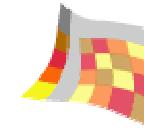


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A Column Expansion Construction for Optimal and Near-Optimal Mixed Covering Arrays

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Mixed covering array construction by column expansion.



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Introduction:

Construction for strength 2 mixed covering array from an orthogonal array and one or more ordered designs.

Construction emerged during development of commercial covering test case generator. Important characteristics for practical applications:

- Generates *mixed* arrays of strength 2

Arrays may have several different alphabet sizes => flexibility

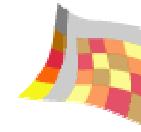
- MCAs have optimal or nearly optimal sizes

Arrays with optimal sizes ($N = v_1 v_2$) are predictable

Sizes of near-optimal arrays are bounded ($N < v_1 v_2 + v_{oa}$)

- Construction is not a computational challenge

Mixed covering array construction
by column expansion.



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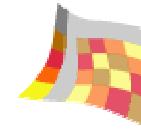
A few references:

M. A. Chateauneuf and D. Kreher, On the State of Strength-Three Covering Arrays, *J Combin Designs*, 10(4):217-238, 2002.

Charles J. Colbourn, Sosina S. Martirosyan, Gary L. Mullen, Dennis Shasha, George B. Sherwood, Joseph L. Yucas, Products of mixed covering arrays of strength two, *J Combin Designs*, 14(2):124-138, 2006.

Lucia Moura, John Stardom, Brett Stevens, Alan Williams, Covering Arrays with Mixed Alphabet Sizes, *J Combin Designs*, 11(6):413-432, 2003.

Covering Array & Orthogonal Array.
Definitions 1.



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Covering array

CA($N; t, k, v$) size N , strength t , degree k , order v

$N \times k$ array of v symbols

Every subarray of t distinct columns contains every t -tuple of v symbols in at least one row.

$$N \geq v^t$$

Orthogonal array (index 1)

OA($N_{oa}; t, k_{oa}, v_{oa}$) size N_{oa} , strength t , degree k_{oa} , order v_{oa}

$N_{oa} \times k_{oa}$ array of v_{oa} symbols

Every subarray of t distinct columns contains every t -tuple of v_{oa} symbols in exactly one row.

$$N_{oa} = v_{oa}^t$$

Mixed Covering Array & Ordered Design.
Definitions 2.



Mixed covering array

$\text{MCA}(N; t, k, \prod v_j^{k_j})$ size N , strength t , degree k

$N \times k$ array. Different columns have different alphabet sizes indicated by the product

$v_1^{k_1} v_2^{k_2} \dots v_j^{k_j} \dots$ in which k_j columns have v_j symbols. $k = \sum k_j$.

Every subarray of t distinct columns contains every t -tuple of symbols from the corresponding alphabets in at least one row.

$N \geq$ product of t largest alphabet sizes.

Ordered design

$\text{OD}(N_{od}; t, k_{od}, v_{od})$ size N_{od} , strength t , degree k_{od} , order v_{od}

$N_{od} \times k_{od}$ array of v_{od} symbols

Every subarray of t distinct columns contains every t -tuple of distinct symbols once.

$$N_{od} = \binom{v_{od}}{t} t !$$

Conventions.



Construction for strength 2 mixed covering array:

$$\text{MCA}(N; 2, k, \prod v_j^{k_j})$$

Alphabet sizes v_1, v_2, \dots are in decreasing order.

$$N \geq v_1^2 \quad \text{if } k_1 > 1$$

$$N \geq v_1 v_2 \quad \text{if } k_1 = 1$$

Equality => optimal size

$$\text{OA}(N_{oa}; 2, k_{oa}, v_{oa})$$

$$N_{oa} = v_{oa}^2$$

$$\text{OD}(N_{od}; 2, k_{od}, v_{od})$$

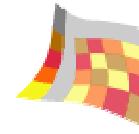
$$N_{od} = v_{od}(v_{od} - 1)$$

Symbols for alphabet size v_j are taken from Z_{v_j} specifically $\{0, 1, \dots, v_j - 1\}$

Unassigned values are denoted by *

MCA(25;2,8,5⁵3³).

Example 1, part 1 of 2.



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OA(25;2,6,5)

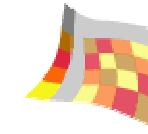
0	0	0	0	0	0
0	1	1	1	1	1
0	2	2	2	2	2
0	3	3	3	3	3
0	4	4	4	4	4
1	0	1	2	3	4
1	1	2	3	4	0
1	2	3	4	0	1
1	3	4	0	1	2
1	4	0	1	2	3
2	0	2	4	1	3
2	1	3	0	2	4
2	2	4	1	3	0
2	3	0	2	4	1
2	4	1	3	0	2
3	0	3	1	4	2
3	1	4	2	0	3
3	2	0	3	1	4
3	3	1	4	2	0
3	4	2	0	3	1
4	0	4	3	2	1
4	1	0	4	3	2
4	2	1	0	4	3
4	3	2	1	0	4
4	4	3	2	1	0

OD(6;2,3,3)

0	1	2
1	2	0
2	0	1
0	2	1
1	0	2
2	1	0

MCA(25;2,8,5⁵3³).

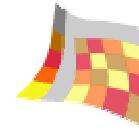
Example 1, part 2 of 2.



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expansion column	ordered design	unexpanded columns	expanded columns
0 0 0 0 0 0	0 1 2	0 0 0 0 0	0 0 0
0 1 1 1 1 1	1 2 0	0 1 1 1 1	1 1 1
0 2 2 2 2 2	2 0 1	0 2 2 2 2	2 2 2
0 3 3 3 3 3	*	0 3 3 3 3	0 1 2
0 4 4 4 4 4	*	0 4 4 4 4	1 2 0
1 0 1 2 3 *	2 1 0	1 0 1 2 3	2 0 1
1 1 2 3 4 0		1 1 2 3 4	0 0 0
1 2 3 4 0 1		1 2 3 4 0	1 1 1
1 3 4 0 1 2		1 3 4 0 1	2 2 2
1 4 0 1 2 *		1 4 0 1 2	0 2 1
2 0 2 4 1 *		2 0 2 4 1	1 0 2
2 1 3 0 2 *		2 1 3 0 2	2 1 0
2 2 4 1 3 0		2 2 4 1 3	0 0 0
2 3 0 2 4 1		2 3 0 2 4	1 1 1
2 4 1 3 0 2		2 4 1 3 0	2 2 2
3 0 3 1 4 2		3 0 3 1 4	2 2 2
3 1 4 2 0 *		3 1 4 2 0	* * *
3 2 0 3 1 *		3 2 0 3 1	* * *
3 3 1 4 2 0		3 3 1 4 2	0 0 0
3 4 2 0 3 1		3 4 2 0 3	1 1 1
4 0 4 3 2 1		4 0 4 3 2	1 1 1
4 1 0 4 3 2		4 1 0 4 3	2 2 2
4 2 1 0 4 *		4 2 1 0 4	* * *
4 3 2 1 0 *		4 3 2 1 0	* * *
4 4 3 2 1 0		4 4 3 2 1	0 0 0

Single column expansion, $N = N_{oa}$.
Theorem 1, part 1 of 7.



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Given:

$$\text{OA}(N_{oa}; 2, k_{oa}, v_{oa})$$

$$\text{OD}(N_{od}; 2, k_{od}, v_{od})$$

$$mN_{od} \leq N_{oa} - v_{oa}v_{od}$$

column expansion degree m is a positive integer.

