Chapter 5 The Discrete-Time Fourier Transform

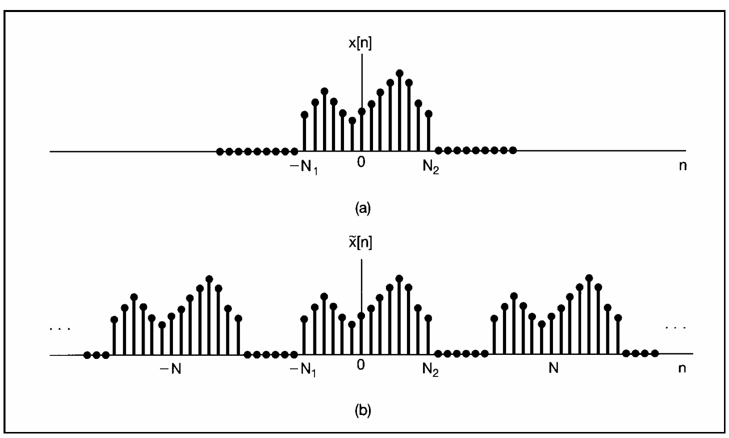
5.0 Introduction

- There are many similarities and strong parallels in analyzing continuous-time and discrete-time signals.
- There are also important differences. For example, the Fourier series representation of a discrete-time periodic signal is finite series, as opposed to the infinite series representation required for continuous-time period signal.
- In this chapter, the analysis will be carried out by taking advantage of the similarities between continuous-time and discrete-time Fourier analysis.

5.1 Representation of Aperiodic Signals: The discrete-Time Fourier Transform

5.1.1 Development of the Discrete-Time Fourier Transform

Consider a general sequence that is a finite duration. That is, for some integers N_1 and N_2, $x[n]$ equals to zero outside the range $N_1 \le n \le N_2$, as shown in the figure below.



We can construct a periodic sequence $\tilde{x}[n]$ using the aperiodic sequence $x[n]$ as one period. As we choose the period N to be larger, $\tilde{x}[n]$ is identical to $x[n]$ over a longer interval, as $N \to \infty$, $\tilde{x}[n] = x[n]$.

Based on the Fourier series representation of a periodic signal given in Eqs. (3.80) and (3.81), we have

$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk(2p/N)n} , \quad (5.1)$$

$$a_k = \sum_{k=\langle N \rangle} \tilde{x}[n] e^{-jk(2p/N)n} . \quad (5.2)$$

If the interval of summation is selected to include the interval $N_1 \le n \le N_2$, so $\tilde{x}[n]$ can be replaced by $x[n]$ in the summation,

$$a_k = \frac{1}{N} \sum_{k=N_1}^{N_2} x[n] e^{-jk(2p/N)n} = \frac{1}{N} \sum_{k=-\infty}^{\infty} x[n] e^{-jk(2p/N)n} , \quad (5.3)$$

Defining the function

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn} , \quad (5.4)$$

So a_k can be written as

$$a_k = \frac{1}{N} X(e^{jkw_0}), \quad (5.5)$$

Then $\tilde{x}[n]$ can be expressed as

$$\tilde{x}[n] = \sum_{k=<N>} \frac{1}{N} X(e^{jkw_0}) e^{jk(2p/N)n} = \frac{1}{2p} \sum_{k=<N>} X(e^{jkw_0}) e^{jk(2p/N)n} w_0 . \quad (5.6)$$

As $N \to \infty$ $\tilde{x}[n] = x[n]$, and the above expression passes to an integral,

$$x[n] = \frac{1}{2p} \int_{2p} X(e^{jw}) e^{jwn} dw , \quad (5.7)$$

The Discrete-time Fourier transform pair:

$$x[n] = \frac{1}{2p} \int_{2p} X(e^{jw}) e^{jwn} dw , \quad (5.8)$$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn} . \quad (5.9)$$

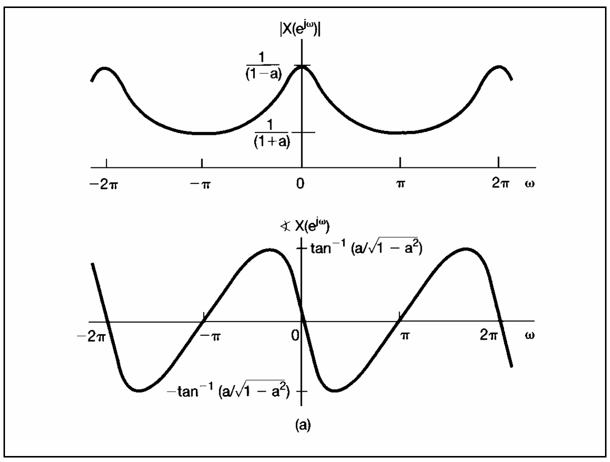
Eq. (5.8) is referred to as ***synthesis equation***, and Eq. (5.9) is referred to ***as analysis equation*** and $X(e^{jkw_0})$ is referred to as the ***spectrum*** of $x[n]$.

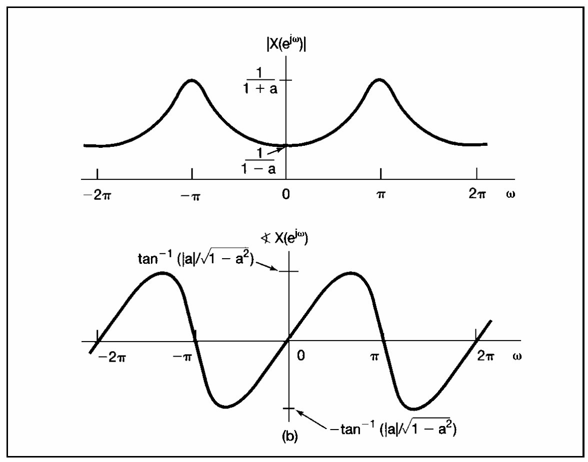
5.1.2 Examples of Discrete-Time Fourier Transforms

Example: Consider $x[n] = a^n u[n], \quad |a| < 1$. (5.10)

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn} = \sum_{n=-\infty}^{\infty} a^n u[n]e^{-jwn} = \sum_{n=0}^{\infty} \left(ae^{-jw}\right)^{-n} = \frac{1}{1-ae^{-jw}}. \qquad (5.11)$$

The magnitude and phase for this example are show in the figure below, where $a > 0$ and $a < 0$ are shown in (a) and (b).



