Covering Arrays and Extremal Set-Partition Systems

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Covering arrays

Definition: Covering Array

A *covering array* of strength t, k factors, g levels for each factor and size N, denoted by $CA(N; t, k, g)$, is an $k \times N$ array with symbols from a g-ary alphabet G such that in every $t \times N$ subarray, every t-tuple in G^t is covered at least once.

$$N = 10, t = 2, k = 4, g = 3$$

1	2	3	4	5	6	7	8	9	10
0	0	0	1	1	1	2	2	2	0
0	1	2	0	1	2	0	1	2	0
0	1	2	2	0	1	1	2	0	1
0	0	2	1	2	0	2	0	1	1

Covering array optimization questions

Fix t and g.

Minimizing N for fixed k (number of tests)

$$CAN(t,k,g) = \min\{N : \textit{ there exists a } CA(N;t,k,g)\}.$$

Maximizing k for fixed N (number of factors)

$$CAK(N,t,g) = \max\{k : \textit{ there exists a } CA(N;t,k,g)\}.$$

Relationship between min-max problems

$$CAN(t,k,g) = \min\{N : CAK(N,t,g) \geq k\}.$$

Methodology

We will focus on the following problem:

Maximizing k for fixed N (number of factors)

$$CAK(N,t,g) = \max\{k : \textit{ there exists a } CA(N;t,k,g)\}.$$

General methodology for an optimization problem (maximization)

1. Relaxed problem: relax constraints to find upper bounds
2. Hard solution matching upper bound: build a feasible solution to hard problem that matches the upper bound (or close to upper bound)

More structure: sometimes it is worth adding more structure to the objects saught.

Binary covering arrays and set systems

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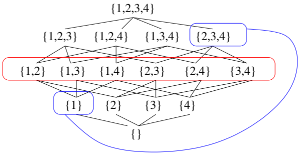
- Maximization problem: $CAK(N, t, g) = \max\{k :$ $there\ exists\ a\ CA(N; t, k, g)\}$
- Maximization problem: **strongly intersecting**
  Given N, find a set system $\mathcal{A}$ with maximum $|\mathcal{A}|$ such that for all $A, B \in \mathcal{A}$ we have:

$$A \cap B \neq \emptyset, \qquad A \cap \overline{B} \neq \emptyset,$$
$$\overline{A} \cap B \neq \emptyset, \qquad \overline{A} \cap \overline{B} \neq \emptyset.$$

Sperner theorem for set systems

A system of subsets of an n-set has the *Sperner property* if no two subsets in the system are comparable.



Sperner's Theorem (1928)

If $\mathcal{A}$ has the Sperner property, then $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

The upper bound is only acchieved by the set of all $(\lfloor \frac{n}{2} \rfloor)$-subsets of the n-set, or by its (subsetwise) complement.

Erdos-Ko-Rado theorem for set systems

A system of subsets of an n-set is t-*intersecting* if every two subsets in the system have intersection cardinality at least t.

Example: $n = 6, k = 3$

$(t = 2)$ $\mathcal{A} = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$

$(t = 1)$ $\mathcal{B} = \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,3,4\},$
$\{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},\{1,5,6\}\}$

Erdos-Ko-Rado Theorem (1961)

Let $t < k < n$. Let $\mathcal{A}$ be a t-intersecting system of subsets of an n-set, such that each subset has cardinality at most k.
If $n \geq (t+1)(k-t+1)$, then $|\mathcal{A}| \leq \binom{n-t}{k-t}$.

Moreover, if $n > (t+1)(k-t+1)$, then equality holds if and only if $\mathcal{A}$ is a k-uniform trivially t-intersecting system.

Applying the methodology to covering arrays

Methodology for binary covering arrays

1. Relaxed problem: strongly intersecting set system (plus complement) is Sperner - use Sperner upper bound.
2. Hard solution matching upper bound: build a strongly intersecting set system that matches the upper bound.

More structure: use point-balanced covering arrays (uniform set systems with sets of cardinality around $N/2$).

Solving the binary covering array problem

Pick all $\lfloor n/2 \rfloor$-subsets of $[1, n]$ that contain a common element.

n odd:

1	2	3	4	5
1	1	0	0	0
1	0	1	0	0
1	0	0	1	0
1	0	0	0	1

n even:

1	2	3	4	5	6
1	1	1	0	0	0
1	1	0	1	0	0
1	1	0	0	1	0
1	1	0	0	0	1
1	0	1	1	0	0
1	0	1	0	1	0
1	0	1	0	0	1
1	0	0	1	1	0
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Note: Systems are strongly Sperner and 1-intersecting.

