Chapter 3 Fourier Series Representation of Period Signals

3.0 Introduction

- Signals can be represented using complex exponentials – ***continuous-time and discrete-time Fourier series and transform***.
- If the input to an LTI system is expressed as a linear combination of periodic complex exponentials or sinusoids, the output can also be expressed in this form.

3.1 A Historical Perspective

By 1807, Fourier had completed a work that series of harmonically related sinusoids were useful in representing temperature distribution of a body. He claimed that any periodic signal could be represented by such series – **Fourier Series**. He also obtained a representation for aperidic signals as weighted integrals of sinusoids **– Fourier Transform**.

Jean Baptiste Joseph Fourier

3.2 The Response of LTI Systems to Complex Exponentials

It is advantageous in the study of LTI systems to represent signals as linear combinations of basic signals that possess the following two properties:

- The set of basic signals can be used to construct a broad and useful class of signals.

- The response of an LTI system to each signal should be simple enough in structure to provide us with a convenient representation for the response of the system to any signal constructed as a linear combination of the basic signal.

Both of these properties are provided by Fourier analysis.

The importance of complex exponentials in the study of LTI system is that the response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude; that is

Continuous time: $e^{st} \rightarrow H(s)e^{st}$, (3.1)

Discrete-time: $z^n \rightarrow H(z)z^n$, (3.2)

where the complex amplitude factor $H(s)$ or $H(z)$ will be in general be a function of the complex variable s or z.

A signal for which the system output is a (possible complex) constant times the input is referred to as an ***eigenfunction*** of the system, and the amplitude factor is referred to as the system's ***eigenvalue***. Complex exponentials are eigenfunctions.

For an input $x(t)$ applied to an LTI system with impulse response of $h(t)$, the output is

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau \\ &= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau \end{aligned}, \qquad (3.3)$$

where we assume that the integral $\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$ converges and is expressed as

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau, \qquad (3.4)$$

the response to e^{st} is of the form

$$y(t) = H(s)e^{st}, \qquad (3.5)$$

It is shown the ***complex exponentials are eigenfunctions*** of LTI systems and $H(s)$ for a specific value of s is then the eigenvalues associated with the eigenfunctions.

Complex exponential sequences are eigenfunctions of discrete-time LTI systems. That is, suppose that an LTI system with impulse response $h[n]$ has as its input sequence

$$x[n] = z^n, \tag{3.6}$$

where z is a complex number. Then the output of the system can be determined from the convolution sum as

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} . \tag{3.7}$$

Assuming that the summation on the right-hand side of Eq. (3.7) converges, the output is the same complex exponential multiplied by a constant that depends on the value of z. That is,

$$y[n] = H(z)z^n, \tag{3.8}$$

where $H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$. (3.9)

It is shown the ***complex exponentials are eigenfunctions*** of LTI systems and $H(z)$ for a specific value of z is then the eigenvalues associated with the eigenfunctions z^n.

The example here shows the usefulness of decomposing general signals in terms of eigenfunctions for LTI system analysis:

Let $x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$, (3.10)

from the eigenfunction property, the response to each separately is

$$a_1 e^{s_1 t} \rightarrow a_1 H_1(s_1) e^{s_1 t}$$

$$a_2 e^{s_2 t} \rightarrow a_2 H_2(s_2) e^{s_2 t}$$

$$a_3 e^{s_3 t} \rightarrow a_3 H_3(s_3) e^{s_3 t}$$

and from the superposition property the response to the sum is the sum of the responses,

$$y(t) = a_1 H_1(s_1) e^{s_1 t} + a_2 H_2(s_2) e^{s_2 t} + a_3 H_3(s_3) e^{s_3 t}, \tag{3.11}$$

Generally, if the input is a linear combination of complex exponentials,

$$x(t) = \sum_k a_k e^{s_k t}, \tag{3.12}$$

the output will be

$$y(t) = \sum_k a_k H(s_k) e^{s_k t} \ , \tag{3.13}$$

Similarly for discrete-time LTI systems, if the input is

$$x[n] = \sum_k a_k z_k^n \ , \tag{3.14}$$

the output is

$$y[n] = \sum_k a_k H(z_k) z_k^n \ , \tag{3.15}$$

3.3 Fourier Series representation of Continuous-Time Periodic Signals

3.31 Linear Combinations of harmonically Related Complex Exponentials

A periodic signal with period of T,

$$x(t) = x(t+T) \text{ for all } t \ , \tag{3.16}$$

We introduced two basic periodic signals in Chapter 1, the sinusoidal signal

$$x(t) = \cos w_0 t \ , \tag{3.17}$$

and the periodic complex exponential

$$x(t) = e^{jw_0 t} \ , \tag{3.18}$$

Both these signals are periodic with fundamental frequency w_0 and fundamental period $T = 2p / w_0$. Associated with the signal in Eq. (3.18) is the set of *harmonically related* complex exponentials

$$f_k(t) = e^{jkw_0 t} = e^{jk(2p/T)t} \ , \qquad k = 0, \pm 1, \pm 2, \ldots\ldots \tag{3.19}$$

Each of these signals is periodic with period of T (although for $|k| \geq 2$, the fundamental period of $f_k(t)$ is a fraction of T). Thus, a linear combination of harmonically related complex exponentials of the form

$$x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2p/T)t}\ , \tag{3.20}$$

is also periodic with period of T.

- $k=0$, $x(t)$ is a constant.
- $k=+1$ and $k=-1$, both have fundamental frequency equal to w_0 and are collectively referred to as ***the fundamental components*** or ***the first harmonic components***.
- $k=+2$ and $k=-2$, the components are referred to as ***the second harmonic components***.
- $k=+N$ and $k=-N$, the components are referred to as ***the Nth harmonic components***.

Eq. (3.20) can also be expressed as

$$x(t)=x^*(t)=\sum_{k=-\infty}^{+\infty} a^*_k\, e^{-jkw_0 t}\ , \tag{3.21}$$

where we assume that $x(t)$ is real, that is, $x(t)=x^*(t)$.

Replacing k by $-k$ in the summation, we have

$$x(t)=\sum_{k=-\infty}^{+\infty} a^*_{-k}\, e^{jkw_0 t}\ , \tag{3.22}$$

which , by comparison with Eq. (3.20), requires that $a_k = a^*_{-k}$, or equivalently

$$a^*_k = a_{-k}\ . \tag{3.23}$$

To derive the alternative forms of the Fourier series, we rewrite the summation in Eq. (2.20) as

$$x(t)=a_0+\sum_{k=1}^{+\infty}\left[a_k e^{jkw_0 t} + a_{-k} e^{-jk(2p/T)t}\right]. \tag{3.24}$$

Substituting a^*_k for a_{-k}, we have

$$x(t)=a_0+\sum_{k=1}^{+\infty}\left[a_k e^{jkw_0 t} + a^*_k\, e^{-jk(2p/T)t}\right]. \tag{3.25}$$

Since the two terms inside the summation are complex conjugate of each other, this can be expressed as

$$x(t)=a_0+\sum_{k=1}^{+\infty} 2\,\mathrm{Re}\left\{a_k e^{jkw_0 t}\right\}. \tag{3.26}$$

If a_k is expressed in polar from as

$$a_k = A_k e^{jq_k},$$

then Eq. (3.26) becomes

$$x(t) = a_0 + \sum_{k=1}^{+\infty} 2\,\mathrm{Re}\left\{A_k e^{j(kw_0 t + q_k)}\right\}.$$

That is

$$x(t) = a_0 + 2\sum_{k=1}^{+\infty} A_k \cos(kw_0 t + q_k)\,. \qquad (3.27)$$

It is one commonly encountered form for the Fourier series of real periodic signals in continuous time.

Another form is obtained by writing a_k in rectangular form as

$$a_k = B_k + jC_k$$

then Eq. (3.26) becomes

$$x(t) = a_0 + 2\sum_{k=1}^{+\infty} \left[B_k \cos kw_0 t - C_k \sin kw_0 t\right]. \qquad (3.28)$$

For real periodic functions, the Fourier series in terms of complex exponential has the following *three* equivalent forms:

$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2p/T)t}$
$x(t) = a_0 + 2\sum_{k=1}^{+\infty} A_k \cos(kw_0 t + q_k)$
$x(t) = a_0 + 2\sum_{k=1}^{+\infty} \left[B_k \cos kw_0 t - C_k \sin kw_0 t\right]$

3.3.2 Determination of the Fourier Series Representation of a Continuous-Time Periodic Signal

Multiply both side of $x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$ by $e^{-jnw_0 t}$, we obtain

$$x(t)e^{-jnw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} e^{-jnw_0 t}\ , \tag{3.29}$$

Integrating both sides from 0 to $T=2p/w_0$, we have

$$\int_0^T x(t)e^{-jnw_0 t}dt = \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{jkw_0 t} e^{-jnw_0 t}dt\right] = \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{j(k-n)w_0 t}dt\right], \tag{3.30}$$

Note that

$$\int_0^T e^{j(k-n)w_0 t}dt = \begin{cases} T, & k=n \\ 0, & k\neq n \end{cases}$$

So Eq. (3.30) becomes

$$a_n = \frac{1}{T}\int_0^T x(t)e^{-jnw_0 t}dt\ , \tag{3.31}$$

The Fourier series of a periodic continuous-time signal

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2p/T)t} \tag{3.32}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2p/T)t}dt \tag{3.33}$$

Eq. (3.32) is referred to as the ***Synthesis equation***, and Eq. (3.33) is referred to as ***analysis equation***. The set of coefficient $\{a_k\}$ are often called the Fourier series coefficients of the spectral coefficients of $x(t)$.

The coefficient a_0 is the ***dc*** or ***constant component*** and is given with $k=0$, that is

$$a_0 = \frac{1}{T}\int_T x(t)dt\,, \tag{3.34}$$

Example: consider the signal $x(t) = \sin w_0 t$.

$$\sin w_0 t = \frac{1}{2j}e^{jw_0 t} - \frac{1}{2j}e^{-jw_0 t}\,.$$

Comparing the right-hand sides of this equation and Eq. (3.32), we have

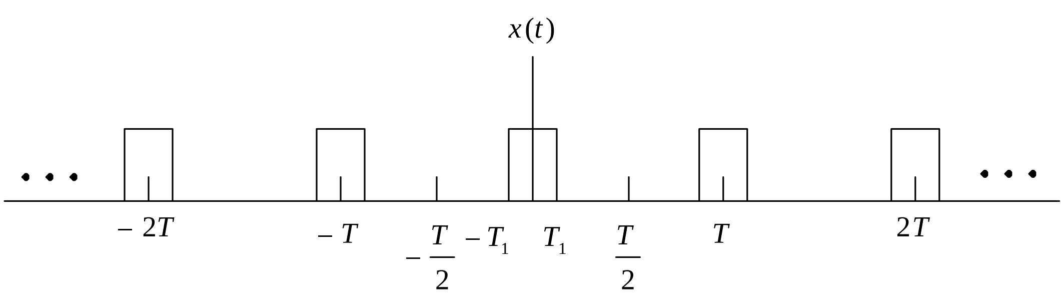
$a_1 = \frac{1}{2j}$, $\qquad a_{-1} = -\frac{1}{2j}$

$a_k = 0$, $\qquad k \neq +1 \;\; or \;\; -1$

Example: The periodic square wave, sketched in the figure below and define over one period is

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}, \tag{3.35}$$

The signal has a fundamental period T and fundamental frequency $w_0 = 2p / T$.