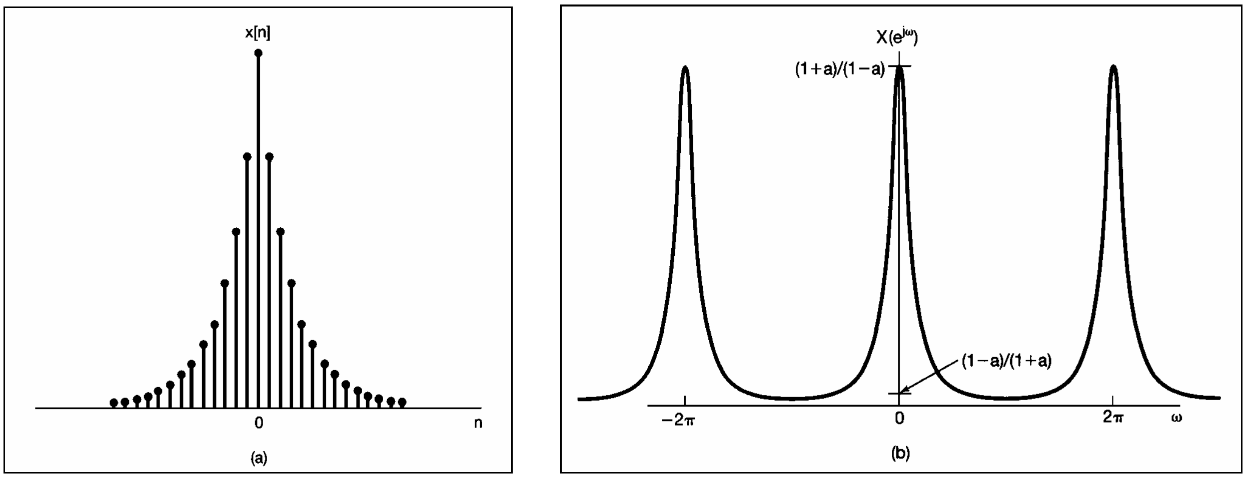


Example: $x[n] = a^{|n|}, \ |a| < 1$. (5.12)

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} a^{|n|} u[n]e^{-jwn} = \sum_{n=-\infty}^{-1} a^{-n} e^{-jwn} + \sum_{n=0}^{\infty} a^n e^{-jwn}$$

Let $m = -n$ in the first summation, we obtain

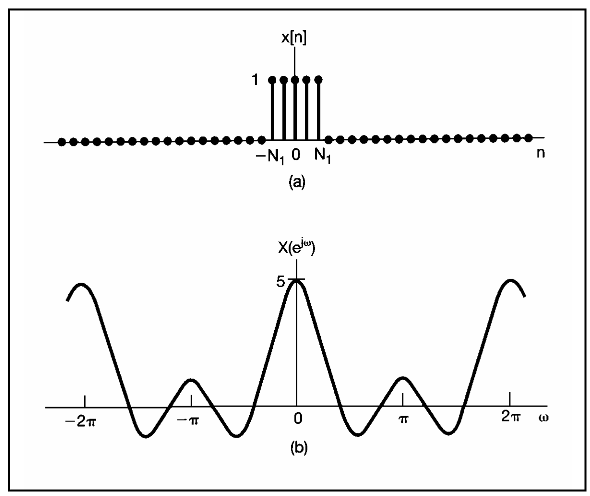
$$\begin{aligned} X(e^{jw}) &= \sum_{n=-\infty}^{\infty} a^{|n|} u[n]e^{-jwn} = \sum_{m=1}^{\infty} a^m e^{jwm} + \sum_{n=0}^{\infty} a^n e^{-jwn} \\ &= \frac{ae^{jw}}{1-ae^{jw}} + \frac{1}{1-ae^{-jw}} = \frac{1-a^2}{1-2a\cos w + a^2} \end{aligned} \qquad (5.13)$$



Example: Consider the rectangular pulse

$$x[n]=\begin{cases}1, & |n|\le N_1\\ 0, & |n|> N_1\end{cases}, \qquad (5.14)$$

$$X(jw)=\sum_{n=-N_1}^{N_1} e^{-jwn}=\frac{\sin w(N_1+1/2)}{\sin(w/2)}. \qquad (5.15)$$



This function is the discrete counterpart of the sic function, which appears in the Fourier transform of the continuous-time pulse.

The difference between these two functions is that the discrete one is periodic (see figure) with period of $2p$, whereas the sinc function is aperiodic.

5.1.3 Convergence

The equation $X(e^{jw})=\sum_{n=-\infty}^{\infty} x[n]e^{-jwn}$ converges either if $x[n]$ is absolutely summable, that is

$$\sum_{n=-\infty}^{\infty}|x[n]|<\infty, \qquad (5.16)$$

or if the sequence has finite energy, that is

$$\sum_{n=-\infty}^{\infty}|x[n]|^2<\infty. \qquad (5.17)$$

And there is no convergence issues associated with the synthesis equation (5.8).