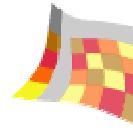
There is:

$$\text{MCA}(N_{oa}; 2, k_{oa} + k_{od}^m - 1, v_{oa}^{k_{oa}-1} v_{od}^{k_{od}^m})$$

with optimal size when $k_{oa} > 2$.

Single column expansion, $N = N_{oa}$.
Theorem 1, part 2 of 7. Proof.

Construction (1st expansion):



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Choose any column from the OA to expand.

Reduce the alphabet size for this *expansion column* from v_{oa} to v_{od} by keeping only the $v_{oa}v_{od}$ symbols from $Z_{v_{od}}$.

The remaining $N_{oa} - v_{oa}v_{od}$ unassigned positions designate *expansion rows*.

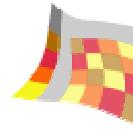
Replace the expansion column with the $N_{oa} \times k_{od}$ subarray which repeats the column k_{od} times.

In the subarray replace any N_{od} of the expansion rows with the OD rows.

There are enough expansion rows because $N_{od} \leq N_{oa} - v_{oa}v_{od}$.

Single column expansion, $N = N_{oa}$.
Theorem 1, part 3 of 7. Proof.

Coverage (1st expansion):



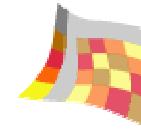
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The unchanged OA columns form a covering array with each of the expanded (subarray) columns individually. The only other pairs of columns to consider are pairs within the subarray. Therefore the expanded array is a covering array if the subarray is a covering array.

The subarray is a covering array because

- (a) each symbol is paired with itself in a row from the OA, and
- (b) each symbol is paired with all the other symbols in an expansion row from the OD.

Single column expansion, $N = N_{oa}$.
Theorem 1, part 4 of 7. Proof.



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Construction (recursion):

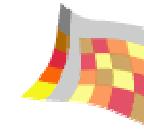
If $m > 1$, replace each of the expanded columns with the $N_{oa} \times k_{od}$ subarray which repeats the column k_{od} times.

In each subarray replace any N_{od} of the remaining expansion rows with the OD rows.

Repeat this process recursively a total of m times.

There are enough expansion rows because $mN_{od} \leq N_{oa} - v_{oa}v_{od}$.

Single column expansion, $N = N_{oa}$.
Theorem 1, part 5 of 7. Proof.



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Coverage (recursion):

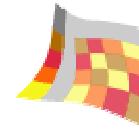
In each expansion, a new subarray replaces each of the previously expanded columns. Consider any pair of different columns according to four cases:

- (1) 2 columns not expanded from the OA.
- (2) 1 not expanded & 1 expanded in a new subarray.
- (3) 2 columns from different new subarrays.
- (4) 2 columns from the same new subarray.

Pairs for (1) & (2) are included because the unchanged OA columns form a CA with each new subarray column individually. Pairs of columns from different new subarrays
(3) cover as follows:

- (a) each symbol is paired with itself in a row from the OA, and
- (b) each symbol is paired with all the other symbols in an OD row from a previous expansion.

Single column expansion, $N = N_{oa}$.
Theorem 1, part 6 of 7. Proof.



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Coverage (recursion):

The only other pairs of columns to consider are pairs within the same new subarray

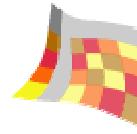
(4). Therefore the expanded array is a covering array if the new subarrays are covering arrays.

Each new subarray is a covering array because

- (a) each symbol is paired with itself in a row from the OA, and
- (b) each symbol is paired with all the other symbols in an OD row for the current expansion.

Thus the array resulting from m expansions is a covering array.

Single column expansion, $N = N_{oa}$.
Theorem 1, part 7 of 7. Proof.



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Parameters:

The number of expanded columns is multiplied by k_{od} in each expansion. There are

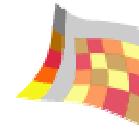
k_{od}^m columns with alphabet size v_{od} , and $k_{oa} - 1$ columns with alphabet size v_{oa} .

The size of the array is unchanged: $N = N_{oa} = v_{oa}^2$.

For $k_{oa} > 2$, there are at least 2 columns with v_{oa} symbols. Thus the size $N = v_{oa}^2$ is optimal.

□

Single column expansion, $N = N_{oa}$.
Boundaries.



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Theorem 1 inequality =>

Maximum column expansion degree: $m \leq [(N_{oa} - v_{oa}v_{od})/N_{od}]$

Approximate maximum v_{od} given v_{oa} : $v_{od} \lesssim 0.62v_{oa} + 0.28$

$(v_{oa} \gg m = 1)$.

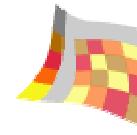
Generalize expansion to more columns.

Covering pairs for different expansion columns occur in OA.

Alphabet sizes for different expansion columns can be same or different.

Expansion degree depends on alphabet size for that column only.

Multiple column expansion, $N = N_{oa}$.
Theorem 2, part 1 of 3.



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Given:

OA($N_{oa}; 2, k_{oa}, v_{oa}$)

column expansion number n, $1 \leq n \leq k_{oa}$

n ordered designs OD($N_{odj}; 2, k_{odj}, v_{odj}$), $1 \leq j \leq n$

$$m_j N_{odj} \leq N_{oa} - v_{oa} v_{odj}$$

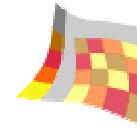
column expansion degree m_j is a positive integer.

There is:

$$\text{MCA}(N_{oa}; 2, k_{oa} - n + \sum_{j=1}^n k_{odj}^{m_j}, v_{oa}^{k_{oa} - n} \prod_{j=1}^n v_{odj}^{k_{odj}^{m_j}})$$

with optimal size when $n \leq k_{oa} - 2$.

Multiple column expansion, $N = N_{oa}$.
Theorem 2, part 2 of 3. Proof.



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Construction:

Apply the construction of Theorem 1 to any n columns of the OA.

There are enough expansion rows because $m_j N_{odj} \leq N_{oa} - v_{oa} v_{odj}$.

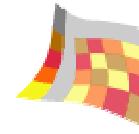
Coverage:

Consider any pair of different columns according to four cases:

- (1) 2 columns not expanded from OA: All symbol pairs remain from OA.
- (2) 1 not expanded & 1 expanded: As in Theorem 1, all pairs remain from OA.
- (3) 2 columns expanded from the same OA column j : As in Theorem 1,
 - (a) each symbol paired with itself in row from OA, and
 - (b) each symbol paired with all other symbols in row from OD.
- (4) 2 columns expanded from different OA columns: All symbol pairs remain from OA.

These are the only choices for a pair, so the expanded array is a MCA.

Multiple column expansion, $N = N_{oa}$.
Theorem 2, part 3 of 3. Proof.



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Parameters:

For each expansion column j , there are $k_{odj}^{m_j}$ resulting columns. There are $k_{oa} - n$

unchanged columns from the OA with alphabet size v_{oa} . $k = k_{oa} - n + \sum_{j=1}^n k_{odj}^{m_j}$.

The size of the array is unchanged: $N = N_{oa} = v_{oa}^2$.

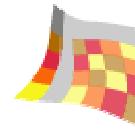
For $n \leq k_{oa} - 2$, there are at least 2 columns with v_{oa} symbols. Thus the size $N = v_{oa}^2$ is optimal.