To determine the Fourier series coefficients for $x(t)$, we use Eq. (3.33). Because of the symmetry of $x(t)$ about $t = 0$, we choose $-T/2 \leq t \leq T/2$ as the interval over which the integration is performed, although any other interval of length T is valid the thus lead to the same result.

For $k = 0$,

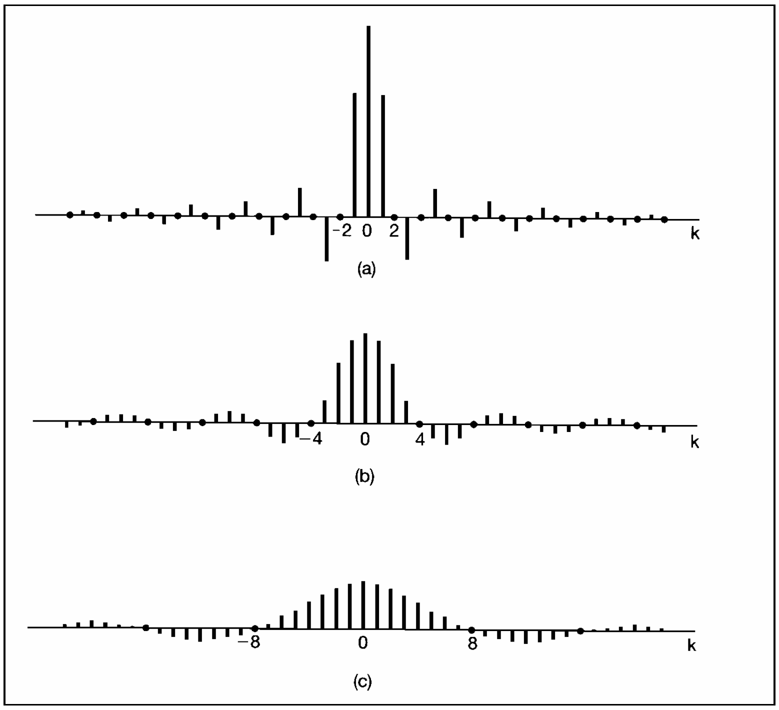
$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} x(t)dt = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}\,, \tag{3.36}$$

For $k \neq 0$, we obtain

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jkw_0 t}dt = -\frac{1}{jkw_0 T}e^{-jkw_0 t}\Big|_{-T_1}^{T_1}$$

$$= \frac{2}{kw_0 T}\left[\frac{e^{jkw_0 T_1} - e^{-jkw_0 T_1}}{2j}\right] \tag{3.37}$$

$$= \frac{2\sin(kw_0 T_1)}{kw_0 T} = \frac{\sin(kw_0 T_1)}{kp}$$



The above figure is a bar graph of the Fourier series coefficients for a fixed T_1 and several values of T. For this example, the coefficients are real, so they can be depicted with a single graph. For complex coefficients, two graphs corresponding to the real and imaginary parts or amplitude and phase of each coefficient, would be required.

3.4 Convergence of the Fourier Series

If a periodic signal $x(t)$ is approximated by a linear combination of finite number of harmonically related complex exponentials

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jkw_0 t} \ . \tag{3.38}$$

Let $e_N(t)$ denote the approximation error,

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jkw_0 t} \ . \qquad (3.39)$$

The criterion used to measure quantitatively the approximation error is the energy in the error over one period:

$$E_N = \int_T |e_N(t)|^2 dt \ . \qquad (3.40)$$

It is shown (problem 3.66) that the particular choice for the coefficients that minimize the energy in the error is

$$a_k = \frac{1}{T}\int_T x(t) e^{-jkw_0 t} dt \ . \qquad (3.41)$$

It can be seen that Eq. (3.41) is identical to the expression used to determine the Fourier series coefficients. Thus, if $x(t)$ has a Fourier series representation, the best approximation using only a finite number of harmonically related complex exponentials is obtained by truncating the Fourier series to the desired number of terms.

The limit of E_N as $N \to \infty$ is zero.

One class of periodic signals that are representable through Fourier series is those signals which have finite energy over a period,

$$\int_T |x(t)|^2 dt < \infty \ , \qquad (3.42)$$

When this condition is satisfied, we can guarantee that the coefficients obtained from Eq. (3.33) are finite. We define

$$e(t) = x(t) - \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t} \ , \qquad (3.43)$$

then

$$\int_T |e(t)|^2 dt = 0 \ , \qquad (3.44)$$

The convergence guaranteed when $x(t)$ has finite energy over a period is very useful. In this case, we may say that $x(t)$ and its Fourier series representation are ***indistinguishable***.

Alternative set of conditions developed by Dirichlet that guarantees the equivalence of the signal and its Fourier series representation:

Condition 1: Over any period, $x(t)$ must be absolutely integrable, that is

$$\int_T |x(t)|dt < \infty , \tag{3.45}$$

This guarantees each coefficient a_k will be finite, since

$$|a_k| = \frac{1}{T}\int_T \left|x(t)e^{-jkw_0 t}\right|dt = \frac{1}{T}\int_T |x(t)|dt < \infty . \tag{3.46}$$

A periodic function that violates the first Dirichlet condition is

$$x(t) = \frac{1}{t}, \qquad 0 < t < 1.$$

Condition 2: In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during a single period of the signal.

An example of a function that meets Condition1 but not Condition 2:

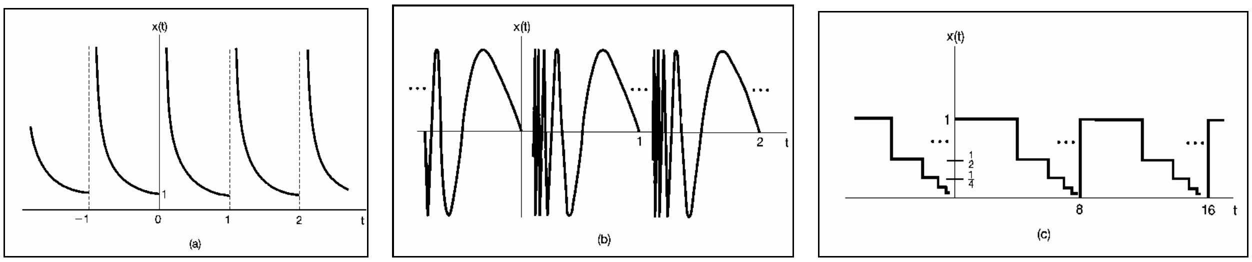
$$x(t) = \sin\left(\frac{2p}{t}\right), \qquad 0 < t \le 1, \tag{3.47}$$

Condition 3: In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

An example that violates this condition is a function defined as

$x(t) = 1$, $0 \le t < 4$, $x(t) = 1/2$, $4 \le t < 6$, $x(t) = 1/4$, $6 \le t < 7$, $x(t) = 1/8$, $7 \le t < 7.5$, etc.

The above three examples are shown in the figure below.



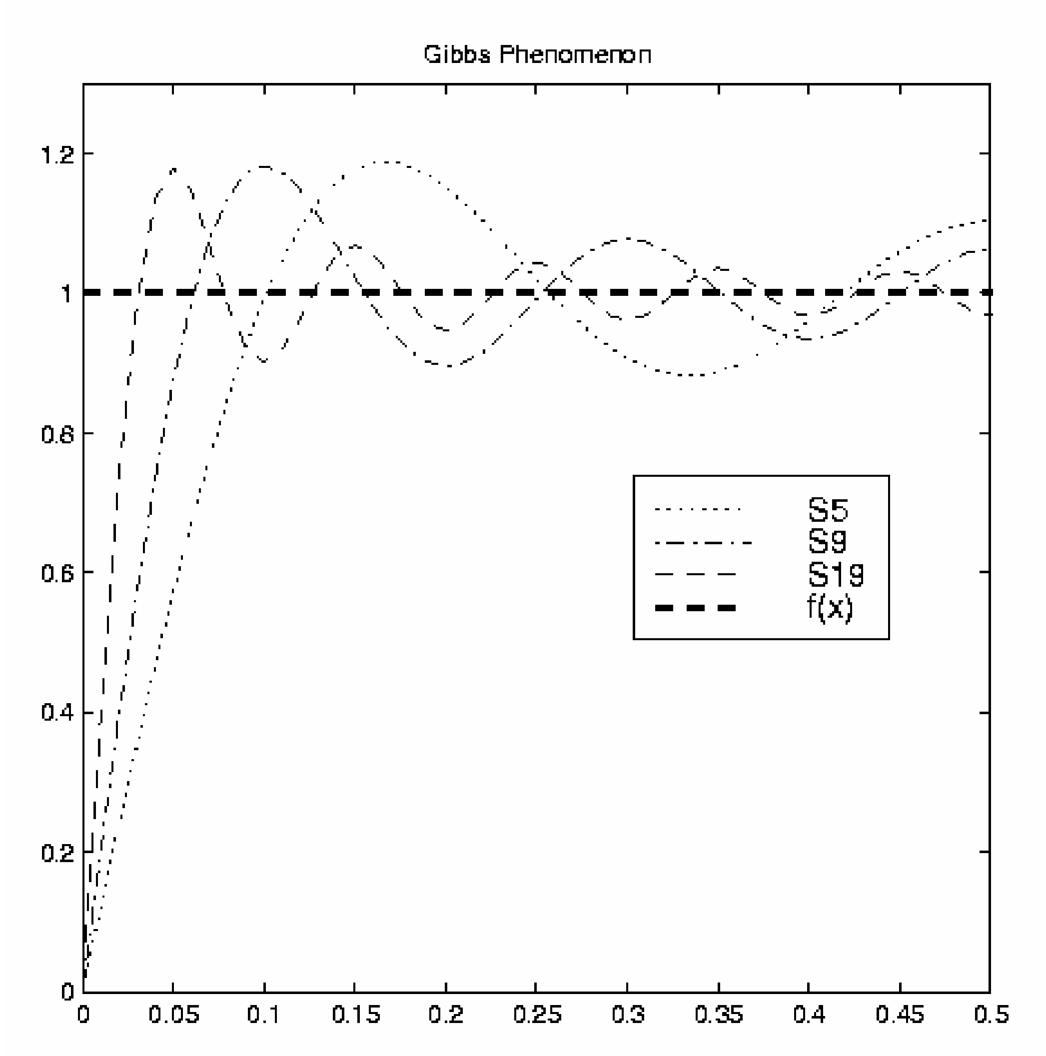
The above are generally pathological in nature and consequently do not typically arise in practical contexts.

Summary:

- For a periodic signal that has no discontinuities, the Fourier series representation converges and equals to the original signal at all the values of t.
- For a periodic signal with a finite number of discontinuities in each period, the Fourier series representation equals to the original signal at all the values of t except the isolated points of discontinuity.

Gibbs Phenomenon:

Near a point, where $x(t)$ has a jump discontinuity, the partial sums $x_N(t)$ of a Fourier series exhibit a substantial overshoot near these endpoints, and an increase in N will not diminish the amplitude of the overshoot, although with increasing N the overshoot occurs over smaller and smaller intervals. This phenomenon is called Gibbs phenomenon.



A large enough value of N should be chosen so as to guarantee that the total energy in these ripples is insignificant.