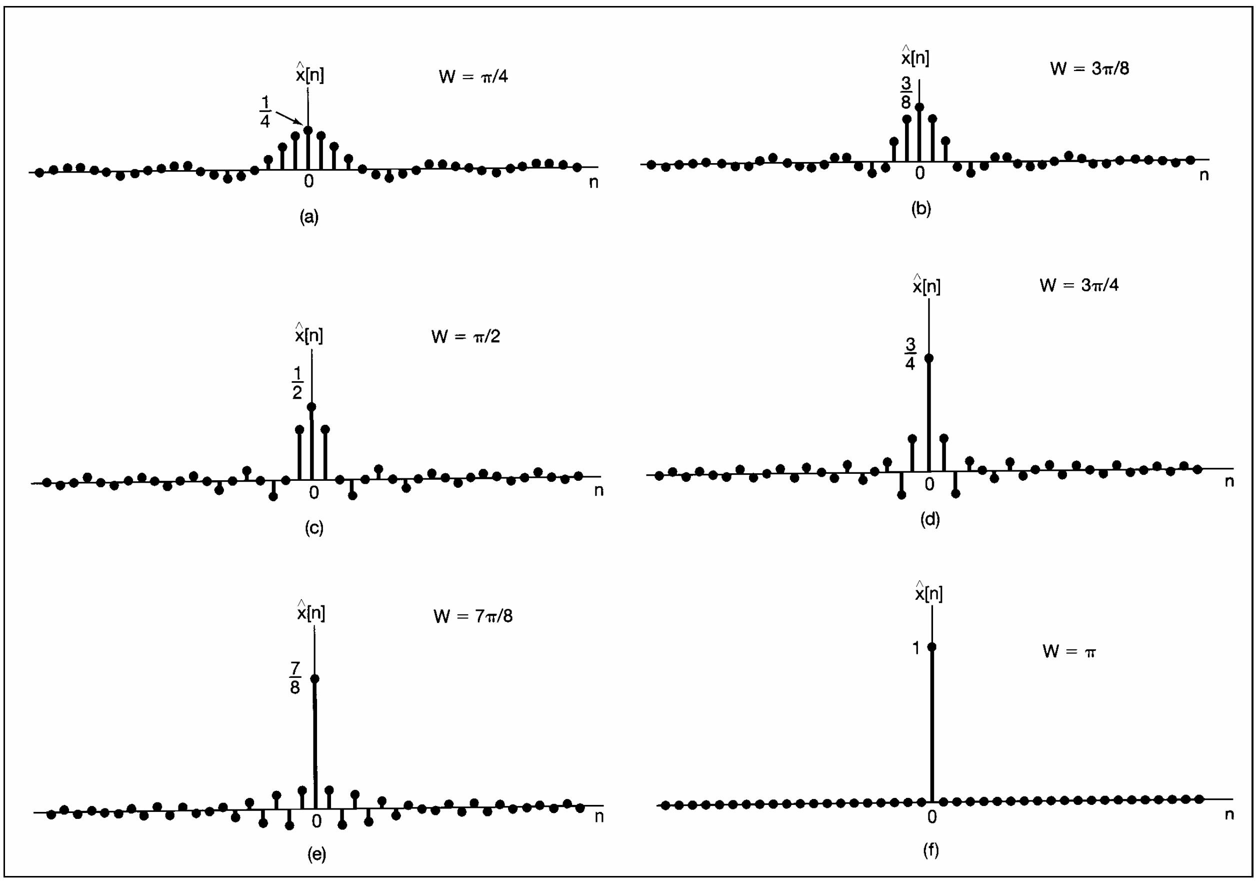
If we approximate an aperidic signal $x[n]$ by an integral of complex exponentials with frequencies taken over the interval $|w| \le W$,

$$\hat{x}[n] = \frac{1}{2p}\int_{-W}^{W} X(e^{jw})e^{jwn}dw \ , \qquad (5.18)$$

and $\hat{x}[n] = x[n]$ for $W = p$. Therefore, the Gibbs phenomenon does not exist in the discrete-time Fourier transform.

Example: the approximation of the impulse response with different values of W .

For $W = p/4,\ 3p/8,\ p/2,\ 3p/4,\ 7p/8, p$, the approximations are plotted in the figure below. We can see that when $W = p$, $\hat{x}[n] = x[n]$.



5.2 Fourier transform of Periodic Signals

For a periodic discrete-time signal,

$$x[n] = e^{jw_0 n}, \tag{5.19}$$

its Fourier transform of this signal is periodic in w with period $2p$, and is given

$$X(e^{jw}) = \sum_{l=-\infty}^{+\infty} 2pd(w - w_0 - 2pl). \tag{5.20}$$

Now consider a periodic sequence $x[n]$ with period N and with the Fourier series representation

$$x[n] = \sum_{k=<N>} a_k e^{jk(2p/N)n}. \tag{5.21}$$

The Fourier transform is

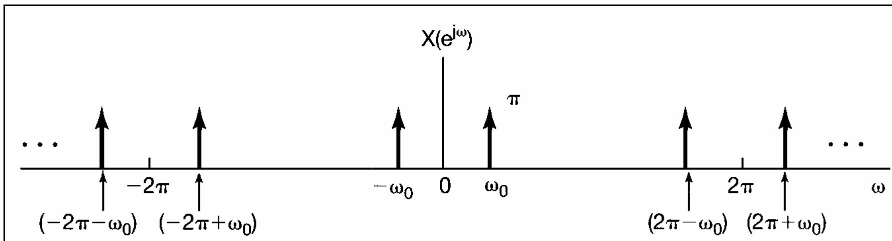
$$X(e^{jw}) = \sum_{k=-\infty}^{+\infty} 2pa_k d(w - \frac{2pk}{N}). \tag{5.22}$$

Example: The Fourier transform of the periodic signal

$$x[n] = \cos \omega_0 n = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}, \text{ with } w_0 = \frac{2p}{3}, \tag{5.23}$$

is given as

$$X(e^{jw}) = pd\left(w - \frac{2p}{3}\right) + pd\left(w + \frac{2p}{3}\right), \qquad -p \le w < p. \tag{5.24}$$



Discrete-time Fourier transform of $x[n] = \cos \omega_0 n$.

Example: The periodic impulse train

$$x[n] = \sum_{k=-\infty}^{+\infty} d[n-kN]. \tag{5.25}$$

The Fourier series coefficients for this signal can be calculated

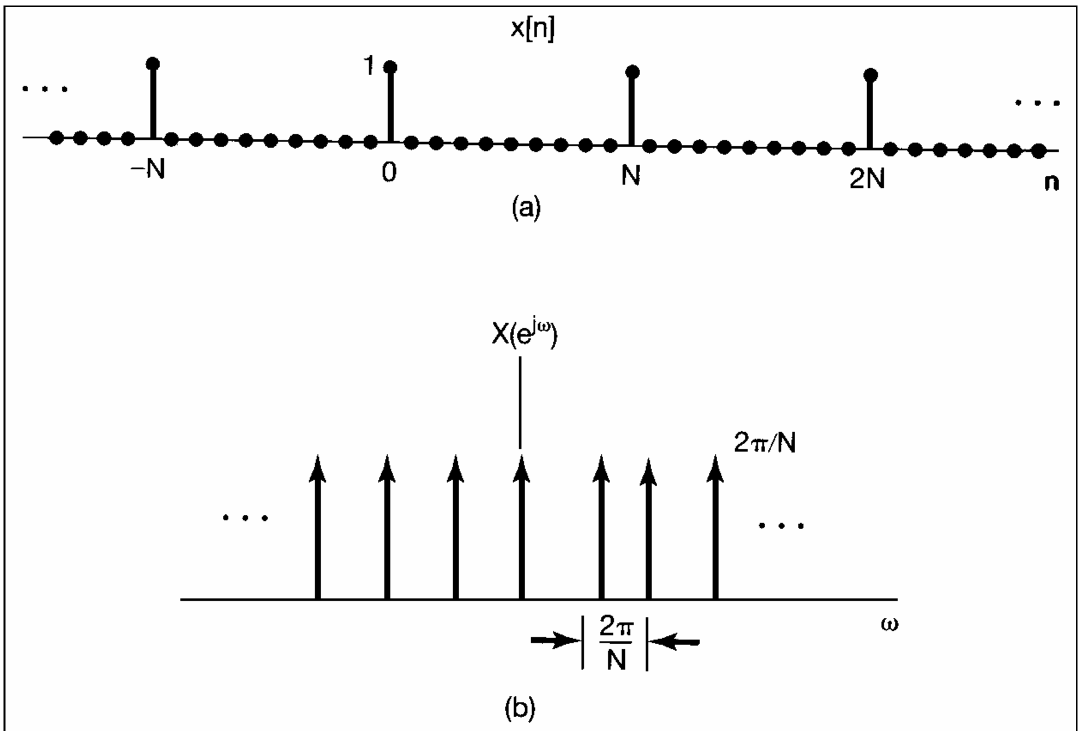
$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2p/N)n}. \tag{5.26}$$