□

MCA(25;2,137,5³3⁶2¹²⁸).

Example 2.

Theorem 2 => Construction for MCA(25;2,137,5³3⁶2¹²⁸).



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3 OA columns remain with alphabet size 5.

2 OA columns expanded with alphabet size 3 and expansion degree 1.

1 OA column expanded with alphabet size 2 and expansion degree 7.

Many more arrays possible by symbol fusion.

MCA(25;2,137,5³3⁶2¹²⁸) => MCA(25;2,137,5²4¹3⁴2¹³⁰).

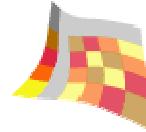
Tabulate examples with 2 alphabet sizes

for largest expansion degree m

for $1 \leq n \leq k_{oa}$.

Size is optimal when $n \leq k_{oa} - 2$.

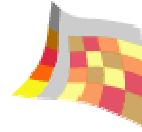
Parameters for mixed covering arrays of size $N = v_{oa}^2$ with 2 alphabet sizes. Table 1, part 1 of 2.



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v_{oa}	k_{oa}	v_{od}	k_{od}	m	N	v_1	k_1	v_2	k_2
3	4	2	2	1	9	3	4-n	2	$2n$
4	5	2	2	4	16	4	5-n	2	$2^4 n$
5	6	2	2	7	25	5	6-n	2	$2^7 n$
		3	3	1	25	5	6-n	3	$3n$
6	3	2	2	12	36	6	3-n	2	$2^{12} n$
		3	3	3	36	6	3-n	3	$3^3 n$
		4	4	1	36	6	3-n	4	$4n$
7	8	2	2	17	49	7	8-n	2	$2^{17} n$
		3	3	4	49	7	8-n	3	$3^4 n$
		4	4	1	49	7	8-n	4	$4n$

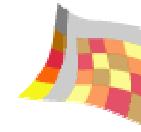
Parameters for mixed covering arrays of size $N = v_{oa}^2$ with 2 alphabet sizes. Table 1, part 2 of 2.



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v_{oa}	k_{oa}	v_{od}	k_{od}	m	N	v_1	k_1	v_2	k_2
8	9	2	2	24	64	8	9-n	2	$2^{24}n$
		3	3	6	64	8	9-n	3	3^6n
		4	4	2	64	8	9-n	4	4^2n
		5	5	1	64	8	9-n	5	$5n$
	10	2	2	31	81	9	10-n	2	$2^{31}n$
9	10	3	3	9	81	9	10-n	3	3^9n
		4	4	3	81	9	10-n	4	4^3n
		5	5	1	81	9	10-n	5	$5n$
	3	2	2	40	100	10	3-n	2	$2^{40}n$
10	3	3	3	11	100	10	3-n	3	$3^{11}n$
		4	4	5	100	10	3-n	4	4^5n
		5	5	2	100	10	3-n	5	5^2n
		6	3	1	100	10	3-n	6	$3n$

Column expansion, $N > N_{oa}$.
Rationale.



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Can construct MCA with $v_{odj} \leq v_{oa}$ and m_j with no upper bound.

The size will grow: $N > N_{oa}$.

Each expansion column will need $N_j = m_j N_{odj} + v_{oa} v_{odj}$ rows.

There is:

$$\text{MCA}(N_j; 2, k_{oa} - n + \sum_{j=1}^n k_{odj}^{m_j}, v_{oa}^{k_{oa} - n} \prod_{i=1}^n v_{odj}^{k_{odj}^{m_j}})$$

with size $N = N_j$.

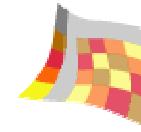
J designates an expansion column where N_j takes its maximum value.

Typically size is not optimal.

If one column of the OA has its alphabet size increased, there are MCAs for larger values of v_{odj} and m_j which may be optimal.

Construction requires an OD with a special property...

Latin Ordered Design.
Definitions 3.



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Latin ordered design

$\text{LOD}(N_{od}; t, k_{od}, v_{od})$ is an ordered design $\text{OD}(N_{od}; t, k_{od}, v_{od})$ partitioned into $\binom{v_{od}-1}{t-1}(t-1)!$

$v_{od} \times k_{od}$ Latin rectangles

Every column of each Latin rectangle contains all v_{od} symbols.

$\text{LOD}(N_{od}; 2, k_{od}, v_{od})$ has $v_{od}-1$ Latin rectangles.

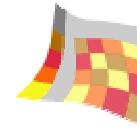
v_{od} a prime power $\Rightarrow \text{LOD}(N_{od}; 2, v_{od}, v_{od})$ consisting of Latin squares

Choose element of Latin square h in row i and column j to be $i + h \times j$ with addition
and multiplication from $\text{GF}(v_{od})$ and $h \neq 0$.

Otherwise \Rightarrow trivial $\text{LOD}(N_{od}; 2, 2, v_{od})$

In Latin rectangle h elements of row i are $(i, i + h \bmod v_{od})$, $h \neq 0$.

MCA(32;2,17,6¹5³4⁴3⁹).
Example 3, part 1 of 2.



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OA(25;2,6,5)

0	0	0	0	0	0
0	1	1	1	1	1
0	2	2	2	2	2
0	3	3	3	3	3
0	4	4	4	4	4
1	0	1	2	3	4
1	1	2	3	4	0
1	2	3	4	0	1
1	3	4	0	1	2
1	4	0	1	2	3
2	0	2	4	1	3
2	1	3	0	2	4
2	2	4	1	3	0
2	3	0	2	4	1
2	4	1	3	0	2
3	0	3	1	4	2
3	1	4	2	0	3
3	2	0	3	1	4
3	3	1	4	2	0
3	4	2	0	3	1
4	0	4	3	2	1
4	1	0	4	3	2
4	2	1	0	4	3
4	3	2	1	0	4
4	4	3	2	1	0

LOD(12;2,4,4)

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0
0	2	3	1
1	3	2	0
2	0	1	3
3	1	0	2
0	3	1	2
1	2	0	3
2	1	3	0
3	0	2	1

LOD(6;2,3,3)

0	1	2
1	2	0
2	0	1
0	2	1
1	0	2
2	1	0

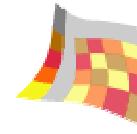
MCA(32;2,17,6¹5³4⁴3⁹).