3.5 Properties of the Continuous-Time Fourier Series

Notation: suppose $x(t)$ is a periodic signal with period T and fundamental frequency w_0. Then if the Fourier series coefficients of $x(t)$ are denoted by a_k, we use the notation

$$x(t) \xleftrightarrow{FS} a_k,$$

to signify the pairing of a periodic signal with its Fourier series coefficients.

3.5.1 Linearity

Let $x(t)$ and $y(t)$ denote two periodic signals with period T and which have Fourier series coefficients denoted by a_k and b_k, that is

$x(t) \xleftrightarrow{FS} a_k$ and $y(t) \xleftrightarrow{FS} b_k$,

then we have

$$z(t) = Ax(t) + By(t) \xleftrightarrow{FS} c_k = Aa_k + Bb_k . \quad (3.48)$$

3.5.2 Time Shifting

When a time shift to a periodic signal $x(t)$, the period T of the signal is preserved.

If $x(t) \xleftrightarrow{FS} a_k$, then we have

$$x(t - t_0) \xleftrightarrow{FS} e^{-jkw_0 t} a_k . \quad (3.49)$$

The magnitudes of its Fourier series coefficients remain unchanged.

3.4.3 Time Reversal

If $x(t) \xleftrightarrow{FS} a_k$, then

$$x(-t) \xleftrightarrow{FS} a_{-k} . \quad (3.50)$$

Time reversal applied to a continuous-time signal results in a time reversal of the corresponding sequence of Fourier series coefficients.

If $x(t)$ is even, that is $x(t) = x(-t)$, the Fourier series coefficients are also even, $a_{-k} = a_k$. Similarly, if $x(t)$ is odd, that is $x(-t) = -x(t)$, the Fourier series coefficients are also odd, $a_{-k} = -a_k$.

3.5.4 Time Scaling

If $x(t)$ has the Fourier series representation $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$, then the Fourier series representation of the time-scaled signal $x(at)$ is

$$x(at) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(a w_0)t} . \quad (3.51)$$

The Fourier series coefficients have not changes, the Fourier series representation has changed because of the change in the fundamental frequency.

3.5.5 Multiplication

Suppose $x(t)$ and $y(t)$ are two periodic signals with period *T* and that

$$x(t) \xleftrightarrow{FS} a_k ,$$

$$y(t) \xleftrightarrow{FS} b_k .$$

Since the product $x(t)y(t)$ is also periodic with period *T*, its Fourier series coefficients h_k is

$$x(t)y(t) \xleftrightarrow{FS} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l} . \quad (3.52)$$

The sum on the right-hand side of Eq. (3.52) may be interpreted as the discrete-time convolution of the sequence representing the Fourier coefficients of $x(t)$ and the sequence representing the Fourier coefficients of $y(t)$.

3.5.6 Conjugate and Conjugate Symmetry

Taking the complex conjugate of a periodic signal $x(t)$ has the effect of complex conjugation and time reversal on the corresponding Fourier series coefficients. That is, if

$$x(t) \xleftrightarrow{FS} a_k ,$$

then

$$x^*(t) \xleftrightarrow{FS} a^*_{-k} . \quad (3.53)$$

If $x(t)$ is real, that is, $x(t) = x^*(t)$, the Fourier series coefficients will be ***conjugate symmetric***, that is

$$a_{-k} = a^*_k . \quad (3.54)$$

From this expression, we may get various symmetry properties for the magnitude, phase, real parts and imaginary parts of the Fourier series coefficients of real signals. For example:

- From Eq. (3.54), we see that if $x(t)$ is real, a_0 is real and $|a_{-k}| = |a_k|$.
- If $x(t)$ is real and even, we have $a_k = a_{-k}$, from Eq. (3.54) $a_{-k} = a^*_k$, so $a_k = a^*_k \Rightarrow$ the Fourier series coefficients are real and even.
- If $x(t)$ is real and odd, the Fourier series coefficients are real and odd.

3.5.7 Parseval's Relation for Continuous-Time periodic Signals

Parseval's Relation for Continuous-Time periodic Signals is

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2 , \qquad (3.55)$$

Since

$$\frac{1}{T}\int_T \left|a_k e^{jkw_0 t}\right|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt = |a_k|^2 ,$$

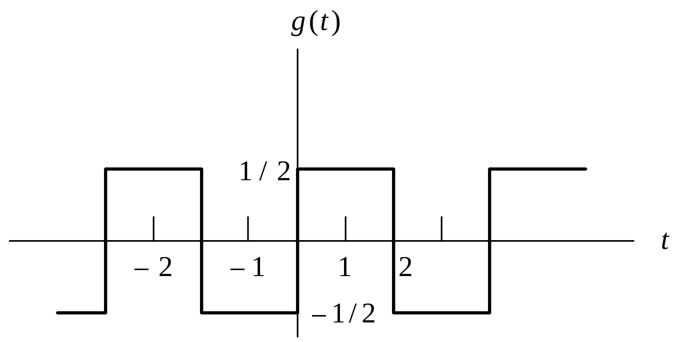
so that $|a_k|^2$ is the average power in the *k*th harmonic component.

Thus, Parseval's Relation states that the total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

3.5.8 Summary of Properties of the Continuous-Time Fourier Series

Property	**Periodic Signal**	**Fourier Series Coefficients**
	$\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\}$ Periodic with period T and fundamental frequency $w_0 = 2p / T$	a_k b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	$x(t - t_0)$	$e^{-jkw_0 t} a_k$
Frequency shifting	$e^{jMw_0 t} x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time Reversal	$x(-t)$	a_{-k}
Time Scaling	$x(at)$, $a > 0$ (Periodic with period T / a)	a_k
Periodic Convolution	$\int_T x(t) y(t - t) dt$	$Ta_k b_k$
Multiplication	$x(t) y(t)$	$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$
Differentiation	$\frac{dx(t)}{dt}$	$jkw_0 a_k = jk \frac{2p}{T} a_k$
Integration	$\int_{-\infty}^{t} x(t) dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jkw_0}\right) a_k = \left(\frac{1}{jk(2p / T)}\right) a_k$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathrm{Re}\{a_k\} = \mathrm{Re}\{a_{-k}\} \\ \mathrm{Im}\{a_k\} = -\mathrm{Im}\{a_{-k}\} \\ \|a_k\| = \|a_{-k}\| \\ \angle a_k = -\angle a_{-k} \end{cases}$
Real and Even Signals Real and Odd Signals Even-Odd Decomposition of Real Signals	$x(t)$ real and even $x(t)$ real and odd $\begin{cases} x_e(t) = Ev\{x(t)\} & [x(t)\ real] \\ x_e(t) = Od\{x(t)\} & [x(t)\ real] \end{cases}$	a_k real and even a_k purely imaginary and odd $\mathrm{Re}\{a_k\}$ $j\,\mathrm{Im}\{a_k\}$
	Parseval's Relation for Periodic Signals $\frac{1}{T} \int_T \|x(t)\|^2 dt = \sum_{k=-\infty}^{\infty} \|a_k\|^2$	

Example: Consider the signal $g(t)$ with a fundamental period of 4.



The Fourier series representation can be obtained directly using the analysis equation (3.33). We may also use the relation of $g(t)$ to the symmetric periodic square wave $x(t)$ discussed on page 8. Referring to that example, $T = 4$ and $T_1 = 1$,

$$g(t) = x(t-1) - 1/2 \,. \tag{3.56}$$

The time-shift property indicates that if the Fourier series coefficients of $x(t)$ are denoted by a_k the Fourier series coefficients of $x(t-1)$ can be expressed as

$$b_k = a_k e^{-jkp/2} \,. \tag{3.57}$$

The Fourier coefficients of the *dc* offset in $g(t)$, that is the term –1/2 on the right-hand side of Eq. (3.56) are given by

$$c_k = \begin{cases} 0, & for\ k \neq 0 \\ -\frac{1}{2}, & for\ k = 0 \end{cases} . \tag{3.58}$$

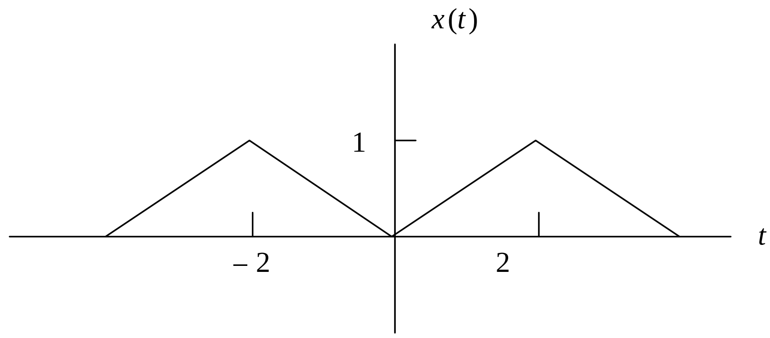
Applying the linearity property, we conclude that the coefficients for $g(t)$ can be expressed as

$$d_k = \begin{cases} a_k e^{-jkp/2}, & for\ k \neq 0 \\ a_0 - \frac{1}{2}, & for\ k = 0 \end{cases} , \tag{3.59}$$

replacing $a_k = \dfrac{\sin(pk/2)}{kp} e^{jkp/2}$, then we have

$$d_k = \begin{cases} \dfrac{\sin(pk/2)}{pk} e^{-jkp/2}, & for\ k \neq 0 \\ 0, & for\ k = 0 \end{cases} . \tag{3.60}$$

Example: The triangular wave signal $x(t)$ with period $T = 4$, and fundamental frequency $\omega_0 = \pi / 2$ is shown in the figure below.



The derivative of this function is the signal $g(t)$ in the previous preceding example. Denoting the Fourier series coefficients of $g(t)$ by d_k, and those of $x(t)$ by e_k, based on the differentiation property, we have

$$d_k = jk(\pi / 2)e_k . \qquad (3.61)$$

This equation can be expressed in terms of e_k except when $k = 0$. From Eq. (3.60),

$$e_k = \frac{2d_k}{jk\pi} = \frac{2\sin(\pi k / 2)}{j(k\pi)^2} e^{-jk\pi / 2} . \qquad (3.62)$$

For $k = 0$, e_0 can be simply calculated by calculating the area of the signal under one period and divide by the length of the period, that is

$$e_0 = 1/2 . \qquad (3.63)$$

Example: The properties of the Fourier series representation of periodic train of impulse,