Choosing the interval of summation as $0 \le n \le N-1$, we have

$$a_k = \frac{1}{N}. \tag{5.27}$$

The Fourier transform is

$$X(e^{jw}) = \frac{2p}{N}\sum_{k=-\infty}^{\infty} d\left(w - \frac{2pk}{N}\right). \tag{5.28}$$



(a) Discrete-time periodic impulse train; (b) its Fourier transform.

5.3 Properties of the Discrete-Time Fourier Transform

Notations to be used

$X(e^{jw}) = F\{x[n]\}$,

$x[n] = F^{-1}\{X(e^{jw})\}$,

$x[n] \stackrel{F}{\longleftrightarrow} X(e^{jw})$.

5.3.1 Periodicity of the Discrete-Time Fourier Transform

The discrete-time Fourier transform is always periodic in w with period $2p$, i.e.,

$$X\left(e^{j(w+2p)}\right) = X\left(e^{jw}\right). \qquad (5.29)$$

5.3.2 Linearity

If $x_1[n] \stackrel{F}{\longleftrightarrow} X_1(e^{jw})$, and $x_2[n] \stackrel{F}{\longleftrightarrow} X_2(e^{jw})$,

then

$$\boxed{ax_1[n] + bx_2[n] \stackrel{F}{\longleftrightarrow} aX_1(e^{jw}) + bX_2(e^{jw})} \qquad (5.30)$$

5.3.3 Time Shifting and Frequency Shifting

If $x[n] \stackrel{F}{\longleftrightarrow} X(e^{jw})$,

then

$$\boxed{x[n-n_0] \stackrel{F}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})} \qquad (5.31)$$

and

$$\boxed{e^{j\omega_0 n} x[n] \stackrel{F}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})} \qquad (5.32)$$

5.3.4 Conjugation and Conjugate Symmetry

If $x[n] \overset{F}{\longleftrightarrow} X(e^{jw})$,

then

$$\boxed{x^*[n] \overset{F}{\longleftrightarrow} X^*(e^{-jw})} \tag{5.33}$$

If $x[n]$ is real valued, its transform $X(e^{jw})$ is conjugate symmetric. That is

$$\boxed{X(e^{jw}) = X^*(e^{-jw})} \tag{5.34}$$

From this, it follows that $\mathrm{Re}\{X(e^{jw})\}$ is an even function of w and $\mathrm{Im}\{X(e^{jw})\}$ is an odd function of w. Similarly, the **magnitude** of $X(e^{jw})$ is an even function and the phase angle is an odd function. Furthermore,

$$Ev\{x[n]\} \overset{F}{\longleftrightarrow} \mathrm{Re}\{X(e^{jw}\}, \tag{5.35}$$

and

$$Od\{x[n]\} \overset{F}{\longleftrightarrow} j\,\mathrm{Im}\{X(e^{jw}\}. \tag{5.36}$$

5.3.5 Differencing and Accumulation

If $x[n] \overset{F}{\longleftrightarrow} X(e^{jw})$,

then

$$\boxed{x[n] - x[n-1] \overset{F}{\longleftrightarrow} \left(1 - e^{-jw}\right) X(e^{jw})}. \tag{5.37}$$

For signal

$$y[n] = \sum_{m=-\infty}^{n} x[m], \tag{5.38}$$

its Fourier transform is given as

$$\boxed{\sum_{m=-\infty}^{n} x[m] \overset{F}{\longleftrightarrow} \frac{1}{1-e^{-jw}} X(e^{jw}) + pX(e^{j0}) \sum_{m=-\infty}^{+\infty} d(w-2pk)}. \qquad (5.39)$$

The impulse train on the right-hand side reflects the dc or average value that can result from summation.

For example, the Fourier transform of the unit step $x[n]=u[n]$ can be obtained by using the accumulation property.

We know $g[n]=d[n] \overset{F}{\longleftrightarrow} G(e^{jw})=1$, so

$$x[n]=\sum_{m=-\infty}^{n} g[m] \overset{F}{\longleftrightarrow} \frac{1}{\left(1-e^{-jw}\right)} G(e^{jw}) + pG(e^{j0}) \sum_{k=-\infty}^{+\infty} d(w-2pk) = \frac{1}{\left(1-e^{-jw}\right)} + p \sum_{k=-\infty}^{+\infty} d(w-2pk). \qquad (5.40)$$

5.3.6 Time Reversal

If $x[n] \overset{F}{\longleftrightarrow} X(e^{jw})$,

then

$$\boxed{x[-n] \overset{F}{\longleftrightarrow} X(-e^{jw})}. \qquad (5.41)$$

5.3.7 Time Expansion

For continuous-time signal, we have

$$x(at) \overset{F}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{jw}{a}\right). \qquad (5.42)$$

For discrete-time signals, however, a should be an integer. Let us define a signal with k a positive integer,

$$x_{(k)}[n]=\begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}. \qquad (5.43)$$

$x_{(k)}[n]$ is obtained from $x[n]$ by placing $k-1$ zeros between successive values of the original signal.

The Fourier transform of $x_{(k)}[n]$ is given by

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]e^{-j\omega n} = \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]e^{-j\omega rk} = \sum_{r=-\infty}^{+\infty} x[r]e^{-j(k\omega)r} = X(e^{jk\omega}). \qquad (5.44)$$

That is,

$$\boxed{x_{(k)}[n] \xleftrightarrow{F} X(e^{jkw})}. \qquad (5.45)$$

For $k > 1$, the signal is spread out and slowed down in time, while its Fourier transform is compressed.