The binary covering array theorem

Theorem (Katona 1973, Kleitman and Spencer 1973)

$CAK(n, t = 2, g = 2) = \binom{n-1}{\lfloor n/2 \rfloor - 1}$. *Moreover, this bound is attained by a* $\lfloor n/2 \rfloor$*-uniform trivially* 1*-intersecting set system.*

Proof: Let $\mathcal{A}$ be the set system corresponding to the CA.

- **(Case 1)** n **even.**

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- **(Case 2)** n **odd.**
- Wlog assume $|A| \leq \lfloor n/2 \rfloor$, for all $A \in \mathcal{A}$.
- $\mathcal{A}$ is 1-intersecting, so by the EKR theorem, $|\mathcal{A}| \leq \binom{n-1}{\lfloor n/2 \rfloor - 1}$.

Covering arrays are systems of set partitions

- A covering array (strength 2) is a system of set-partitions:

1	2	3	4	5	6	7	8	9	10	
0	0	0	1	1	1	2	2	2	0	{1,2,3,10} {4,5,6} {7,8,9}
0	1	2	0	1	2	0	1	2	0	{1,4,7,10} {2,5,8} {3,6,9}
0	1	2	2	0	1	1	2	0	1	{1,5,9} {2,6,7,10} {3,4,8}
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0	1	2	2	0	1	1	2	0	1	{1,5,9}	{2,6,7,10}	{3,4,8}
0	0	2	1	2	0	2	0	1	1	{1,2,6,8}	{4,9,10}	{3,5,7}

- Maximization problem:
  Given N, find a set partition system $\mathcal{P}$ with maximum $|\mathcal{P}|$ that is **(pairwise) strongly intersecting**:
  For all $P, Q \in \mathcal{P}$ we have

$$\text{for all } P_i \in P, Q_j \in Q, \quad P_i \cap Q_j \neq \emptyset.$$

Strongly intersecting condition: upper bound via 2-parts

Theorem (Stevens, Moura and Mendelsohn 1998)

$CAK(n, 2, g) \leq \frac{1}{2}\binom{\lfloor \frac{2n}{g} \rfloor}{\lfloor \frac{n}{g} \rfloor}$.

This theorem only uses the two smallest parts of each partition, and the following fact:
Consider a pair of set systems, $A_1, A_2, \ldots, A_k$ and $B_1, B_2, \ldots, B_k$, with $|A_i| + |B_i| \leq c$ and $|A_i| \leq a \leq c/2$, and such that $A_i \cap B_i = \emptyset$, and all other sets intersect. Then, $k \leq \frac{1}{2}\binom{c}{a}$.
It is possible to relabel symbols of the covering array so that $|P_{1j}| \leq \lfloor \frac{n}{g} \rfloor$ and $|P_{1j}| + |P_{2j}| \leq \lfloor \frac{2n}{g} \rfloor$

Stronly intersecting versus Sperner formulation

Strongly intersecting formulation:
Partitions P and Q corresponding to two rows of a covering array must satisfy:

$$\text{for all } P_i \in P, Q_j \in Q, \quad P_i \cap Q_j \neq \emptyset.$$

Strongly Sperner formulation:
Partitions P and Q corresponding to two rows of a covering array must satisfy:

$$\text{for all } P_i \in P, Q_j \in Q, \quad P_i \not\subseteq \overline{Q}_j \text{ and } Q_j \not\subseteq \overline{P}_i$$

Higher level extremal problems

- Framework by Ahlswede, Cai and Zhang:
  System of "clouds": $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k\}$.

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    Comparable, iNcomparable, Disjoint, Intersecting.

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    Comparable, iNcomparable, Disjoint, Intersecting.
  - $I_n(\exists, \forall)$: largest cardinality k of a system of clouds of $[1, n]$ such that for all $\mathcal{A}_i, \mathcal{A}_j, \in \mathcal{A}$:

$$\exists A \in \mathcal{A}_i, \forall A' \in \mathcal{A}_j (A \cap A' \neq \emptyset).$$

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- For us each cloud is a g-partition of $[1, n]$:
  - $I_n(\forall, \forall) = CAK(n, 2, g)$.