Example 3, part 2 of 2.



incremented column ↓	unexpanded columns	expanded columns	expanded columns
0	0 0 0 0	0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0	1 1 1	1 1 1 1 1	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
0	2 2 2	2 2 2 2 2	2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2
0	3 3 3	3 3 3 3 3	0 0 0 2 2 2 2 1 1 1
0	4 4 4	0 2 3 1 1 1 1 0 0 0 2 2 2	
1	0 1 2	3 3 3 3 3	2 2 2 1 1 1 0 0 0
1	1 2 3	1 3 2 0	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1	2 3 4	0 0 0 0 0	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1	3 4 0	1 1 1 1 1	2 2 2 2 2 2 2 2 2 2 2 2 2 2 2
1	4 0 1	2 2 2 2 2	0 1 2 0 1 2 0 1 2 1 2
2	0 2 4	1 1 1 1 1	1 2 0 1 2 0 1 2 0
2	1 3 0	2 2 2 2 2	2 0 1 2 0 1 2 0 1
2	2 4 1	3 3 3 3 3	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
2	3 0 2	2 0 1 3	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
2	4 1 3	0 0 0 0 0	2 2 2 2 2 2 2 2 2 2 2 2 2 2 2
3	0 3 1	3 1 0 2	2 2 2 2 2 2 2 2 2 2 2 2 2 2 2
3	1 4 2	0 0 0 0 0	0 2 1 0 2 1 0 2 1
3	2 0 3	1 1 1 1 1	1 0 2 1 0 2 1 0 2
3	3 1 4	2 2 2 2 2	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
3	4 2 0	3 3 3 3 3	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
4	0 4 3	2 2 2 2 2	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
4	1 0 4	3 3 3 3 3	2 2 2 2 2 2 2 2 2 2 2 2 2 2 2
4	2 1 0	0 3 1 2 2 1 0 2 1 0 2 1 0	
4	3 2 1	0 0 0 0 0	* * * * * * * * * * * *
4	4 3 2	1 1 1 1 1	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
pairing subarray	5 0 0 0	0 1 2 3 0 0 0 1 1 1 2 2 2	
	5 1 1 1	1 0 3 2 1 1 1 2 2 2 0 0 0	
	5 2 2 2	2 3 0 1 2 2 2 0 0 0 1 1 1	
	5 3 3 3	3 2 1 0	* * * * * * * * * *
leftover subarray	5 4 4 4	1 2 0 3	* * * * * * * * *
	* * * *	2 1 3 0	* * * * * * * * *
	* * * *	3 0 2 1	* * * * * * * * *

Multiple column expansion, $N > N_{oa}$.
 Theorem 3, part 1 of 6.



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Given:

$$OA(N_{oa}; 2, k_{oa}, v_{oa})$$

n Latin ordered designs $LOD(N_{odj}; 2, k_{odj}, v_{odj})$, $v_{odj} \leq v_{oa}$, $1 \leq j \leq n \leq k_{oa}$

$m_j N_{odj} > N_{oa} - v_{oa} v_{odj}$ for one or more j , $m_j > 0$.

There is:

$$MCA(N_J; 2, k_{oa} - n + \sum_{j=1}^n k_{odj}^{m_j}, v_1^{-1} v_{oa}^{k_{oa}-n-1} \prod_{j=1}^n v_{odj}^{k_{odj}^{m_j}})$$

$$N_J = \max_{j=1}^n m_j N_{odj} + v_{oa} v_{odj}$$

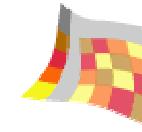
$$v_1 = [N_J / v_{oa}]$$

N_J optimal size when

either $n \leq k_{oa} - 2$ or $v_{odj} = v_{oa}$ for some j ,

and v_{oa} divides N_J .

Multiple column expansion, $N > N_{oa}$.
Theorem 3, part 2 of 6. Proof.



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Construction:

$N_J - N_{oa}$ rows will be appended to the OA. Consider two cases:

- (1) $N_J - N_{oa} \geq v_{oa}$: Choose any column from the OA. The alphabet size of this *incremented column* will be increased to v_1 .

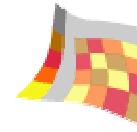
Partition the appended rows into $v_{oa} \times k_{oa}$ *pairing subarrays* and

(if v_{oa} does not divide N_J) one *leftover subarray* with fewer than v_{oa} rows.

In each pairing subarray fill the incremented column with an unused symbol from Z_{v_1} . Symbols in the incremented column of the leftover subarray are left unspecified.

If $n < k_{oa} - 1$, there are one or more unexpanded columns. In each unexpanded column in each pairing subarray, assign each of the symbols from $Z_{v_{oa}}$ to one of the rows.

Multiple column expansion, $N > N_{oa}$.
Theorem 3, part 3 of 6. Proof.



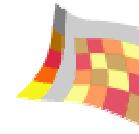
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Construction:

Apply the construction of Theorem 1 to any other n columns of the OA, selecting the expansion rows as follows. For each expansion column place one Latin rectangle from the LOD into each of the pairing subarrays. This step can assign up to $(v_1 - v_{oa})v_{odj}$ expansion rows of the $m_j N_{odj}$ needed.

- (a) $m_j N_{odj} > (v_1 - v_{oa})v_{odj}$: Use any remaining expansion rows to assign the remaining rows from the LOD.
 - (b) $m_j N_{odj} = (v_1 - v_{oa})v_{odj}$: The assignment of the LOD rows is complete.
 - (c) $m_j N_{odj} < (v_1 - v_{oa})v_{odj}$: Place one of the previously assigned Latin rectangles into each of the remaining pairing subarrays.
- (2) $N_J - N_{oa} < v_{oa}$: The alphabet size will not be increased beyond v_{oa} . The appended rows form a leftover subarray. Expand the OA by applying the construction of Theorem 1 to any n columns.

Multiple column expansion, $N > N_{oa}$.
Theorem 3, part 4 of 6. Proof.



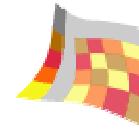
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Coverage:

Consider any pair of different columns according to six cases:

- (1) 2 columns not expanded from OA: All symbol pairs remain from OA.
- (2) 1 not expanded & 1 expanded: As in Theorem 1, all pairs remain from OA.
- (3) 2 columns expanded from the same OA column j : As in Theorem 1,
 - (a) each symbol paired with itself in row from OA, and
 - (b) each symbol paired with all other symbols in row from OD.
- (4) 2 columns expanded from different OA columns: All symbol pairs remain from OA.

Multiple column expansion, $N > N_{oa}$.
Theorem 3, part 5 of 6. Proof.



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Coverage:

- (5) 1 incremented column & 1 not expanded:
 - (a) Pairs with symbols from $Z_{v_{oa}}$ in incremented column remain from OA.
 - (b) Each additional symbol from Z_{v_1} in incremented column is paired with all v_{oa} symbols in pairing subarray.
- (6) 1 incremented column & 1 expanded:
 - (a) Pairs with symbols from $Z_{v_{oa}}$ in incremented column remain from OA.
 - (b) Each additional symbol from Z_{v_1} in incremented column is paired with all v_{odj} symbols in column of a Latin rectangle in pairing subarray.

These are the only choices for a pair, so the expanded array is a MCA.

Multiple column expansion, $N > N_{oa}$.
 Theorem 3, part 6 of 6. Proof.



Parameters:

For each expansion column j , there are $k_{odj}^{m_j}$ resulting columns. There are k_{oa} - n unexpanded columns from the OA with alphabet size v_1 or v_{oa} . $k = k_{oa}$ - $n + \sum_{j=1}^n k_{odj}^{m_j}$.

$$N = N_J = \max_{j=1}^n m_j N_{odj} + v_{oa} v_{odj}.$$

$n \leq k_{oa}$ -2 or $v_{odj} = v_{oa}$ for some $j \Rightarrow$ at least one column has alphabet size v_{oa} .

$N_J > v_{oa}^2 \Rightarrow v_1 = [N_J/v_{oa}] \geq v_{oa} \Rightarrow$ 2 largest alphabet sizes are v_1 & v_{oa} .

v_{oa} divides $N_J \Rightarrow v_1 = N_J/v_{oa}$, so the size $N = N_J = v_1 v_{oa}$ is optimal.

Thus the size is optimal when

either $n \leq k_{oa}$ -2 or $v_{odj} = v_{oa}$ for some j ,

and v_{oa} divides N_J .