Example: Consider the sequence $x[n]$ displayed in the figure (a) below. This sequence can be related to the simpler sequence $y[n]$ as shown in (b).

$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1],$$

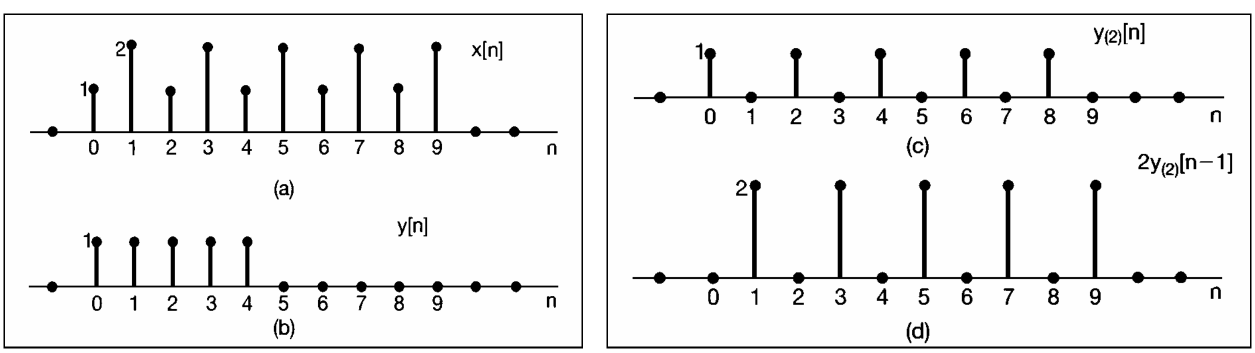
where

$$y_2[n] = \begin{cases} y[n/2], & if\ \ n\ is\ even \\ 0, & if\ \ n\ is\ odd \end{cases}$$

The signals $y_{(2)}[n]$ and $2y_{(2)}[n-1]$ are depicted in (c) and (d).

As can be seen from the figure below, $y[n]$ is a rectangular pulse with $N_1 = 2$, its Fourier transform is given by

$$Y(e^{jw}) = e^{-j2w}\frac{\sin(5w/2)}{\sin(w/2)}.$$



Using the time-expansion property, we then obtain

$$y_{(2)}[n] \overset{F}{\longleftrightarrow} e^{-j4w} \frac{\sin(5w)}{\sin(w)}$$

$$2y_{(2)}[n-1] \overset{F}{\longleftrightarrow} 2e^{-j5w} \frac{\sin(5w)}{\sin(w)}$$

Combining the two, we have

$$X(e^{jw}) = e^{-j4w}(1+2e^{-jw})\left(\frac{\sin(5w)}{\sin(w)}\right).$$

5.3.8 Differentiation in Frequency

If $x[n] \overset{F}{\longleftrightarrow} X(e^{jw})$,

Differentiate both sides of the analysis equation $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

$$\frac{dX(e^{jw})}{dw} = \sum_{n=-\infty}^{+\infty} -jnx[n]e^{-jwn} \quad (5.46)$$

The right-hand side of the Eq. (5.46) is the Fourier transform of $-jnx[n]$. Therefore, multiplying both sides by j, we see that

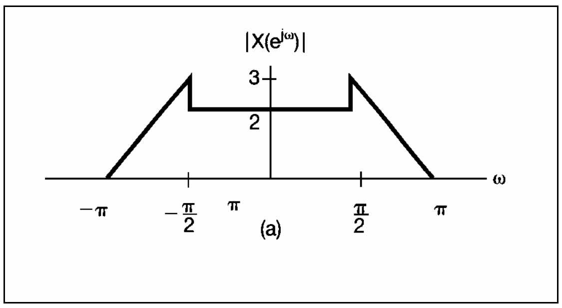
$$\boxed{nx[n] \overset{F}{\longleftrightarrow} j\frac{dX(e^{jw})}{dw}} \quad (5.47)$$

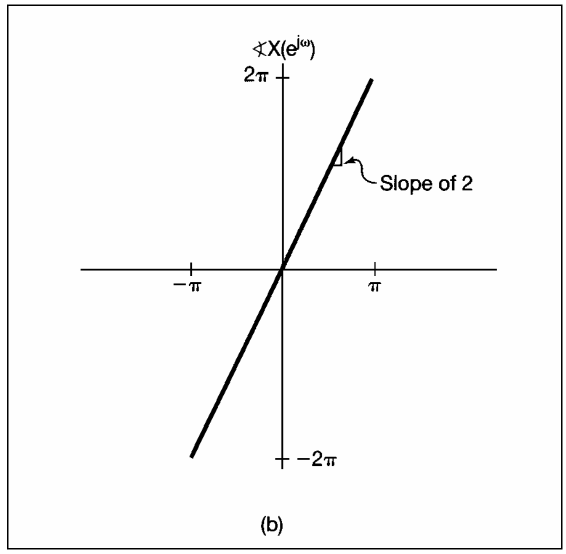
5.3.9 Parseval's Relation

If $x[n] \overset{F}{\longleftrightarrow} X(e^{jw})$, then we have

$$\boxed{\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2p}\int_{2p} \left|X(e^{jw})\right|^2 dw} \quad (5.48)$$

Example: Consider the sequence $x[n]$ whose Fourier transform $X(e^{j\omega})$ is depicted for $-\pi \le \omega \le \pi$ in the figure below. Determine whether or not, in the time domain, $x[n]$ is periodic, real, even, and /or of finite energy.





- The periodicity in time domain implies that the Fourier transform has only impulses located at various integer multiples of the fundamental frequency. This is not true for $X(e^{j\omega})$. We conclude that $x[n]$ is not periodic.
- Since real-valued sequence should have a Fourier transform of even magnitude and a phase function that is odd. This is true for $\left|X(e^{j\omega})\right|$ and $\angle X(e^{j\omega})$. We conclude that $x[n]$ is real.
- If $x[n]$ is real and even, then its Fourier transform should be real and even. However, since $X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{-j2\omega}$, $X(e^{j\omega})$ is not real, so we conclude that $x[n]$ is not even.
- Based on the Parseval's relation, integrating $\left|X(e^{j\omega})\right|^2$ from $-\pi$ to π will yield a finite quantity. We conclude that $x[n]$ has finite energy.