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- For us each cloud is a g-partition of $[1, n]$:
  - $I_n(\forall, \forall) = CAK(n, 2, g)$.
  - $N_n(\forall, \forall) =$ Sperner g-partition system
    Note: $I_n(\forall, \forall) \leq N_n(\forall, \forall)$

Sperner's theorem for set-partition systems

$N_n(\forall,\forall)$: largest cardinality k of a system $\mathcal{P}$ of g-partitions of $[1,n]$ such that for all $\mathcal{P}_i, \mathcal{P}_j \in \mathcal{P}$:

$$\forall P \in \mathcal{P}_i, \forall P' \in \mathcal{P}_j (P \not\subseteq P' \text{and} P' \not\subseteq P).$$ (Weakly) Sperner

Theorem (Meagher, Moura and Stevens 2005)

Let g, n such that $n = cg + r$ and $0 \leq r < g$. Then,

$$N_n(\forall,\forall) \leq \frac{1}{(g-r) + \frac{r(c+1)}{n-1}} \binom{n}{c}.$$

Theorem (Meagher, Moura and Stevens 2005)

Let g, n such that $g|n$. Then, $N_n(\forall,\forall) = \binom{n-1}{\frac{n}{g}-1}$. Moreover, this bound is met if and only if the g-partitions are uniform (all parts with cardinality $\frac{n}{g}$).

Example: weakly Sperner property

n=2g	n=3g
{1,2,3},{4,5,6}	{1,2,3},{4,5,6},{7,8,9}
{1,2,4},{3,5,6}	{1,2,4},....
{1,2,5},{3,4,6}	{1,2,5},...
{1,2,6},{3,4,5}	{1,2,6},...
{1,3,4},{2,5,6}	.
{1,3,5},{2,4,6}	.
{1,3,6},{2,4,5}	.
{1,4,5},{2,3,6}	.
{1,4,6},{2,3,5}	{1,7,8},...
{1,5,6},{2,3,4}	{1,8,9},{2,3,4}, **{5,6,7}**

Comparison of two bounds obtained

Theorem (Stevens, Moura and Mendelsohn 1998)

$CAK(n, 2, g) \leq \frac{1}{2}\binom{\lfloor \frac{2n}{g} \rfloor}{\lfloor \frac{n}{g} \rfloor}$.

Theorem (Meagher, Moura and Stevens 2005)

If $g|n$, *then* $CAK(n, 2, g) \leq \binom{n-1}{\frac{n}{g}-1}$.

if $g > 2$, $g|n$, then

$$\frac{1}{2}\binom{\frac{2n}{g}}{\frac{n}{g}} < \binom{n-1}{\frac{n}{g}-1}$$

Erdos-Ko-Rado theorem for set-partition systems

- We are interested on: $(\exists, \exists)$ with property p-intersecting.
- This is useful for bounds on "anti-covering-arrays" for certain uniform cases. For example: $n = g^2$, $p = 2$

Conjecture

Suppose $g|n$, and let $c = n/g$ be the size of each part of the (uniform) partition system. $p - I_n(\exists, \exists) = \binom{n-p}{c-p} U(n-c, g-1)$.

This has been proven for $p = c$:

Theorem (Meagher and Moura 2005)

Let $n \geq g \geq 1$ and let $\mathcal{P} \subseteq U_g^n$ be a partition system in which every two partitions share at least one class. Let $c = n/g$. Then, $|\mathcal{P}| \leq U(n-c, g-1)$

Higher strength: strongly intersecting and Sperner

Strongly intersecting formulation:

Partitions $P^{i_1}, P^{i_2}, \ldots, P^{i_t}$ corresponding to t rows of a covering array must satisfy:

$$\text{for all } A_{k_1} \in P^{i_1}, A_{k_2} \in P^{i_2}, \ldots, A_{k_t} \in P^{i_t}, \quad A_{k_1} \cap A_{k_2} \cap \cdots \cap A_{k_t} \neq \emptyset.$$

Generalization of t-wise intersecting set systems.

Strongly Sperner formulation:

Partitions $P^{i_1}, P^{i_2}, \ldots P^{i_t}$ corresponding to t rows of a covering array must satisfy:

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...

Conclusions

- Binary case was solved via systems of sets.
  The case $g > 2$ can be studied via systems of g-partition:
  - Stronger upper bounds for set-partition systems= stronger lower bounds for covering arrays.
  - Some cases might be amenable to complete solution (for example, binary covering arrays with strength 3.)
- Special attention should be given to point-balanced case (Meagher conjectures there is always an optimal covering array that is (almost) point-balanced).
- Systems of set-partitions are interesting on their own right, and other extremal problems could be investigated.

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