MCA(45;2,131342,9¹5¹⁰4¹⁶3²⁴³2¹³¹⁰⁷²).

Example 4.

Theorem 3 => Construction for MCA(45;2,131342,9¹5¹⁰4¹⁶3²⁴³2¹³¹⁰⁷²).



$v_1 = 9$ (smallest value for optimal array with $v_{oa} = 5$ & $v_{odj} = 5$).

2 OA columns expanded with alphabet size 5 and expansion degree 1 ($N_j = 45$).

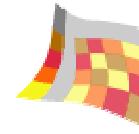
1 OA column expanded with alphabet size 4 and expansion degree 2 ($N_j = 44$).

1 OA column expanded with alphabet size 3 and expansion degree 5 ($N_j = 45$).

1 OA column expanded with alphabet size 2 and expansion degree 17 ($N_j = 44$).

Larger allowed values for v_{odj} and m_j enlarge parameter space of column expansion arrays, increase flexibility of construction.

Multiple column expansion, $N > N_{oa}$.
Condition for optimal size, part 1 of 2.



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Notation convention:

$$N_J = \max_{j=1}^n m_j N_{odj} + v_{oa} v_{odj} = m_J N_{odJ} + v_{oa} v_{odJ}$$

Subscript J denotes values associated with $\max N_j$.

$N_J > N_{oa} \Rightarrow m_J > (N_{oa} - v_{oa} v_{odJ})/N_{odJ}$. m_J has lower bound, but not upper bound.

Necessary condition for optimal size:

v_{oa} divides $N_J \Leftrightarrow v_{oa}$ divides $m_J N_{odJ}$. 2 special cases:

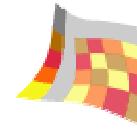
- v_{oa} divides $m_J \Rightarrow$ MCA has optimal size when m_J is a multiple of v_{oa} .

$$m_J = i v_{oa} \Rightarrow i > (v_{oa} - v_{odJ})/N_{odJ} \text{ & } v_1 = v_{odJ} + i N_{odJ}$$

- v_{oa} divides $N_{odJ} \Rightarrow$ MCA has optimal size for all values of m_J .

$$N_{odJ} = i v_{oa} \Rightarrow i > (v_{oa} - v_{odJ})/m_J \text{ & } v_1 = v_{odJ} + i m_J$$

Multiple column expansion, $N > N_{oa}$.
Condition for optimal size, part 2 of 2.



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When does v_{oa} divide N_{odJ} ?

$$N_{odJ} = i v_{oa} =>$$

$$v_{odJ} \leq v_{oa} \leq N_{odJ} \quad (\text{and } i \text{ has an upper bound too: } i \leq v_{odJ}-1).$$

Given $v_{odJ} > 2$, there are at least 2 values of v_{oa} for optimal size, at boundaries of

$$v_{odJ} \leq v_{oa} \leq N_{odJ}.$$

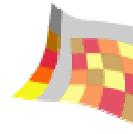
If v_{oa} is a prime or prime power, it will divide N_{odJ} only if $v_{odJ} = v_{oa}$.

Can prove by expressing N_{odJ} as product of prime factors of v_{odJ} and $v_{odJ}-1$, and

observing that $v_{oa} \nmid v_{odJ}$.

$\Rightarrow v_{oa}$ must have distinct prime factors to divide N_{odJ} when $v_{oa} > v_{odJ}$.

Multiple column expansion, $N > N_{oa}$.
Tabulated examples.



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Examples with 2 or 3 alphabet sizes (1 alphabet size for expansion)

for small values of expansion degree m , for $1 \leq n \leq k_{oa}$.

Optimal size is indicated when:

- v_{oa} divides m ,
- v_{oa} divides N_{od} ,
- and otherwise.

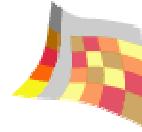
Compare optimal parameters for $v_{oa} = 5$ (typical prime) with those

for $v_{oa} = 6$ and $v_{od} = 3$ or 4 ,

for $v_{oa} = 10$ and $v_{od} = 5$ or 6 .

Note that when $N_{od} > v_{oa}$, v_1 can be increased to yield intermediate sized MCAs without expanding columns. And since N is increased by v_{oa} when v_1 is increased by 1, an optimal sized array yields another.

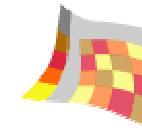
Parameters for mixed covering arrays of size $N > v_{oa}^2$ with 2 or 3 alphabet sizes. Table 2, part 1 of 19.



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v_{oa}	k_{oa}	v_{od}	k_{od}	m	$v_{oa} m$	$v_{oa} N_{od}$	optimal*	N	v_1	k_1	v_2	k_2	v_3	k_3
2	3	2	2	1		•	•	6	3	1	2	2^2		
				2	•	•	•	8	4	1	2	2^3		
				3		•	•	10	5	1	2	2^4		
3	4	2	2	2				10	3	$4-n$	2	$2^2 n$		
				3	•		•	12	4	1	3	$3-n$	2	$2^3 n$
				4				14	4	1	3	$3-n$	2	$2^4 n$
				5				16	5	1	3	$3-n$	2	$2^5 n$
				6	•		•	18	6	1	3	$3-n$	2	$2^6 n$
		3	3	1		•	•	15	5	1	3	3^2		
				2		•	•	21	7	1	3	3^3		
				3	•	•	•	27	9	1	3	3^4		

Parameters for mixed covering arrays of size $N > v_{oa}^2$
with 2 or 3 alphabet sizes. Table 2, part 2 of 19.

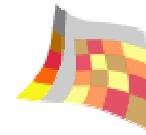


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v_{oa}	k_{oa}	v_{od}	k_{od}	m	$v_{oa} m$	$v_{oa} N_{od}$	optimal*	N	v_1	k_1	v_2	k_2	v_3	k_3
4	5	2	2	5				18	4	5-n	2	$2^5 n$		
				6			•	20	5	1	4	$4-n$	2	$2^6 n$
				7				22	5	1	4	$4-n$	2	$2^7 n$
				8	•		•	24	6	1	4	$4-n$	2	$2^8 n$
	3	3	1					18	4	5-n	3	$3n$		
			2					24	6	1	4	$4-n$	3	$3^2 n$
			3					30	7	1	4	$4-n$	3	$3^3 n$
			4	4	•		•	36	9	1	4	$4-n$	3	$3^4 n$
	4	4	1		•	•		28	7	1	4	4^2		
			2		•	•		40	10	1	4	4^3		
			3		•	•		52	13	1	4	4^4		

* for $n \leq k_{oa}-2$

Parameters for mixed covering arrays of size $N > v_{oa}^2$
with 2 or 3 alphabet sizes. Table 2, part 3 of 19.