5.4 The convolution Property

If $x[n]$, $h[n]$ and $y[n]$ are the input, impulse response, and output, respectively, of an LTI system, so that

$$y[n] = x[n] * h[n], \tag{5.49}$$

then,

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}), \tag{5.50}$$

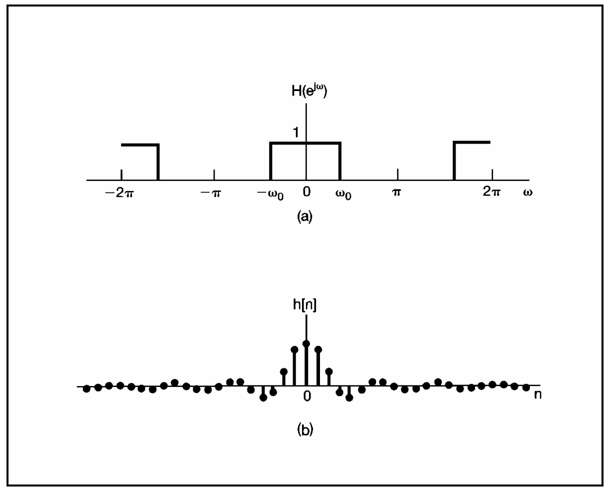
where $X(e^{jw})$, $H(e^{jw})$ and $Y(e^{jw})$ are the Fourier transforms of $x[n]$, $h[n]$ and $y[n]$, respectively.

Example: Consider the discrete-time ideal lowpass filter with a frequency response $H(e^{jw})$ illustrated in the figure below. Using $-p \le w \le p$ as the interval of integration in the synthesis equation, we have

$$h[n] = \frac{1}{2p}\int_{-p}^{p} H(e^{jw})e^{jwn}dw$$

$$= \frac{1}{2p}\int_{-p}^{p} e^{jwn}dw = \frac{\sin w_c n}{pn}$$

The frequency response of the discrete-time ideal lowpass filter is shown in the right figure.



Example: Consider an LTI system with impulse response

$$h[n] = a^n u[n], \qquad |a| < 1,$$

and suppose that the input to the system is

$$x[n] = b^n u[n], \qquad |b| < 1.$$

The Fourier transforms for $h[n]$ and $x[n]$ are

$$H(e^{jw}) = \frac{1}{1 - a e^{-jw}},$$

and

$$X(e^{jw}) = \frac{1}{1 - b e^{-jw}},$$

so that

$$Y(e^{jw}) = H(e^{jw})X(e^{jw}) = \frac{1}{(1 - a e^{-jw})(1 - b e^{-jw})}.$$

If $a \neq b$, the partial fraction expansion of $Y(e^{jw})$ is given by

$$Y(e^{jw}) = \frac{A}{(1-a\,e^{-jw})} + \frac{B}{(1-b\,e^{-jw})} = \frac{\frac{a}{a-b}}{(1-a\,e^{-jw})} + \frac{-\frac{b}{a-b}}{(1-b\,e^{-jw})},$$

We can obtain the inverse transform by inspection:

$$y[n] = \frac{a}{a-b}a^n u[n] - \frac{b}{a-b}b^n u[n] = \frac{1}{a-b}\left(a^{n+1}u[n] - bb^{n+1}u[n]\right).$$

For $a = b$,

$$Y(e^{jw}) = \frac{1}{(1-a\,e^{-jw})^2},$$ which can be expressed as

$$Y(e^{jw}) = \frac{j}{a}e^{jw}\frac{d}{dw}\left(\frac{1}{1-a\,e^{-jw}}\right).$$

Using the frequency differentiation property, we have

$$na^n u[n] \overset{F}{\longleftrightarrow} j\frac{d}{dw}\left(\frac{1}{1-a\,e^{-jw}}\right),$$

To account for the factor e^{jw}, we use the time-shifting property to obtain

$$(n+1)a^{n+1}u[n+1] \overset{F}{\longleftrightarrow} je^{jw}\frac{d}{dw}\left(\frac{1}{1-a\,e^{-jw}}\right),$$

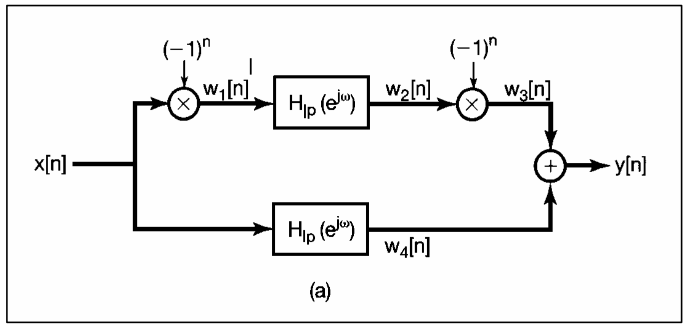
Finally, accounting for the factor $1/a$, we have

$$y[n] = (n+1)a^n u[n+1].$$

Since the factor $n+1$ is zero at $n=-1$, so $y[n]$ can be expressed as

$$y[n] = (n+1)a^n u[n].$$

Example: Consider the system shown in the figure below. The LTI systems with frequency response $H_{lp}(e^{jw})$ are ideal lowpass filters with cutoff frequency $p/4$ and unity gain in the passband.



(a)

- $w_1[n] = (-1)^n x[n] = e^{j\pi n} x[n]$

$\Rightarrow \qquad W_1(e^{j\omega}) = X(e^{j(\omega-\pi)})$.

- $W_2(e^{j\omega}) = H_{lp}(e^{j\omega}) X(e^{j(\omega-\pi)})$.

- $w_3[n] = (-1)^n w_2[n] = e^{j\pi n} w_2[n]$

$\Rightarrow \qquad W_3(e^{j\omega}) = W_2(e^{j(\omega-\pi)}) = H_{lp}(e^{(j\omega-\pi)}) X(e^{j(\omega-2\pi)})$.

$\Rightarrow \qquad W_3(e^{j\omega}) = W_2(e^{j(\omega-\pi)}) = H_{lp}(e^{j\omega-\pi}) X(e^{j\omega})$ (Discrete-Fourier transforms are always periodic with period of 2π).

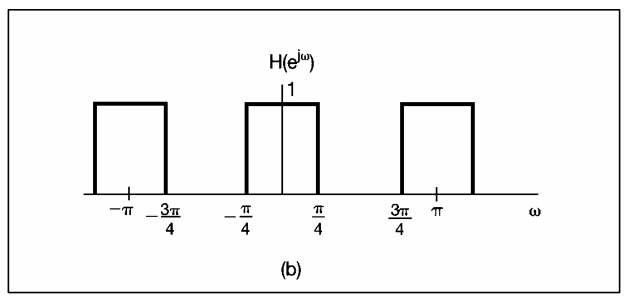
- $W_4(e^{j\omega}) = H_{lp}(e^{j\omega)}) X(e^{j\omega})$.