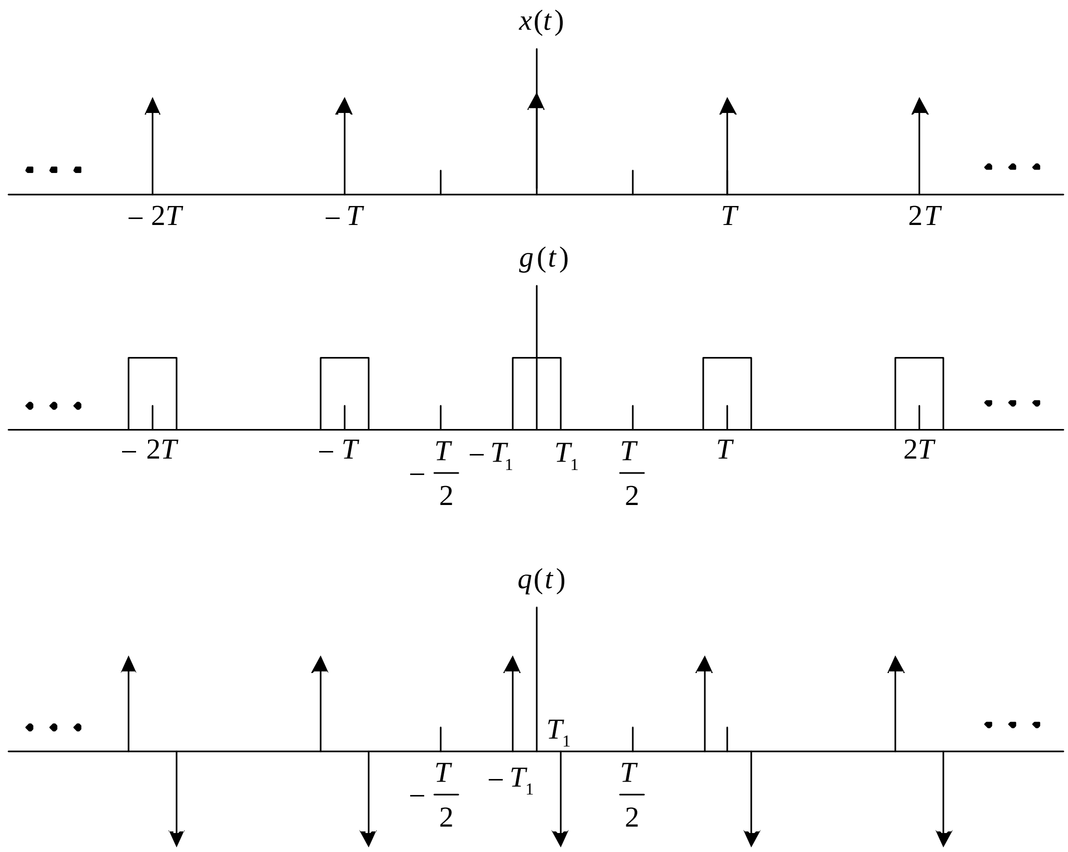
$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) . \qquad (3.64)$$

We use Eq. (3.33) and select the integration interval to be $-T/2 \le t \le T/2$, avoiding the placement of impulses at the integration limits.

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk(2\pi / T)t} dt = \frac{1}{T} . \qquad (3.65)$$

All the Fourier series coefficients of this periodic train of impulse are identical, real and even.

The periodic train of impulse has a straightforward relation to square-wave signals such as $g(t)$ on page 8. The derivative of $g(t)$ is the signal $q(t)$ shown in the figure below,



which can also interpreted as the difference of two shifted versions of the impulse train $x(t)$. That is,

$$q(t) = x(t+T_1) - x(t-T_1)\,. \tag{3.66}$$

Based on the time-shifting and linearity properties, we may express the Fourier coefficients b_k of $q(t)$ in terms of the Fourier series coefficient of a_k; that is

$$b_k = e^{jkw_0T_1}a_k - e^{-jkw_0T_1}a_k = \frac{1}{T}\left[e^{jkw_0T_1} - e^{-jkw_0T_1}\right], \tag{3.67}$$

Finally we use the differentiation property to get

$$b_k = jkw_0 c_k\,, \tag{3.68}$$

where c_k is the Fourier series coefficients of $g(t)$. Thus

$$c_k = \frac{b_k}{jkw_0} = \frac{2j\sin(kw_0T_1)}{jkw_0T} = \frac{2\sin(kw_0T_1)}{kw_0T},\ k \neq 0, \tag{3.69}$$

c_0 can be solve by inspection from the figure:

$$c_0 = \frac{2T_1}{T}. \tag{3.70}$$

Example: Suppose we are given the following facts about a signal $x(t)$

1. $x(t)$ is a real signal.
2. $x(t)$ is periodic with period $T = 4$, and it has Fourier series coefficients a_k.
3. $a_k = 0$ for $k > 1$.
4. The signal with Fourier coefficients $b_k = e^{-jpk/2}a_{-k}$ is odd.
5. $\frac{1}{4}\int_4 |x(t)|^2 dt = \frac{1}{2}$

Show that the information is sufficient to determine the signal $x(t)$ to within a sign factor.

- According to Fact 3, $x(t)$ has at most three nonzero Fourier series coefficients a_k: a_{-1}, a_0 and a_1. Since the fundamental frequency $w_0 = 2p / T = 2p / 4 = p / 2$, it follows that

$$x(t) = a_0 + a_1 e^{jpt/2} + a_{-1}e^{-jpt/2}. \tag{3.71}$$

- Since $x(t)$ is real (Fact 1), based on the symmetry property a_0 is real and $a_1 = a^*_{-1}$. Consequently,

$$x(t) = a_0 + a_1 e^{jpt/2} + \left(a_1 e^{jpt/2}\right)^* = a_0 + 2\,\mathrm{Re}\left\{a_1 e^{jpt/2}\right\}. \tag{3.72}$$

- Based on the Fact 4 and considering the time-reversal property, we note that a_{-k} corresponds to $x(-t)$. Also the multiplication property indicates that multiplication of kth Fourier series by $e^{-jkp/2}$ corresponds to the signal being shifted by 1 to the right. We conclude that the coefficients b_k correspond to the signal $x(-(t-1)) = x(-t+1)$, which according to Fact 4 must be odd. Since $x(t)$ is real, $x(-t+1)$ must also be real. So based the property, the Fourier series coefficients must be purely imaginary and odd. Thus, $b_0 = 0$, $b_{-1} = -b_1$.
- Since time reversal and time shift cannot change the average power per period, Fact 5 holds even if $x(t)$ is replaced by $x(-t+1)$. That is

$$\frac{1}{4}\int_4 |x(-t+1)|^2 dt = \frac{1}{2}. \tag{3.73}$$

Using Parseval's relation,

$$|b_1|^2 + |b_{-1}|^2 = 1/2. \qquad (3.74)$$

Since $b_{-1} = -b_1$, we obtain $|b_1| = 1/2$. Since b_1 is known to be purely imaginary, it must be either $b_1 = j/2$ or $b_1 = -j/2$.

- Finally we translate the conditions on b_0 and b_1 into the equivalent statement on a_0 and a_1. First, since $b_0 = 0$, Fact 4 implies that $a_0 = 0$. With $k = 1$, this condition implies that $a_1 = e^{-jp/2} b_{-1} = -jb_{-1} = jb_1$. Thus, if we take $b_1 = j/2$, $a_1 = -1/2$, from Eq. (3.72), $x(t) = -\cos(pt/2)$. Alternatively, if we take $b_1 = -j/2$, the $a_1 = 1/2$, and therefore, $x(t) = \cos(pt/2)$.

3.6 Fourier Series Representation of Discrete-Time Periodic Signals

The Fourier series representation of a discrete-time periodic signal is ***finite***, as opposed to the ***infinite*** series representation required for continuous-time periodic signals

3.6.1 Linear Combination of Harmonically Related Complex Exponentials

A discrete-time signal $x[n]$ is periodic with period N if

$$x[n] = x[n+N]. \qquad (3.75)$$

The fundamental period is the smallest positive N for which Eq. (3.75) holds, and the fundamental frequency is $w_0 = 2p/N$.

The set of all discrete-time complex exponential signals that are periodic with period N is given by

$$f_k[n] = e^{jkw_0 n} = e^{jk(2p/N)n}, \qquad k = 0, \pm 1, \pm 2, ..., \qquad (3.76)$$

All of these signals have fundamental frequencies that are multiples of $2p/N$ and thus are harmonically related.

There are only N distinct signals in the set given by Eq. (3.76); this is because the discrete-time complex exponentials which differ in frequency by a multiple of $2p$ are identical, that is,

$$f_k[n] = f_{k+rN}[n]. \qquad (3.77)$$

The representation of periodic sequences in terms of linear combinations of the sequences $f_k[n]$ is

$$x[n] = \sum_k a_k f_k[n] = \sum_k a_k e^{jkw_0 n} = \sum_k a_k e^{jk(2p/N)n} . \qquad (3.78)$$

Since the sequences $f_k[n]$ are distinct over a range of N successive values of k, the summation in Eq. (3.78) need include terms over this range. We indicate this by expressing the limits of the summation as $k = \langle N \rangle$. That is,

$$x[n] = \sum_{k=\langle N \rangle} a_k f_k[n] = \sum_{k=\langle N \rangle} a_k e^{jkw_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2p/N)n} . \qquad (3.79)$$

Eq. (3.79) is referred to as the discrete-time Fourier series and the coefficients a_k as the Fourier series coefficients.

6.2 Determination of the Fourier Series Representation of a Periodic Signal

The discrete-time Fourier series pair:

$$x[n] = \sum_{k=\langle N \rangle} a_k f_k[n] = \sum_{k=\langle N \rangle} a_k e^{jkw_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2p/N)n} , \qquad (3.80)$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jkw_0 n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2p/N)n} . \qquad (3.81)$$

Eq. (3.80) is called ***synthesis equation*** and Eq. (3.81) is called ***analysis equation***.

Example: Consider the signal $x[n] = \sin w_0 n$, (3.82)

$x[n]$ is periodic only if $2p/w_0$ is ***an integer***, or ***a ratio of integer***. For the case the when $2p/w_0$ is an integer N, that is, when

$$w_0 = \frac{2p}{N} , \qquad (3.83)$$

$x[n]$ is periodic with the fundamental period N. Expanding the signal as a sum of two complex exponentials, we get

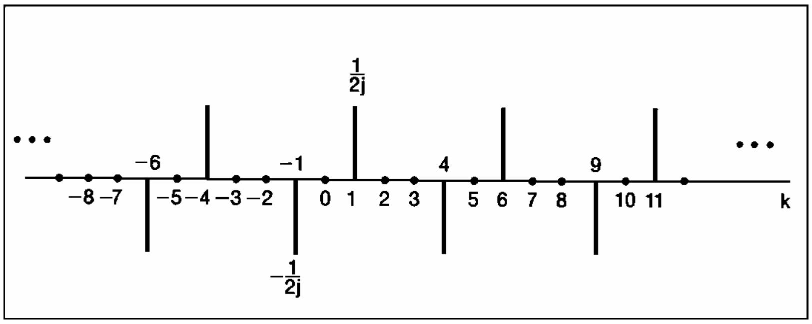
$$x[n] = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n}, \tag{3.84}$$

From Eq. (3.84), we have

$$a_1 = \frac{1}{2j}, \; a_{-1} = -\frac{1}{2j}, \tag{3.85}$$

and the remaining coefficients over the interval of summation are zero. As discussed previously, these coefficients repeat with period *N*.

The Fourier series coefficients for this example with $N = 5$ are illustrated in the figure below.



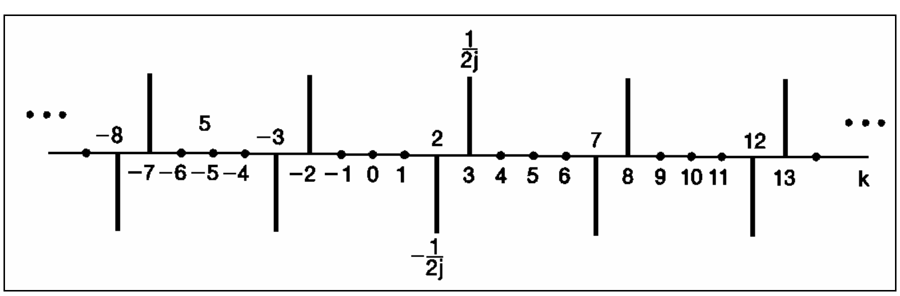
When $2\pi/\omega_0$ is a ratio of integer, that is, when

$$\omega_0 = \frac{2\pi M}{N}, \tag{3.86}$$

Assuming the *M* and *N* do not have any common factors, $x[n]$ has a fundamental period of *N*. Again expanding $x[n]$ as a sum of two complex exponentials, we have

$$x[n] = \frac{1}{2j} e^{jM(2\pi/N)n} - \frac{1}{2j} e^{-jM(2\pi/N)n}, \tag{3.87}$$

From which we determine by inspection that $a_M = (1/2j)$, $a_{-M} = -(1/2j)$, and the remaining coefficients over one period of length *N* are zero. The Fourier coefficients for this example with $M = 3$ and $N = 5$ are depicted in the figure below.