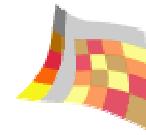


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v_{oa}	k_{oa}	v_{od}	k_{od}	m	$v_{oa} m$	$v_{oa} N_{od}$	optimal*	N	v_1	k_1	v_2	k_2	v_3	k_3
5	6	2	2	8				26	5	6-n	2	$2^8 n$		
				9				28	5	6-n	2	$2^9 n$		
				10	•	•	•	30	6	1	5	$5-n$	2	$2^{10} n$
				3	3	2		27	5	6-n	3	$3^2 n$		
				3				33	6	1	5	$5-n$	3	$3^3 n$
				4				39	7	1	5	$5-n$	3	$3^4 n$
				5	•	•	•	45	9	1	5	$5-n$	3	$3^5 n$
	4	4	4	1				32	6	1	5	$5-n$	4	$4 n$
				2				44	8	1	5	$5-n$	4	$4^2 n$
				3				56	11	1	5	$5-n$	4	$4^3 n$
				4				68	13	1	5	$5-n$	4	$4^4 n$
				5	•	•	•	80	16	1	5	$5-n$	4	$4^5 n$
	5	5	5	1				45	9	1	5	5^2		
				2				65	13	1	5	5^3		
				3				85	17	1	5	5^4		

* for $n \leq k_{oa}-2$

Parameters for mixed covering arrays of size $N > v_{oa}^2$
with 2 or 3 alphabet sizes. Table 2, part 4 of 19.

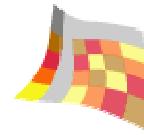


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v_{oa}	k_{oa}	v_{od}	k_{od}	m	$v_{oa} m$	$v_{oa} N_{od}$	optimal*	N	v_1	k_1	v_2	k_2	v_3	k_3
6	3	2	2	13				38	6	3-n	2	$2^{13}n$		
				14				40	6	3-n	2	$2^{14}n$		
		3	3	15			•	42	7	1	6	1	2	2^{15}
				16				44	7	1	6	1	2	2^{16}
				17				46	7	1	6	1	2	2^{17}
				18	•	•	•	48	8	1	6	1	2	2^{18}
	4	4	4	4		•	•	42	7	1	6	1	3	3^4
				5		•	•	48	8	1	6	1	3	3^5
		3	4	6	•	•	•	54	9	1	6	1	3	3^6
				7		•	•	60	10	1	6	1	3	3^7
				8		•	•	66	11	1	6	1	3	3^8
				9		•	•	72	12	1	6	1	3	3^9
	4	4	4	2		•	•	48	8	1	6	1	4	4^2
				3		•	•	60	10	1	6	1	4	4^3
		3	3	4		•	•	72	12	1	6	1	4	4^4
				5		•	•	84	14	1	6	1	4	4^5
				6	•	•	•	96	16	1	6	1	4	4^6
				7		•	•	108	18	1	6	1	4	4^7

* for $n \leq k_{oa}-2$

Parameters for mixed covering arrays of size $N > v_{oa}^2$
with 2 or 3 alphabet sizes. Table 2, part 5 of 19.

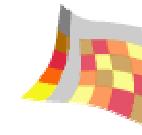


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v_{oa}	k_{oa}	v_{od}	k_{od}	m	$v_{oa} m$	$v_{oa} N_{od}$	optimal*	N	v_1	k_1	v_2	k_2	v_3	k_3
6	3	5	5	1				50	8	1	6	1	5	5
				2			•	70	11	1	6	1	5	5^2
				3			•	90	15	1	6	1	5	5^3
				4			•	110	18	1	6	1	5	5^4
				5			•	130	21	1	6	1	5	5^5
				6	•	•	•	150	25	1	6	1	5	5^6
	6	2	1		•	•	•	66	11	1	6		2^2	
			2		•	•	•	96	16	1	6		2^3	
			3		•	•	•	126	21	1	6		2^4	
	7	8	2	18				50	7	$8-n$	2		$2^{18}n$	
				19				52	7	$8-n$	2		$2^{19}n$	
				20				54	7	$8-n$	2		$2^{20}n$	
		3	3	21	•		•	56	8	1	7	$7-n$	2	$2^{21}n$
				5				51	7	$8-n$	3		3^5n	
				6				57	8	1	7	$7-n$	3	3^6n
				7	•		•	63	9	1	7	$7-n$	3	3^7n

* for $n \leq k_{oa}-2$

Parameters for mixed covering arrays of size $N > v_{oa}^2$
with 2 or 3 alphabet sizes. Table 2, part 6 of 19.

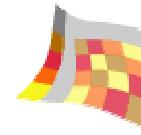


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v_{oa}	k_{oa}	v_{od}	k_{od}	m	$v_{oa} m$	$v_{oa} N_{od}$	optimal*	N	v_1	k_1	v_2	k_2	v_3	k_3
7	8	4	4	2	2			52	7	8-n	4	4^2n		
					3			64	9	1	7	$7-n$	4	4^3n
					4			76	10	1	7	$7-n$	4	4^4n
					5			88	12	1	7	$7-n$	4	4^5n
					6			100	14	1	7	$7-n$	4	4^6n
					7	•	•	112	16	1	7	$7-n$	4	4^7n
				5	1			55	7	8-n	5	$5n$		
	5	2			2			75	10	1	7	$7-n$	5	5^2n
		3			3			95	13	1	7	$7-n$	5	5^3n
		4			4			115	16	1	7	$7-n$	5	5^4n
		5			5			135	19	1	7	$7-n$	5	5^5n
		6			6			155	22	1	7	$7-n$	5	5^6n
		7			7	•	•	175	25	1	7	$7-n$	5	5^7n

* for $n \leq k_{oa}-2$

Parameters for mixed covering arrays of size $N > v_{oa}^2$
with 2 or 3 alphabet sizes. Table 2, part 7 of 19.

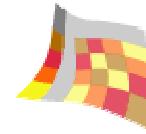


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v_{oa}	k_{oa}	v_{od}	k_{od}	m	$v_{oa} m$	$v_{oa} N_{od}$	optimal*	N	v_1	k_1	v_2	k_2	v_3	k_3
7	8	6	2	1				72	10	1	7	7-n	6	$2n$
				2				102	14	1	7	7-n	6	$2^2 n$
				3				132	18	1	7	7-n	6	$2^3 n$
				4				162	23	1	7	7-n	6	$2^4 n$
				5				192	27	1	7	7-n	6	$2^5 n$
				6				222	31	1	7	7-n	6	$2^6 n$
				7	•	•	•	252	36	1	7	7-n	6	$2^7 n$
	7	7	1					91	13	1	7	7^2		
			2					133	19	1	7	7^3		
			3					175	25	1	7	7^4		
8	9	2	2	25				66	8	9-n	2	$2^{25} n$		
				26				68	8	9-n	2	$2^{26} n$		
				27				70	8	9-n	2	$2^{27} n$		
				28			•	72	9	1	8	8-n	2	$2^{28} n$
				29				74	9	1	8	8-n	2	$2^{29} n$
				30				76	9	1	8	8-n	2	$2^{30} n$
				31				78	9	1	8	8-n	2	$2^{31} n$
				32	•	•	•	80	10	1	8	8-n	2	$2^{32} n$

* for $n \leq k_{oa}-2$

Parameters for mixed covering arrays of size $N > v_{oa}^2$
with 2 or 3 alphabet sizes. Table 2, part 8 of 19.