- $Y(e^{j\omega}) = W_3(e^{j\omega}) + W_4(e^{j\omega}) = \left[H_{lp}(e^{(j\omega-\pi)}) + H_{lp}(e^{j\omega})\right] X(e^{j\omega})$.

The overall system has a frequency response

$$H_{lp}(e^{j\omega}) = \left[H_{lp}(e^{(j\omega-\pi)}) + H_{lp}(e^{j\omega})\right] X(e^{j\omega}),$$

which is shown in figure (b).



(b)

The filter is referred to as bandstop filter, where the stop band is the region $\pi/4 < |\omega| < 3\pi/4$.

It is important to note that not every discrete-time LTI system has a frequency response. If an LTI system is stable, then its impulse response is absolutely summable; that is,

$$\sum_{n=-\infty}^{+\infty} |h[n]| < \infty, \tag{5.51}$$

5.5 The multiplication Property

Consider $y[n]$ equal to the product of $x_1[n]$ and $x_2[n]$, with $Y(e^{j\omega})$, $X_1(e^{j\omega})$, and $X_2(e^{j\omega})$ denoting the corresponding Fourier transforms. Then

$$\boxed{y[n] = x_1[n]x_2[n] \overset{F}{\longleftrightarrow} \frac{1}{2p}\int_{2p} X_1(e^{jw})X_2(e^{j(w-q)})dq} \qquad (5.52)$$

Eq. (5.52) corresponds to a periodic convolution of $X_1(e^{jw})$ and $X_2(e^{jw})$, and the integral in this equation can be evaluated over any interval of length $2p$.

Example: Consider the Fourier transform of a signal $x[n]$ which the product of two signals; that is

$$x[n] = x_1[n]x_2[n]$$

where

$$x_1[n] = \frac{\sin(3pn/4)}{pn}, \text{ and}$$

$$x_2[n] = \frac{\sin(pn/2)}{pn}.$$

Based on Eq. (5.52), we may write the Fourier transform of $x[n]$

$$X(e^{jw}) = \frac{1}{2p}\int_{-p}^{p} X_1(e^{jw})X_2(e^{j(w-q)})dq. \qquad (5.53)$$

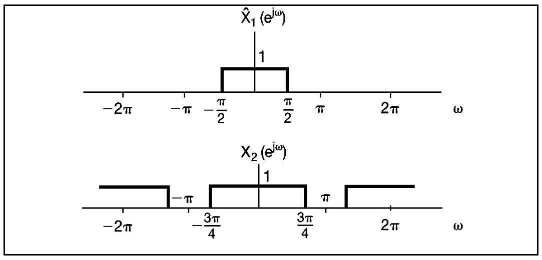
Eq. (5.53) resembles aperiodic convolution, except for the fact that the integration is limited to the interval of $-p < q < p$. The equation can be converted to ordinary convolution with integration interval $-\infty < q < \infty$ by defining

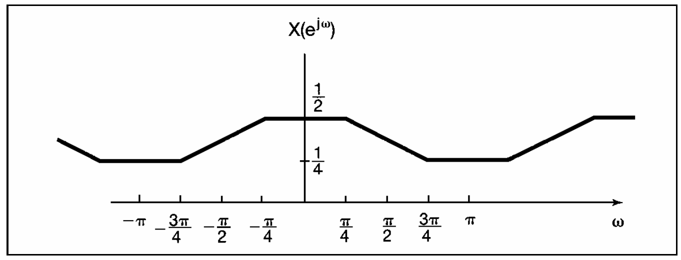
$$\hat{X}_1(e^{jw}) = \begin{cases} X_1(e^{jw}) & \text{for } -p < w < p \\ 0 & \text{otherwise} \end{cases}$$

Then replacing $X_1(e^{jw})$ in Eq. (5.53) by $\hat{X}_1(e^{jw})$, and using the fact that $\hat{X}_1(e^{jw})$ is zero for $-p < w < p$, we see that

$$X(e^{jw}) = \frac{1}{2p}\int_{-p}^{p} X_1(e^{jw})X_2(e^{j(w-q)})dq = \frac{1}{2p}\int_{-\infty}^{\infty} \hat{X}_1(e^{jw})X_2(e^{j(w-q)})dq.$$

Thus, $X(e^{jw})$ is $1/2p$ times the aperiodic convolution of the rectangular pulse $\hat{X}_1(e^{jw})$ and the periodic square wave $X_2(e^{jw})$. The result of thus convolution is the Fourier transform $X(e^{jw})$, as shown in the figure below.





5.6 Tables of Fourier Transform Properties and Basic Fourier Transform Paris

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{matrix}X(e^{j\omega})\\Y(e^{j\omega})\end{matrix}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n-n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4 T	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n]=\begin{cases}x[n/k], & \text{if } n=\text{multiple of } k\\0, & \text{if } n\neq\text{multiple of } k\end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi}X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n]-x[n-1]$	$(1-e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n}x[k]$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega-2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases}X(e^{j\omega})=X^*(e^{-j\omega})\\\Re e\{X(e^{j\omega})\}=\Re e\{X(e^{-j\omega})\}\\\Im m\{X(e^{j\omega})\}=-\Im m\{X(e^{-j\omega})\}\\\lvert X(e^{j\omega})\rvert=\lvert X(e^{-j\omega})\rvert\\\sphericalangle X(e^{j\omega})=-\sphericalangle X(e^{-j\omega})\end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n]=\mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n]=\mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\Re e\{X(e^{j\omega})\}$ $j\Im m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals $\sum_{n=-\infty}^{+\infty}\lvert x[n]\rvert^2=\frac{1}{2\pi}\int_{2\pi}\lvert X(e^{j\omega})\rvert^2d\omega$		

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \ k \neq 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \ k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

5.7 Duality

For continuous-time Fourier transform, we observed a symmetry or duality between the analysis and synthesis equations. For discrete-time Fourier transform, such duality does not exist. However, ***there is a duality in the discrete-time series equations***. In addition, ***there is a duality relationship between the discrete-time Fourier transform and the continuous-time Fourier series***.