Example: Consider the signal

$$x[n] = 1 + \sin\left(\frac{2\pi}{N}\right)n + 3\cos\left(\frac{2\pi}{N}\right)n + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right).$$

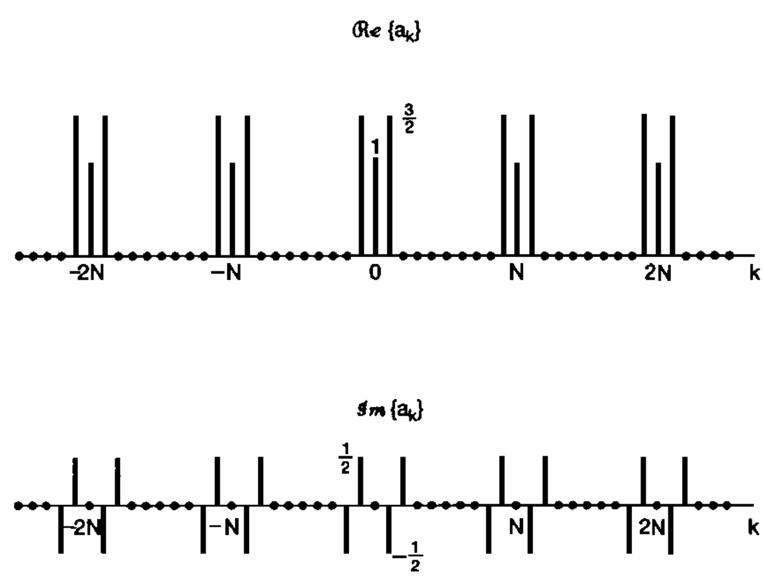
Expanding this signal in terms of complex exponential, we have

$$x[n] = 1 + (\frac{3}{2} + \frac{1}{2j})e^{j(2\pi/N)n} + (\frac{3}{2} - \frac{1}{2j})e^{-j(2\pi/N)n} + \left(\frac{1}{2}e^{j\pi/2}\right)e^{j2(2\pi/N)n} + \left(\frac{1}{2}e^{-j\pi/2}\right)e^{-j2(2\pi/N)n}.$$

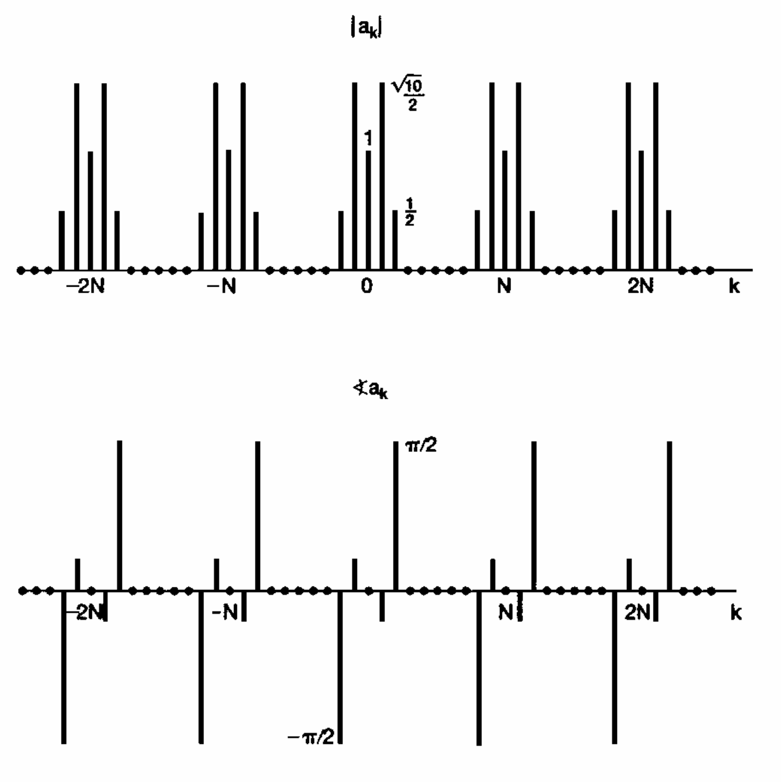
Thus the Fourier series coefficients for this signal are

$$a_0 = 1,$$
$$a_1 = \frac{3}{2} + \frac{1}{2j} = \frac{3}{2} - \frac{1}{2}j,$$
$$a_{-1} = \frac{3}{2} - \frac{1}{2j} = \frac{3}{2} + \frac{1}{2}j,$$
$$a_2 = \frac{1}{2}j,$$
$$a_{-2} = -\frac{1}{2}j.$$

with $a_k = 0$ for other values of k in the interval of summation in the synthesis equation. The real and imaginary parts of these coefficients for $N = 10$, and the magnitude and phase of the coefficients are depicted in the figure below.

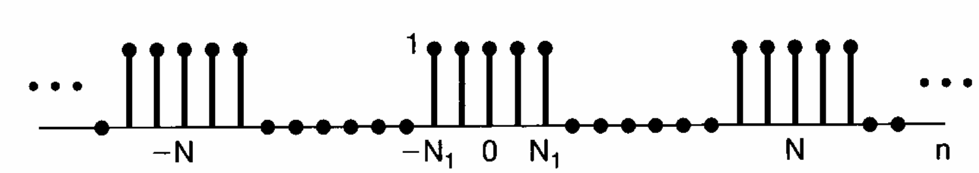


(a)



(b)

Example: Consider the square wave shown in the figure below.



Because $x[n]=1$ for $-N_1 \le n \le N_1$, we choose the length-N interval of summation to include the range $-N_1 \le n \le N_1$. The coefficients are given

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk(2p/N)n} , \qquad (3.88)$$

Let $m = n + N_1$, we observe that Eq. (3.88) becomes

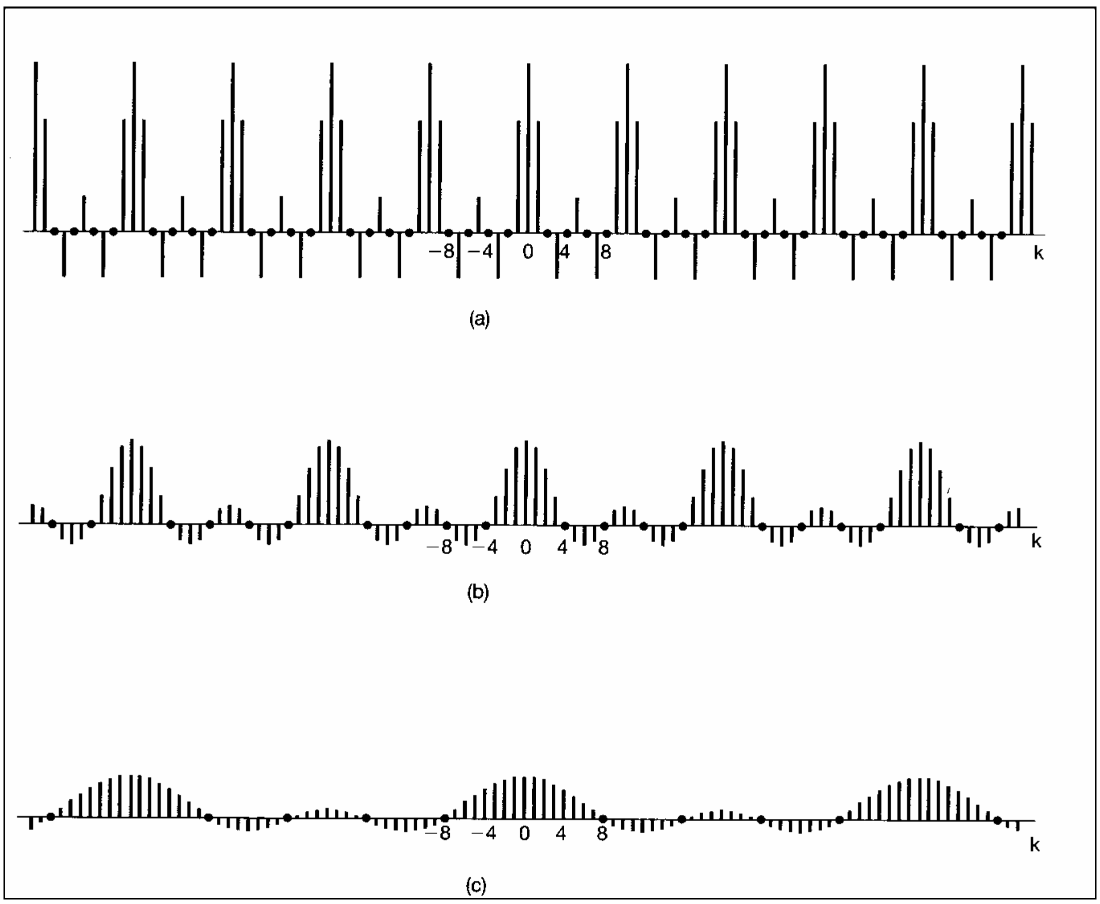
$$a_k = \frac{1}{N}\sum_{n=0}^{2N_1} e^{-jk(2p/N)(m-N_1)} = \frac{1}{N} e^{jk(2p/N)N_1} \sum_{n=0}^{2N_1} e^{-jk(2p/N)m}, \qquad (3.89)$$

$$a_k = \frac{1}{N} e^{jk(2p/N)N_1}\left(\frac{1-e^{jk2p(2N_1+1)/N}}{1-e^{-jk(2p/N)}}\right) = \frac{1}{N}\frac{\sin[2pk(N_1+1/2)/N]}{\sin(pk/N)},\ k \neq 0, \pm N, \pm 2N, ... \qquad (3.90)$$

and

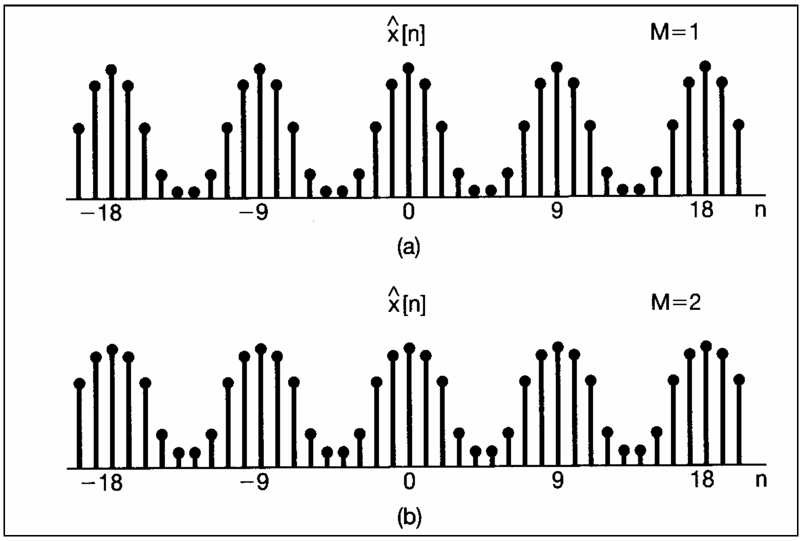
$$a_k = \frac{2N_1+1}{N},\ k = 0, \pm N, \pm 2N, ... \qquad (3.91)$$

