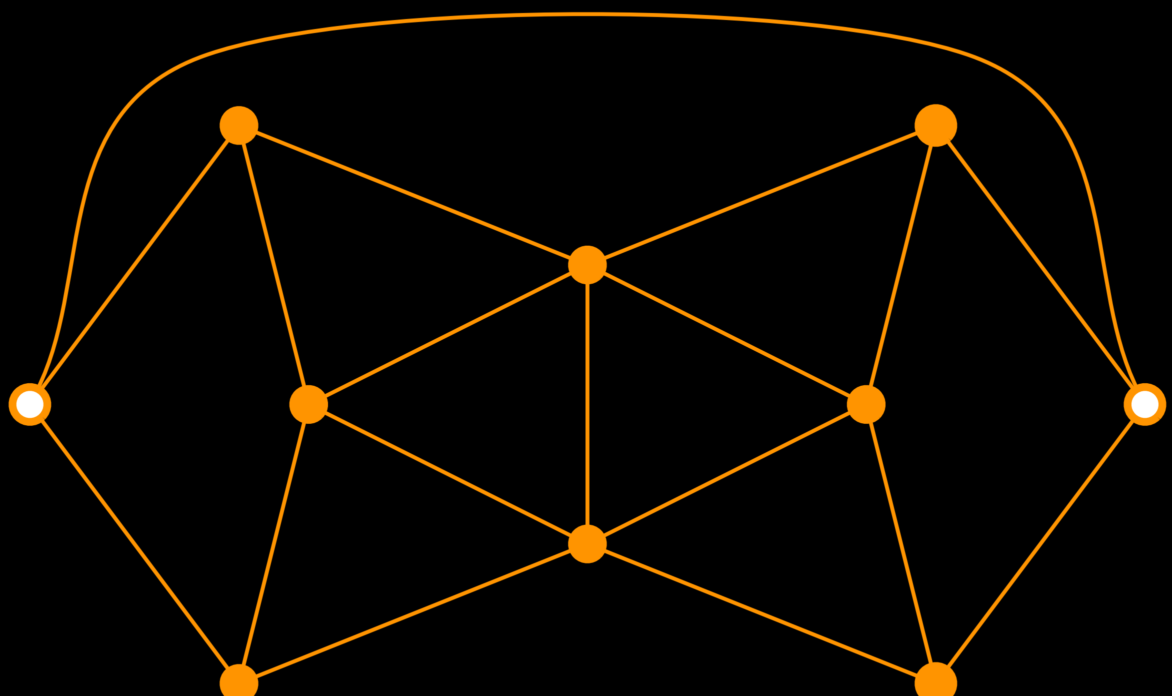
Hedrlín and Pultr, 1965 Every monoid is isomorphic to the endomorphism monoid of a suitable digraph G.

Pultr and Trnková, 1980 Any countable partial order is isomorphic to a suborder of the digraph poset.

Pultr and Trnková, 1980 Every concrete category can be represented in the category of finite graphs.

Complexity Issues

BFHHM (and others) examine retraction complexity and *no-certificates.*



List Homomorphisms

Definition 8 *Let G and H be graphs. Let $L(v)$ be a subset of $V(H)$ for each vertex $v \in V(G)$. A* list homomorphism *$f : G \to H$ is a homomorphism such that $f(v) \in L(v)$ for all v.*