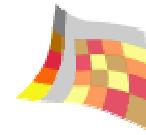


Testcover.com

v_{oa}	k_{oa}	v_{od}	k_{od}	m	$v_{oa} m$	$v_{oa} N_{od}$	optimal*	N	v_1	k_1	v_2	k_2	v_3	k_3
8	9	3	3	7				66	8	9-n	3	3^7n		
				8	•	•	•	72	9	1	8	$8-n$	3	3^8n
				9				78	9	1	8	$8-n$	3	3^9n
				10				84	10	1	8	$8-n$	3	$3^{10}n$
				11				90	11	1	8	$8-n$	3	$3^{11}n$
				12			•	96	12	1	8	$8-n$	3	$3^{12}n$
		4	4	3				68	8	9-n	4	4^3n		
				4			•	80	10	1	8	$8-n$	4	4^4n
				5				92	11	1	8	$8-n$	4	4^5n
				6			•	104	13	1	8	$8-n$	4	4^6n
				7				116	14	1	8	$8-n$	4	4^7n
				8	•	•	•	128	16	1	8	$8-n$	4	4^8n

* for $n \leq k_{oa}-2$

Parameters for mixed covering arrays of size $N > v_{oa}^2$
with 2 or 3 alphabet sizes. Table 2, part 9 of 19.

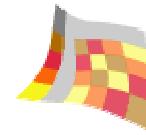


Testcover.com

v_{oa}	k_{oa}	v_{od}	k_{od}	m	$v_{oa} m$	$v_{oa} N_{od}$	*optimal*	N	v_1	k_1	v_2	k_2	v_3	k_3			
8	9	5	5	2	•	•	•	80	10	1	8	8-n	5	$5^2 n$			
								100	12	1	8	8-n	5	$5^3 n$			
				3				120	15	1	8	8-n	5	$5^4 n$			
								140	17	1	8	8-n	5	$5^5 n$			
	6	2	2	4				160	20	1	8	8-n	5	$5^6 n$			
								180	22	1	8	8-n	5	$5^7 n$			
				5				200	25	1	8	8-n	5	$5^8 n$			
								78	9	1	8	8-n	6	$2n$			
	6	3	3	6				108	13	1	8	8-n	6	$2^2 n$			
								138	17	1	8	8-n	6	$2^3 n$			
				7				168	21	1	8	8-n	6	$2^4 n$			
								198	24	1	8	8-n	6	$2^5 n$			
	6	4	4	8				228	28	1	8	8-n	6	$2^6 n$			
								258	32	1	8	8-n	6	$2^7 n$			
				8				288	36	1	8	8-n	6	$2^8 n$			

* for $n \leq k_{oa}-2$

Parameters for mixed covering arrays of size $N > v_{oa}^2$
with 2 or 3 alphabet sizes. Table 2, part 10 of 19.

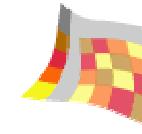


Testcover.com

v_{oa}	k_{oa}	v_{od}	k_{od}	m	$v_{oa} m$	$v_{oa} N_{od}$	optimal*	N	v_1	k_1	v_2	k_2	v_3	k_3
8	9	7	7	1			•	98	12	1	8	8-n	7	$7n$
								140	17	1	8	8-n	7	7^2n
								182	22	1	8	8-n	7	7^3n
								224	28	1	8	8-n	7	7^4n
								266	33	1	8	8-n	7	7^5n
								308	38	1	8	8-n	7	7^6n
								350	43	1	8	8-n	7	7^7n
								392	49	1	8	8-n	7	7^8n
								120	15	1	8			8^2
								176	22	1	8			8^3
9	10	2	2	32			•	232	29	1	8			8^4
								82	9	10-n	2			$2^{32}n$
								84	9	10-n	2			$2^{33}n$
								86	9	10-n	2			$2^{34}n$
								88	9	10-n	2			$2^{35}n$
				36			•	90	10	1	9	9-n	2	$2^{36}n$

* for $n \leq k_{oa}-2$

Parameters for mixed covering arrays of size $N > v_{oa}^2$
with 2 or 3 alphabet sizes. Table 2, part 11 of 19.

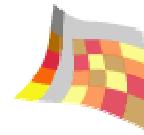


Testcover.com

v_{oa}	k_{oa}	v_{od}	k_{od}	m	$v_{oa} m$	$v_{oa} N_{od}$	* optimal*	N	v_1	k_1	v_2	k_2	v_3	k_3
9	10	3	3	10				87	9	10-n	3	$3^{10}n$		
				11				93	10	1	9	$9-n$	3	$3^{11}n$
				12			•	99	11	1	9	$9-n$	3	$3^{12}n$
		4	4	13				105	11	1	9	$9-n$	3	$3^{13}n$
				14				111	12	1	9	$9-n$	3	$3^{14}n$
				15			•	117	13	1	9	$9-n$	3	$3^{15}n$
				16				123	13	1	9	$9-n$	3	$3^{16}n$
				17				129	14	1	9	$9-n$	3	$3^{17}n$
				18	•		•	135	15	1	9	$9-n$	3	$3^{18}n$
				4				84	9	10-n	4	4^4n		
				5				96	10	1	9	$9-n$	4	4^5n
				6			•	108	12	1	9	$9-n$	4	4^6n
		9	9	7				120	13	1	9	$9-n$	4	4^7n
				8				132	14	1	9	$9-n$	4	4^8n
				9	•		•	144	16	1	9	$9-n$	4	4^9n

* for $n \leq k_{oa}-2$

Parameters for mixed covering arrays of size $N > v_{oa}^2$
with 2 or 3 alphabet sizes. Table 2, part 12 of 19.

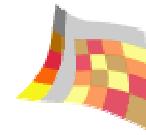


Testcover.com

v_{oa}	k_{oa}	v_{od}	k_{od}	m	$v_{oa} m$	$v_{oa} N_{od}$	*optimal*	N	v_1	k_1	v_2	k_2	v_3	k_3
9	10	5	5	2			•	85	9	10-n	5	5^2n		
								105	11	1	9	$9-n$	5	5^3n
								125	13	1	9	$9-n$	5	5^4n
								145	16	1	9	$9-n$	5	5^5n
								165	18	1	9	$9-n$	5	5^6n
								185	20	1	9	$9-n$	5	5^7n
								205	22	1	9	$9-n$	5	5^8n
								225	25	1	9	$9-n$	5	5^9n
		6	2	1			•	84	9	10-n	6	$2n$		
								114	12	1	9	$9-n$	6	2^2n
								144	16	1	9	$9-n$	6	2^3n
								174	19	1	9	$9-n$	6	2^4n
								204	22	1	9	$9-n$	6	2^5n
								234	26	1	9	$9-n$	6	2^6n
								264	29	1	9	$9-n$	6	2^7n
								294	32	1	9	$9-n$	6	2^8n
								324	36	1	9	$9-n$	6	2^9n

* for $n \leq k_{oa}-2$

Parameters for mixed covering arrays of size $N > v_{oa}^2$
with 2 or 3 alphabet sizes. Table 2, part 13 of 19.