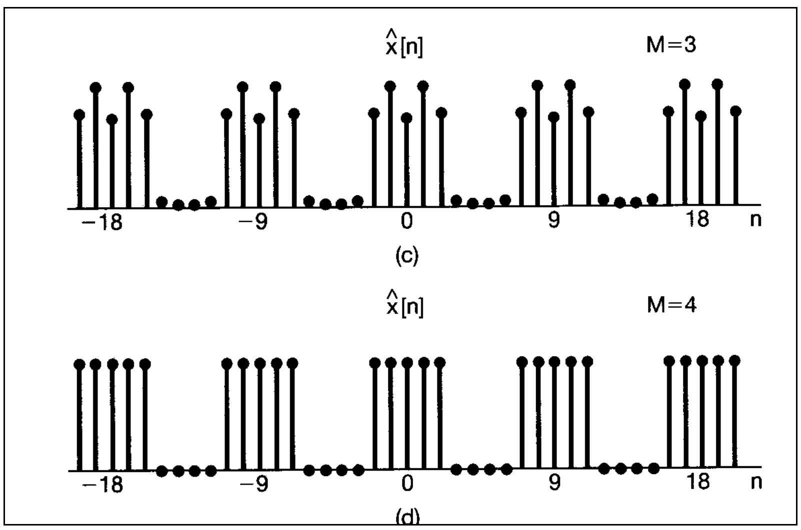
The coefficients a_k for $2N_1+1=5$ are sketched for $N=10,\ 20,$ and 40 in the figure below.



The partial sums for the discrete-time square wave for $M = 1,\ 2,\ 3,$ and 4 are depicted in the figure below, where $N=9$, $2N_1+1=5$.

We see for $M=4$, the partial sum exactly equals to $x[n]$. In contrast to the continuous-time case, **there *are no convergence issues and there is no Gibbs phenomenon***.





3.7 Properties of Discrete-Time Fourier Series

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$ Periodic with period N and fundamenta l frequency $w_0 = 2\pi$	$\begin{matrix} a_k \\ b_k \end{matrix}$ Periodic with period N
Linearity	$Ax[n]+By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n-n_0]$	$e^{-jk(2\pi/N)t}a_k$
Frequency shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}

Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n]=\begin{cases} x[n/m], & if\ n\ is\ a\ multiple\ of\ n \\ 0, & if\ n\ is\ a\ multiple\ of\ n \end{cases}$ (Periodic with period mN)	$\frac{1}{m}a_k\begin{pmatrix} viewed\ as\ periodic \\ with\ period\ mN \end{pmatrix}$
Periodic Convolution	$\sum_{r=[N]} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=<N>} a_l b_{k-l}$
Differentiation	$x[n]-x[n-1]$	$\left(1-e^{-jk(2p/N)}\right)a_k$
Integration	$\sum_{k=-\infty}^{n} x[k]$ (finite valued and periodic only if $a_0=0$)	$\left(\frac{1}{1-e^{-jk(2p/N)}}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathrm{Re}\{a_k\}=\mathrm{Re}\{a_{-k}\} \\ \mathrm{Im}\{a_k\}=-\mathrm{Im}\{a_{-k}\} \\ \lvert a_k\rvert=\lvert a_{-k}\rvert \\ \angle a_k=-\angle a_{-k} \end{cases}$
Real and Even Signals Real and Odd Signals Even-Odd Decomposition of Real Signals	$x[n]$ real and even $x[n]$ real and odd $\begin{cases} x_e[n]=Ev\{x[n]\} & [x[n]\ real] \\ x_e[n]=Od\{x[n]\} & [x[n]\ real] \end{cases}$	a_k real and even a_k purely imaginary and odd $\mathrm{Re}\{a_k\}$ $j\,\mathrm{Im}\{a_k\}$
	Parseval's Relation for Periodic Signals $\frac{1}{T}\sum_{n=<N>}\lvert x[n]\rvert^2=\sum_{n=<N>}\lvert a_k\rvert^2$	

3.7.1 Multiplication

$$x[n]y[n] \stackrel{FS}{\longleftrightarrow} \sum_{l=<N>} a_l b_{k-l} \quad (3.92)$$

Eq. (3.92) is analogous to the convolution, except that the summation variable is now restricted to in interval of N consecutive samples. This type of operation is referred to as a ***Periodic Convolution*** between the two periodic sequences of Fourier coefficients.

The usual form of the convolution sum, where the summation variable ranges from $-\infty$ to $+\infty$, is sometimes referred to as ***Aperiodic Convolution***.

3.7.2 First Difference

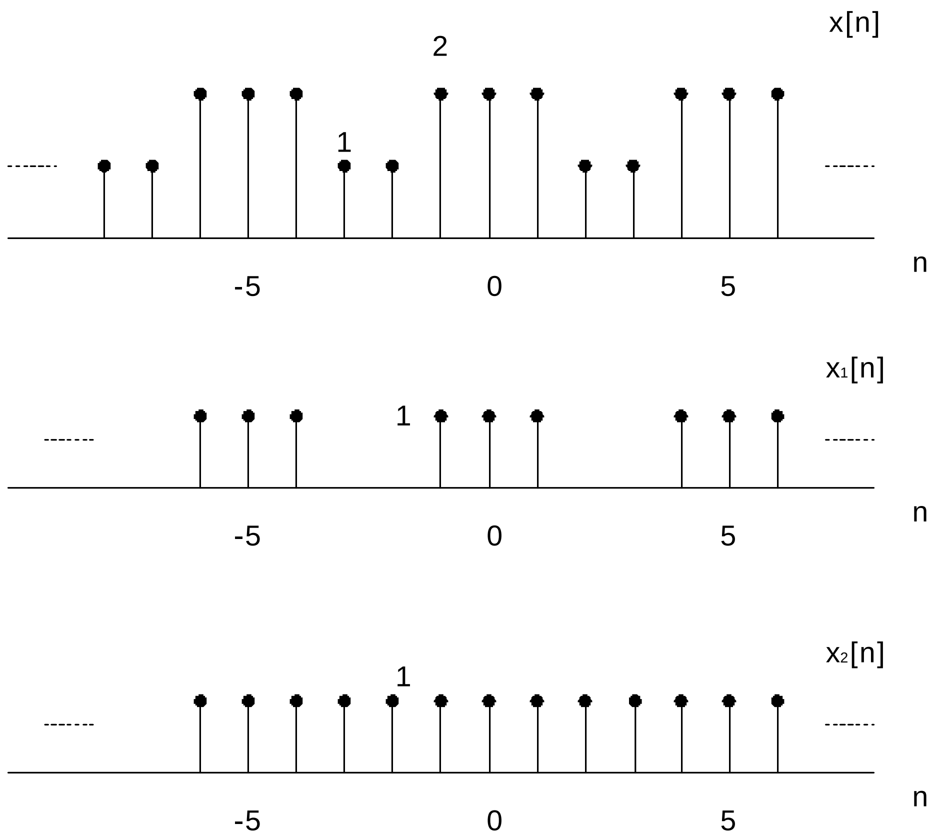
$$x[n]-x[n-1]\xleftrightarrow{FS}\left(1-e^{-jk(2p/N)}\right)a_k. \qquad (3.93)$$

3.7.3 Parseval's Relation

$$\frac{1}{T}\sum_{n=<N>}\left|x[n]\right|^2=\sum_{k=<N>}\left|a_k\right|^2. \qquad (3.94)$$

3.7.4 Examples

Example: Consider the signal shown in the figure below.



The signal $x[n]$ may be viewed as the sum of the square wave $x_1[n]$ with Fourier series coefficients b_k and $x_2[n]$ with Fourier series coefficients c_k.

$$a_k = b_k + c_k, \tag{3.95}$$

The Fourier series coefficients for $x_1[n]$ is

$$b_k = \begin{cases} \frac{1}{5}\frac{\sin(3pk/5)}{\sin(pk/5)}, & for\, k \neq 0, \pm 5, \pm 10, ... \\ \frac{3}{5}, & for\, k = 0, \pm 5, \pm 10, ... \end{cases}. \tag{3.96}$$

The sequence $x_2[n]$ has only a dc value, which is captured by its zeroth Fourier series coefficient:

$$c_0 = \frac{1}{5}\sum_{n=0}^{4} x_2[n] = 1, \tag{3.97}$$

Since the discrete-time Fourier series coefficients are periodic, it follows that $c_k = 1$ whenever k is an integer multiple of 5.

$$a_k = \begin{cases} \frac{1}{5}\frac{\sin(3pk/5)}{\sin(pk/5)}, & for\, k \neq 0, \pm 5, \pm 10, ... \\ \frac{8}{5}, & for\, k = 0, \pm 5, \pm 10, ... \end{cases} \tag{3.98}$$

Example: Suppose we are given the following facts about a sequence $x[n]$:

1. $x[n]$ is periodic with period $N = 6$.
2. $\sum_{n=0}^{5} x[n] = 2$.
3. $\sum_{n=2}^{7} (-1)^n x[n] = 1$.
4. $x[n]$ has minimum power per period among the set of signals satisfying the preceding three conditions.

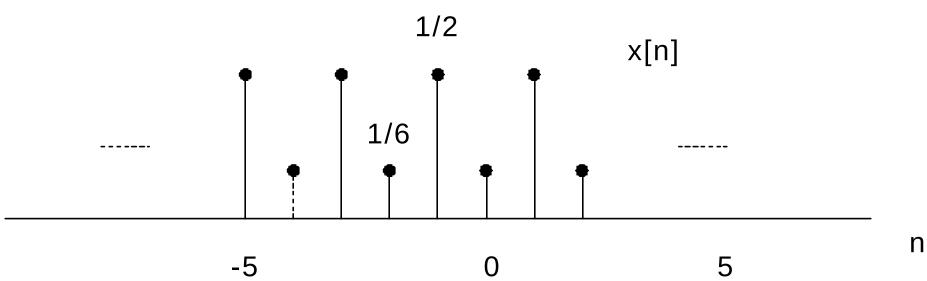
- From Fact 2, we have $a_0 = \frac{1}{6}\sum_{n=0}^{5} x[n] = \frac{1}{3}$.
- Note that $(-1)^n = e^{-jpn} = e^{-j(2p/6)3n}$, we see from Fact 3 that $a_3 = \frac{1}{6}\sum_{2}^{7} x[n]e^{-j3(2p/N)n} = \frac{1}{6}$.
- From Parseval's relation, the average power in $x[n]$ is

$$P = \sum_{k=0}^{5} \left| a_k{}^2 \right| .$$

Since each nonzero coefficient contributes a positive amount to P, and since the values of a_0 and a_3 are specified, the value of P is minimized by choosing $a_1 = a_2 = a_4 = a_5 = 0$. It follows that

$$x[n] = a_0 + a_3 e^{jpn} = \frac{1}{3} + \frac{1}{6}(-1)^n ,$$

which is shown in the figure below.