5.7.1 Duality in the discrete-time Fourier Series

Consider the periodic sequences with period *N*, related through the summation

$$f[m] = \frac{1}{N}\sum_{r=<N>} g(r) e^{-jr(2p/N)m} . \tag{5.54}$$

If we let $m = n$ and $r = -k$, Eq. (5.54) becomes

$$f[n] = \sum_{k=<N>} \frac{1}{N} g(-r) e^{jr(2p/N)n} . \tag{5.55}$$

Compare with the two equations below,

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2p/N)n} , \tag{3.80}$$

$$a_k = \frac{1}{N}\sum_{k=\langle N \rangle} x[n] e^{-jk(2p/N)n} . \tag{3.81}$$

we fond that $\frac{1}{N} g(-r)$ corresponds to the sequence of Fourier series coefficients of $f[n]$. That is

$$f[n] \stackrel{FS}{\longleftrightarrow} \frac{1}{N} g[-k] . \tag{5.56}$$

This duality implies that every property of the discrete-time Fourier series has a dual. For example,

$$x[n - n_0] \stackrel{FS}{\longleftrightarrow} a_k e^{-jk(2p/N)n_0} \tag{5.57}$$

$$e^{jm(2p/N)n} \stackrel{FS}{\longleftrightarrow} a_{k-m} \tag{5.58}$$

are dual.

Example: Consider the following periodic signal with a period of $N = 9$.

$$x[n] = \begin{cases} \frac{1}{9}\frac{\sin(5\pi n/9)}{\sin(\pi n/9)}, & n \neq multiple\ of\ 9 \\ \frac{5}{9}, & n = multiple\ of\ 9 \end{cases} \quad (5.59)$$

We know that a rectangular square wave has Fourier coefficients in a form much as in Eq. (5.59). Duality suggests that the coefficients of $x[n]$ must be in the form of a rectangular square wave.

Let $g[n]$ be a rectangular square wave with period $N = 9$,

$$g[n] = \begin{cases} 1, & |n| \leq 2 \\ 0, & 2 < |n| \leq 4 \end{cases}, \quad (5.60)$$

The Fourier series coefficients b_k for $g[n]$ can be given (refer to example on page 27/3)

$$b_k = \begin{cases} \frac{1}{9}\frac{\sin(5\pi k/9)}{\sin(\pi k/9)}, & k \neq multiple\ of\ 9 \\ \frac{5}{9}, & k = multiple\ of\ 9 \end{cases}. \quad (5.61)$$

The Fourier analysis equation for $g[n]$ can be written

$$b_k = \frac{1}{9}\sum_{n=-2}^{2}(1)e^{-j2\pi nk/9}. \quad (5.62)$$

Interchanging the names of the variable *k* and *n* and noting that $x[n] = b_k$, we find that

$$x[n] = \frac{1}{9}\sum_{k=-2}^{2}(1)e^{-j2\pi nk/9}.$$

Let $k' = -k$ in the sum on the right side, we obtain

$$x[n] = \frac{1}{9}\sum_{k=-2}^{2}(1)e^{+j2\pi nk'/9}.$$

Finally, moving the factor $1/9$ inside the summation, we see that the right side of the equation has the form of the synthesis equation for $x[n]$. Thus, we conclude that the Fourier coefficients for $x[n]$ are given by

$$a_k = \begin{cases} 1/9, & |k| \le 2 \\ 0, & 2 < |k| \le 4 \end{cases},$$

with period of $N = 9$.

5.8 System Characterization by Linear Constant-Coefficient Difference Equations

A general linear constant-coefficient difference equation for an LTI system with input $x[n]$ and output $x[n]$ is of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k], \tag{5.63}$$

which is usually referred to as Nth-order difference equation.

There are two ways to determine $H(e^{jw})$:

- The first way is to apply an input $x[n] = e^{jwn}$ to the system, and the output must be of the form $H(e^{jw})e^{jwn}$. Substituting these expressions into the Eq. (5.63), and performing some algebra allows us to solve for $H(e^{jw})$.
- The second approach is to use discrete-time Fourier transform properties to solve for $H(e^{jw})$.

Based on the convolution property, Eq. (5.63) can be written as

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})}. \tag{5.64}$$

Applying the Fourier transform to both sides and using the linearity and time-shifting properties, we obtain the expression

$$\sum_{k=0}^{N} a_k e^{-jkw} Y(e^{jw}) = \sum_{k=0}^{M} b_k e^{-jkw} X(e^{jw}). \tag{5.65}$$

or equivalently

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{\sum_{k=0}^{M} b_k e^{-jkw}}{\sum_{k=0}^{N} a_k e^{-jkw}} . \qquad (5.66)$$

Example: Consider the causal LTI system that is characterized by the difference equation,

$$y[n] - ay[n-1] = x[n] , \qquad |a| < 1 .$$

The frequency response of this system is

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{1}{1 - ae^{jw}} .$$

The impulse response is given by

$$h[n] = a^n u[n] .$$

Example: Consider a causal LTI system that is characterized by the difference equation

$$y[n] - \frac{3}{4} y[n-1] + \frac{1}{8} y[n-2] = 2x[n] .$$

1. What is the impulse response?
2. If the input to this system is $x[n] = \left(\frac{1}{4}\right)^n u[n]$, what is the system response to this input signal?

The frequency response is

$$H(e^{jw}) = \frac{1}{1 - \frac{1}{2} e^{-jw} + \frac{1}{8} e^{-j2w}} = \frac{2}{(1 - \frac{3}{4} e^{-jw})(1 - \frac{1}{4} e^{-jw})} .$$

After partial fraction expansion, we have

$$H(e^{jw}) == \frac{4}{1 - \frac{1}{2} e^{-jw}} - \frac{2}{1 - \frac{1}{4} e^{-jw}} ,$$

The inverse Fourier transform of each term can be recognized by inspection,

$$h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n] .$$

Using Eq. (5.64) we have

$$Y(e^{jw}) = H(e^{jw})X(e^{jw}) = \left[\frac{2}{(1-\frac{3}{4}e^{-jw})(1-\frac{1}{4}e^{-jw})}\right]\left[\frac{1}{1-\frac{1}{4}e^{-jw}}\right].$$

$$= \frac{2}{(1-\frac{3}{4}e^{-jw})(1-\frac{1}{4}e^{-jw})(1-\frac{1}{4}e^{-jw})}$$

After partial-fraction expansion, we obtain

$$Y(e^{jw}) = H(e^{jw})X(e^{jw}) = -\frac{4}{1-\frac{1}{4}e^{-jw}} - \frac{2}{\left(1-\frac{1}{4}e^{-jw}\right)^2} + \frac{8}{1-\frac{1}{2}e^{-jw}}$$

The inverse Fourier transform is

$$y[n] = \left\{-4\left(\frac{1}{2}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^2\right\}u[n].$$