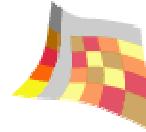


Testcover.com

v_{oa}	k_{oa}	v_{od}	k_{od}	m	$v_{oa} m$	$v_{oa} N_{od}$	optimal*	N	v_1	k_1	v_2	k_2	v_3	k_3
9	10	7	7	1				105	11	1	9	9-n	7	$7n$
				2				147	16	1	9	9-n	7	7^2n
				3			•	189	21	1	9	9-n	7	7^3n
				4				231	25	1	9	9-n	7	7^4n
				5				273	30	1	9	9-n	7	7^5n
				6			•	315	35	1	9	9-n	7	7^6n
				7				357	39	1	9	9-n	7	7^7n
				8			•	399	44	1	9	9-n	7	7^8n
				9	•		•	441	49	1	9	9-n	7	7^9n
				8	8			128	14	1	9	9-n	8	$8n$
				1				184	20	1	9	9-n	8	8^2n
				2				240	26	1	9	9-n	8	8^3n
				3				296	32	1	9	9-n	8	8^4n
				4				352	39	1	9	9-n	8	8^5n
				5				408	45	1	9	9-n	8	8^6n
				6				464	51	1	9	9-n	8	8^7n
				7				520	57	1	9	9-n	8	8^8n
				8				576	64	1	9	9-n	8	8^9n
				9	•		•							

* for $n \leq k_{oa}-2$

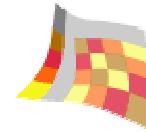
Parameters for mixed covering arrays of size $N > v_{oa}^2$ with 2 or 3 alphabet sizes. Table 2, part 14 of 19.



Testcover.com

v_{oa}	k_{oa}	v_{od}	k_{od}	m	$v_{oa} m$	$v_{od} N_{od}$	optimal*	N	v_1	k_1	v_2	k_2	v_3	k_3
9	10	9	9	1			•	153	17	1	9	9^2		
								225	25	1	9	9^3		
								297	33	1	9	9^4		
10	3	2	2	41			•	102	10	$3-n$	2	$2^{41}n$		
								104	10	$3-n$	2	$2^{42}n$		
								106	10	$3-n$	2	$2^{43}n$		
								108	10	$3-n$	2	$2^{44}n$		
							•	110	11	1	10	1	2	2^{45}
								112	11	1	10	1	2	2^{46}
							•	114	11	1	10	1	2	2^{47}
								116	11	1	10	1	2	2^{48}
								118	11	1	10	1	2	2^{49}
								120	12	1	10	1	2	2^{50}

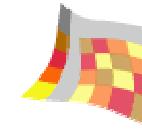
Parameters for mixed covering arrays of size $N > v_{oa}^2$ with 2 or 3 alphabet sizes. Table 2, part 15 of 19.



Testcover.com

v_{oa}	k_{oa}	v_{od}	k_{od}	m	$v_{oa} m$	$v_{oa} N_{od}$	optimal*	N	v_1	k_1	v_2	k_2	v_3	k_3
10	3	3	3	12				102	10	3-n	3	$3^{12}n$		
				13				108	10	3-n	3	$3^{13}n$		
				14				114	11	1	10	1	3	3^{14}
				15			•	120	12	1	10	1	3	3^{15}
				16				126	12	1	10	1	3	3^{16}
				17				132	13	1	10	1	3	3^{17}
				18				138	13	1	10	1	3	3^{18}
				19				144	14	1	10	1	3	3^{19}
				20	•		•	150	15	1	10	1	3	3^{20}
	4	4		6				112	11	1	10	1	4	4^6
				7				124	12	1	10	1	4	4^7
				8				136	13	1	10	1	4	4^8
				9				148	14	1	10	1	4	4^9
				10	•		•	160	16	1	10	1	4	4^{10}

Parameters for mixed covering arrays of size $N > v_{oa}^2$
with 2 or 3 alphabet sizes. Table 2, part 16 of 19.

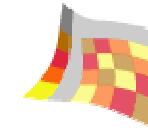


Testcover.com

v_{oa}	k_{oa}	v_{od}	k_{od}	m	$v_{oa} m$	$v_{oa} N_{od}$	optimal*	N	v_1	k_1	v_2	k_2	v_3	k_3
10	3	5	5	3		•	•	110	11	1	10	1	5	5^3
				4		•	•	130	13	1	10	1	5	5^4
				5		•	•	150	15	1	10	1	5	5^5
				6		•	•	170	17	1	10	1	5	5^6
				7		•	•	190	19	1	10	1	5	5^7
				8		•	•	210	21	1	10	1	5	5^8
				9		•	•	230	23	1	10	1	5	5^9
				10	•	•	•	250	25	1	10	1	5	5^{10}
	6	2	2	2		•	•	120	12	1	10	1	6	2^2
				3		•	•	150	15	1	10	1	6	2^3
				4		•	•	180	18	1	10	1	6	2^4
				5		•	•	210	21	1	10	1	6	2^5
				6		•	•	240	24	1	10	1	6	2^6
				7		•	•	270	27	1	10	1	6	2^7
				8		•	•	300	30	1	10	1	6	2^8
				9		•	•	330	33	1	10	1	6	2^9
				10	•	•	•	360	36	1	10	1	6	2^{10}

* for $n \leq k_{oa}-2$

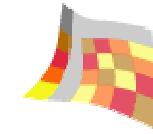
Parameters for mixed covering arrays of size $N > v_{oa}^2$ with 2 or 3 alphabet sizes. Table 2, part 17 of 19.



Testcover.com

v_{oa}	k_{oa}	v_{od}	k_{od}	m	$v_{oa} m$	$N_{oa} N_{od}$	optimal*	N	v_1	k_1	v_2	k_2	v_3	k_3
10	3	7	7	1			•	112	11	1	10	1	7	7
								154	15	1	10	1	7	7^2
								196	19	1	10	1	7	7^3
								238	23	1	10	1	7	7^4
								280	28	1	10	1	7	7^5
								322	32	1	10	1	7	7^6
								364	36	1	10	1	7	7^7
								406	40	1	10	1	7	7^8
								448	44	1	10	1	7	7^9
								490	49	1	10	1	7	7^{10}

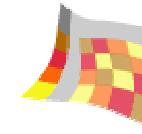
Parameters for mixed covering arrays of size $N > v_{oa}^2$ with 2 or 3 alphabet sizes. Table 2, part 18 of 19.



Testcover.com

v_{oa}	k_{oa}	v_{od}	k_{od}	m	$v_{oa} m$	$v_{oa} N_{od}$	optimal*	N	v_1	k_1	v_2	k_2	v_3	k_3
10	3	8	8	1				136	13	1	10	1	8	8
				2				192	19	1	10	1	8	8^2
				3				248	24	1	10	1	8	8^3
				4				304	30	1	10	1	8	8^4
				5			•	360	36	1	10	1	8	8^5
				6				416	41	1	10	1	8	8^6
				7				472	47	1	10	1	8	8^7
				8				528	52	1	10	1	8	8^8
				9				584	58	1	10	1	8	8^9
				10	•	•	•	640	64	1	10	1	8	8^{10}

Parameters for mixed covering arrays of size $N > v_{oa}^2$
with 2 or 3 alphabet sizes. Table 2, part 19 of 19.



Testcover.com

v_{oa}	k_{oa}	v_{od}	k_{od}	m	$v_{oa} m$	$v_{oa} N_{od}$	optimal*	N	v_1	k_1	v_2	k_2	v_3	k_3
10	3	9	9	1			•	162	16	1	10	1	9	9
								234	23	1	10	1	9	9^2
								306	30	1	10	1	9	9^3
								378	37	1	10	1	9	9^4
								450	45	1	10	1	9	9^5
								522	52	1	10	1	9	9^6
								594	59	1	10	1	9	9^7
								666	66	1	10	1	9	9^8
								738	73	1	10	1	9	9^9
								810	81	1	10	1	9	9^{10}
10	2	10	2	10	•	•	•	190	19	1	10	2^2		
								280	28	1	10	2^3		
								370	37	1	10	2^4		

* for $n \leq k_{oa} - 2$