3.8 Fourier Series and LTI Systems

We have seen that the response of a continuous-time LTI system with impulse response $h(t)$ to a complex exponential signal e^{st} is the same complex exponential multiplied by a complex gain:

$$y(t) = H(s)e^{st} ,$$

where

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st} dt , \qquad (3.99)$$

In particular, for $s = jw$, the output is $y(t) = H(jw)e^{jwt}$. The complex functions $H(s)$ and $H(jw)$ are called the system function (or transfer function) and the frequency response, respectively.

By superposition, the output of an LTI system to a periodic signal represented by a Fourier series

$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2p/T)t}$ is given by

$$y(t)=\sum_{k=-\infty}^{+\infty} a_k H(jkw_0)e^{jkw_0 t} \quad . \tag{3.99}$$

That is, the Fourier series coefficients b_k of the periodic output $y(t)$ are given by

$$b_k = a_k H(jkw_0)\,, \tag{3.100}$$

Similarly, for discrete-time signals and systems, response $h[n]$ to a complex exponential signal e^{jwn} is the same complex exponential multiplied by a complex gain:

$$y[n]=H(jkw_0)e^{jkw_0 n}\,, \tag{3.101}$$

where

$$H(e^{jw})=\sum_{n=-\infty}^{\infty} h[n]e^{-jwn} \quad . \tag{3.102}$$

Example: Suppose that the periodic signal $x(t)=\sum_{k=-3}^{+3} a_k e^{jk2pt}$ with $a_0=1$, $a_1=a_{-1}=\frac{1}{4}$, $a_2=a_{-2}=\frac{1}{2}$, and $a_3=a_{-3}=\frac{1}{3}$ is the input signal to an LTI system with impulse response

$$h(t)=e^{-t}u(t)$$

To calculate the Fourier series coefficients of the output $y(t)$, we first compute the frequency response:

$$H(jw)=\int_0^{\infty} e^{-t}e^{-jwt}\,dt=\frac{1}{1+jw}e^{-t}e^{-wt}\Big|_0^{\infty}=\frac{1}{1+jw}\,, \tag{3.103}$$

The output is

$$y(t)=\sum_{k=-3}^{+3} b_k e^{jk2pt} \quad , \tag{3.104}$$

where $b_k = a_k H(jkw_0)=a_k H(jk2p)$, so that

$$b_0=0\,, \qquad b_1=\frac{1}{4}\left(\frac{1}{1+j2p}\right), \qquad b_{-1}=\frac{1}{4}\left(\frac{1}{1-j2p}\right),$$

$$b_2 = \frac{1}{4}\left(\frac{1}{1+j4p}\right), \qquad b_{-2} = \frac{1}{4}\left(\frac{1}{1-j4p}\right),$$

$$b_3 = \frac{1}{4}\left(\frac{1}{1+j6p}\right), \qquad b_{-3} = \frac{1}{4}\left(\frac{1}{1-j6p}\right).$$

Example: Consider an LTI system with impulse response $h[n] = a^n u[n]$, $-1 < a < 1$, and with the input

$$x[n] = \cos\left(\frac{2pn}{N}\right). \tag{3.105}$$

Write the signal $x[n]$ in Fourier series form as

$$x[n] = \frac{1}{2}e^{j(2p/N)n} + \frac{1}{2}e^{-j(2p/N)n}.$$

Also the transfer function is

$$H(e^{jw}) = \sum_{n=0}^{\infty} a^n e^{-jwn} = \sum_{n=0}^{\infty}\left(a\, e^{-jw}\right)^n = \frac{1}{1-a\, e^{-jw}}. \tag{3.106}$$

The Fourier series for the output

$$\begin{aligned} y[n] &= \frac{1}{2}H\left(e^{j2p/N}\right)e^{j(2p/N)n} + \frac{1}{2}H\left(e^{-j2p/N}\right)e^{-j(2p/N)n} \\ &= \frac{1}{2}\left(\frac{1}{1-a\, e^{-jw}}\right)e^{j(2p/N)n} + \frac{1}{2}\left(\frac{1}{1-a\, e^{-jw}}\right)e^{-j(2p/N)n} \end{aligned} \tag{3.107}$$

3.9 Filtering

Filtering – to change the relative amplitude of the frequency components in a signal or eliminate some frequency components entirely.

Filtering can be conveniently accomplished through the use of LTI systems with an appropriately chosen frequency response.
LTI systems that change the shape of the spectrum of the input signal are referred to as *frequency-shaping filters*.

LTI systems that are designed to pass some frequencies essentially undistorted and significantly attenuate or eliminate others are referred to as *frequency-selective filters*.

Example: A first-order low-pass filter with impulse response $h(t) = e^{-t}u(t)$ cuts off the high frequencies in a periodic input signal, while low frequency harmonics are mostly left intact. The frequency response of this filter

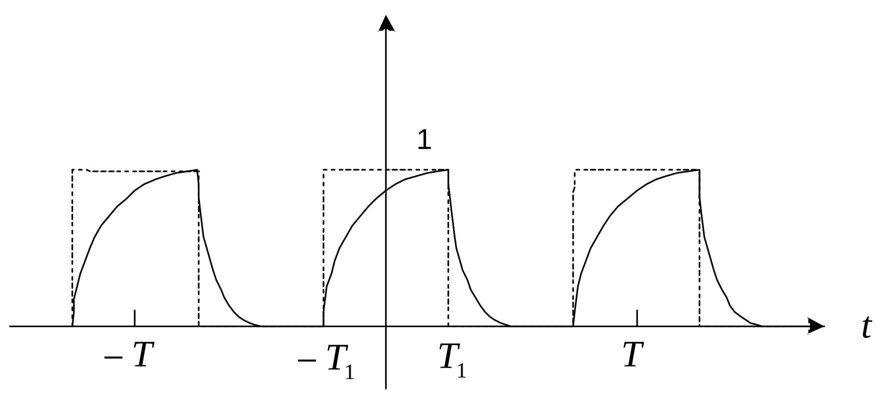
$$H(jw) = \int_0^{+\infty} e^{-t} e^{-jwt}\, dt = \frac{1}{1+jw}. \tag{3.107}$$

We can see that as the frequency w increase, the magnitude of the frequency response of the filter $|H(jw)|$ decreases. If the periodic input signal is a rectangular wave, then the output signal will have its Fourier series coefficients b_k given by

$$b_k = a_k H(jkw_0) = \frac{\sin(kw_0 T_1)}{kp\,(1+jkw_0)}, \qquad k \neq 0 \tag{3.108}$$

$$b_0 = a_0 H(0) = \frac{2T_1}{T}. \tag{3.109}$$

The reduced power at high frequencies produced an output signal that is smother than the input signal.

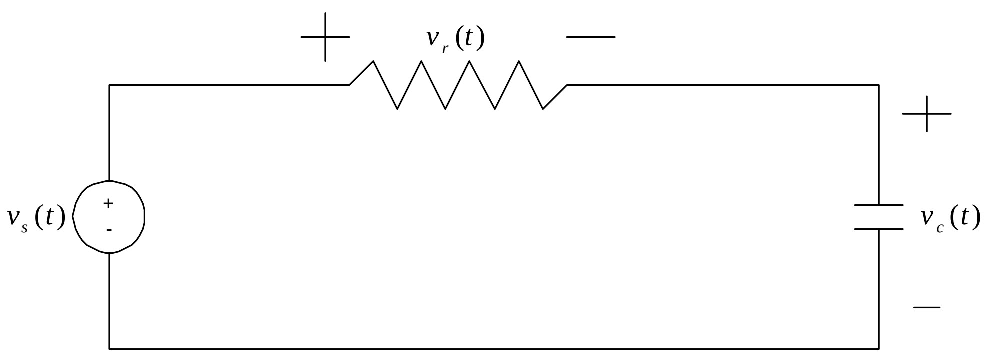


3.10 Examples of continuous-Time Filters Described By Differential Equations

In many applications, frequency-selective filtering is accomplished through the use of LTI systems described by linear constant-coefficient differential or difference equations. In fact, many physical systems that can be interpreted as performing filtering operations are characterized by differential or difference equation.

3.10.1 A simple RC Lowpass Filter

The first-order RC circuit is one of the electrical circuits used to perform continuous-time filtering. The circuit can perform either Lowpass or highpass filtering depending on what we take as the output signal.



If we take the voltage cross the capacitor as the output, then the output voltage is related to the input through the linear constant-coefficient differential equation:

$$RC\frac{dv_c(t)}{dt}+v_c(t)=v_s(t)\,. \tag{3.111}$$

Assuming initial rest, the system described by Eq. (3.111) is LTI. If the input is $v_s(t)=e^{jwt}$, we must have voltage output $v_c(t)=H(jw)e^{jwt}$. Substituting these expressions into Eq. (3.111), we have

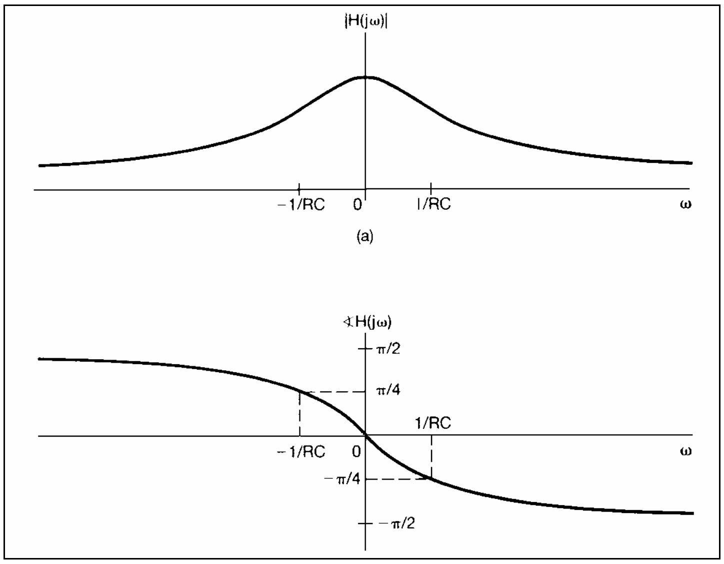
$$RC\frac{d}{dt}\left[H(jw)e^{jwt}\right]+H(jw)e^{jwt}=e^{jwt}\,, \tag{3.112}$$

or

$$RCjwH(jw)e^{jwt}+H(jw)e^{jwt}=e^{jwt}\,, \tag{3.113}$$

Then we have $H(jw)=\frac{1}{1+RCjw}$. (3.114)

Te amplitude and frequency response $H(jw)$ is shown in the figure below.



We can also get the impulse response

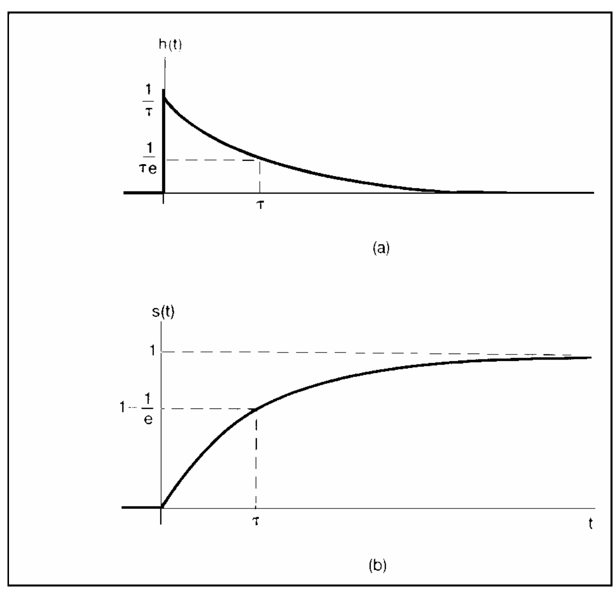
$$h(t)=\frac{1}{RC}e^{-t/RC}u(t), \quad (3.115)$$

and the step response is

$$h(t)=(1-e^{-t/RC})u(t), \quad (3.116)$$

The fundamental trade-off can be found by comparing the figures:

- To pass only very low frequencies, $1/RC$ should be small, or RC should be large.
- To have fast step response, we need a smaller RC.
- The type of trade-off between behaviors in the frequency domain and time domain is typical of the issues arising in the design analysis of LTI systems.



3.10.2 A Simple RC Highpass Filter

If we choose the output from the resistor, then we get an RC highpass filter.

3.11 Examples of Discrete-Time Filter Described by Difference Equations

A discrete-time LTI system described by the first-order difference equation

$$y[n]-ay[n-1]=x[n]\ . \qquad (3.116)$$

Form the eigenfunction property of complex exponential signals, if $x[n]=e^{jwn}$, then $y[n]=H(e^{jw})e^{jwn}$, where $H(e^{jw})$ is the frequency response of the system.

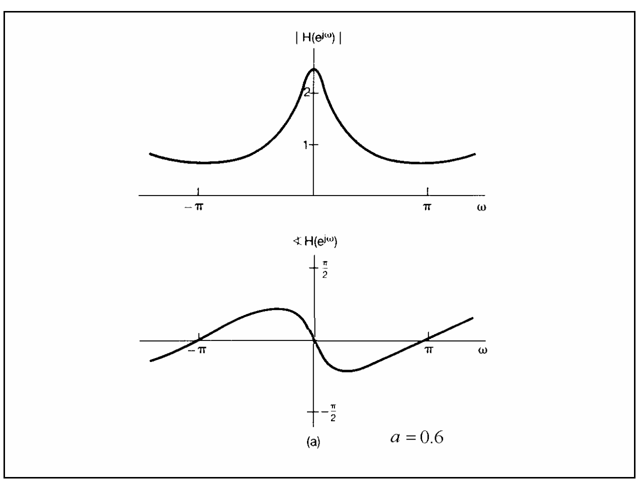
$$H(e^{jw})=\frac{1}{1-ae^{-jw}}\ . \qquad (3.117)$$

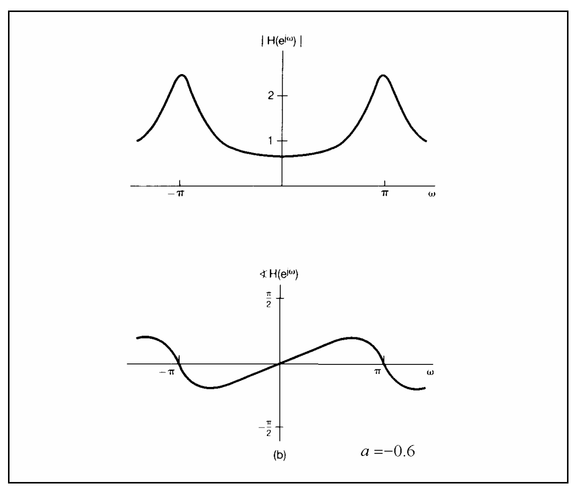
The impulse response of the system is

$$x[n]=a^{n}u[n]. \qquad (3.118)$$

The step response is

$$s[n]=\frac{1-a^{n+1}}{1-a}u[n]. \qquad (3.119)$$





From the above plots we can see that for $a=0.6$ the system acts as a Lowpass filter and $a=-0.6$, the system is a highpass filter. In fact, for any positive value of $a<1$, the system approximates a highpass filter, and for any negative value of $a>-1$, the system approximates a highpass filter, where $|a|$ controls the size of bandpass, with broader pass bands as $|a|$ in decreased.

The trade-off between time domain and frequency domain characteristics, as discussed in continuous time, also exists in the discrete-time systems.

3.11.2.2 Nonrecursive Discrete-Time Filters

The general form of an FIR norecursive difference equation is

$$y[n] = \sum_{k=-N}^{M} b_k x[n-k]. \qquad (3.120)$$

It is a weighted average of the $(\mathrm{N}+\mathrm{M}+1)$ values of $x[n]$, with the weights given by the coefficients b_k.

One frequently used example is a *moving-average filter*, where the output of $y[n]$ is an average of values of $x[n]$ in the vicinity of n_0 - the result corresponding a smooth operation or lowpass filtering.

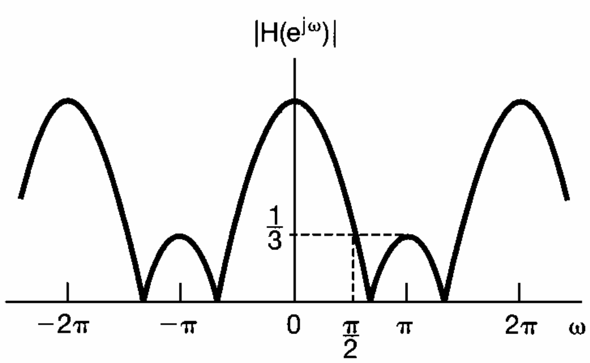
An example: $y[n] = \frac{1}{3}\left(x[n-1] + x[n] + x[n+1]\right)$. (3.121)

The impulse response is

$$h[n] = \frac{1}{3}\left(d[n-1] + d[n] + d[n+1]\right), \qquad (3.122)$$

and the frequency response

$$H(e^{jw}) = \frac{1}{3}\left(e^{jw} + 1 + e^{-jw}\right). \qquad (3.123)$$



Magnitude of the frequency response of a three-point moving-average lowpass filter.

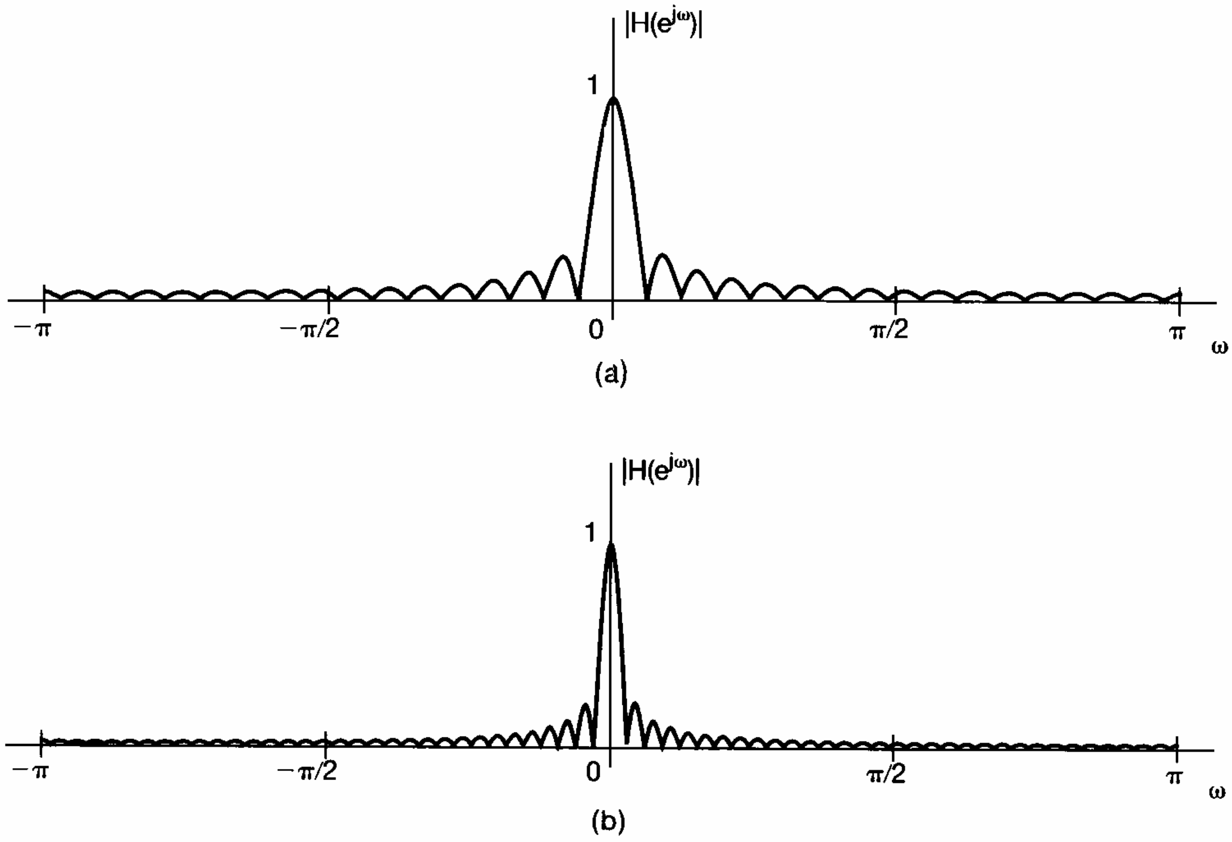
A generalized moving average filter can be expressed as

$$y[n] = \frac{1}{N+M+1}\sum_{k=-N}^{M} b_k x[n-k]\,. \tag{3.124}$$

The frequency response is

$$H(e^{jw}) = \frac{1}{M+N+1}\sum_{k=-N}^{M} e^{-jwk} = \frac{1}{M+N+1} e^{jw[(N-M)/2]} \frac{\sin[w(M+N+1)/2]}{\sin(w/2)}\,. \tag{3.125}$$

The frequency responses with different average window lengths are plotted in the figure below.



Magnitude of the frequency response for the lowpass moving-average filter of eq. (3.162): (a) $M = N = 16$; (b) $M = N = 32$.

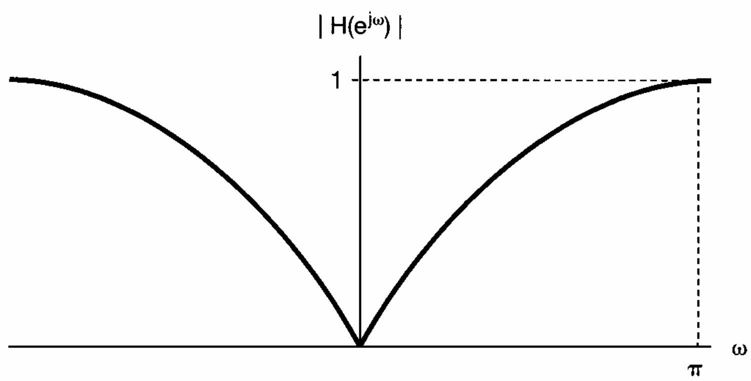
FIR norecursive highpass filter

An example of FIR norecursive highpass filter is

$$y[n] = \frac{x[n] - x[n-1]}{2}. \quad (3.126)$$

The frequency response is

$$H(e^{jw}) = \frac{1}{2}\left(1 - e^{-jw}\right) = je^{jw/2}\sin(w/2). \quad (3.127)$$



Frequency response of a simple highpass filter.