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Radial View of Continuous Cellular Automata *

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Abstract. Continuous cellular automata (or coupled map lattices) are cellular automata where the state of the cells are real values in $[0, 1]$ and the local transition rule is a real function.

The classical observation medium for cellular automata, whether Boolean or continuous, is the space-time diagram, where successive rows correspond to successive configurations in time.

In this paper we introduce a different way to visualize the evolution of continuous cellular automata called *Radial Representation* and we employ it to observe a particular class of continuous cellular automata called fuzzy cellular automata (FCA), where the local rule is the "fuzzification" of the disjunctive normal form that describes the local rule of the corresponding Boolean cellular automata.

Our new visualization method reveals interesting dynamics that are not easily observable with the space-time diagram. In particular, it allows us to detect the quick emergence of spatial correlations among cells and to observe that all circular FCA from random initial configurations appear to converge towards an asymptotic periodic behavior. We propose an empirical classification of FCA based on the length of the observed periodic behavior: interestingly, all the minimum periods that we observe are of lengths one, two, four, or n (where n is the size of a configuration).

Keywords: Continuous Cellular Automata, Fuzzy Cellular Automata, Visualization of dynamics, Evolution, Asymptotic Behavior.

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1. Introduction

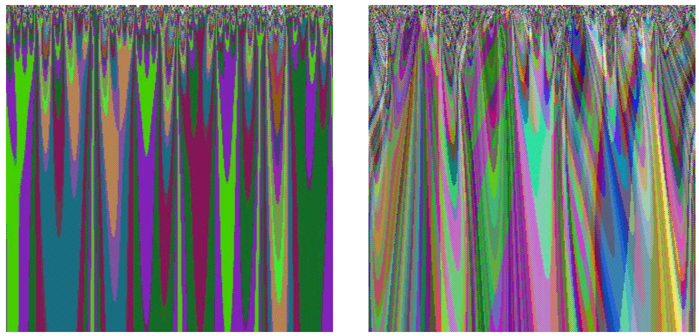
Boolean cellular automata have been introduced by Von Neumann as models of self-organizing and reproducing behaviors [22]. The study of their properties and their behavior has interested various disciplines, as diverse as ecology, biology, and theoretical computer science (e.g., [2, 10, 14, 25]). A one dimensional Boolean cellular automaton (CA) is constituted by a collection of cells arranged in an array. Each cell has a state in $\{0, 1\}$ which changes at successive discrete steps by the iteration of a local transition function that depends on the states of the "neighbouring" cells. The global evolution of a CA is defined by the synchronous update of all states according to the local function applied to each cell. A configuration of the automaton is a description of all cell values at a given time. Cellular automata can be bi-infinite or finite with defined boundary conditions. A typical boundary condition consists of "wrapping around" a finite array (circular CA) or fixing a certain initial size for the array and assuming all the cells beyond the boundaries to be quiescent (zero background).

Continuous cellular automata (or Coupled Map Lattices) are cellular automata where the states of the cells are real values in $[0, 1]$, and the local transition rule is a real function. They have have been introduced by Kaneko as simple models with the features of spatiotemporal chaos, and have now applications in many different areas like fluid dynamics, biology, chemistry, etc. (for the definition of couple map lattices and their applications see [12, 13, 23]). A particular type of continuous cellular automata are *Fuzzy cellular automata* (FCA) where the local transition rule is obtained by "fuzzifying" the disjunctive normal form that describes the local function of the corresponding Boolean CA [1]. In other words, a FCA can be seen as a continuous generalization of a Boolean CA, where the behavior at the extreme of the interval $[0, 1]$ is exactly the same as the behavior of the corresponding Boolean CA; vice-versa, a boolean CA can be considered a discretized version of the corresponding FCA.

Fuzzy cellular automata have been introduced in [3, 4] as a framework for understanding complex behaviors. They have been employed for studying the impact that perturbations (e.g. noisy sources, computation errors, mutations, etc.) can have on the evolution of Boolean CA (e.g., see [9]), they have also been studied in relation to pattern recognition (e.g., see [15, 16]) and as a model to generate interesting images mimicking nature (e.g. [5, 20]). Moreover, dynamical properties of Fuzzy CA rules have been analytically studied (e.g., in [8, 18, 19]) to understand the relationship with Boolean cellular automata and the impact of discretization.

The analytical study of these systems is generally complex; as a consequence visualization plays an important role in the understanding of their dynamics. Space-time diagrams have been traditionally used to display the dynamics of one dimensional CA: the top-most row corresponding to the initial configuration at time $t = 0$ and rows corresponding to configurations at successive time steps (see, for example the space-time diagram of two FCA in Figure 1). When visualizing the space-time diagram of continuous CAs, and in particular of FCAs, the interval [0,1] must be discretized, since only a finite number of states can obviously be represented. The interval is divided in k ranges and each is assigned a different colour. It has been shown in [8] that this discretization process could mislead the observer by showing a totally incorrect dynamics. This is the case, for example, of FCA rule 90 where depending on the parity of k, totally different behaviors are displayed. In fact, if the fixed point ($\frac{1}{2}$) happens to be the extreme of a range, the diagram shows an alternation between the two different values around the fixed point; on the other hand, if the fixed point belongs to a range, the diagram shows a quick convergence to

[1] not to be confused with a variant of cellular automata, also called Fuzzy cellular automata, where the fuzziness refers to the fuzzy choice of a deterministic local rule (e.g., see [1])



(a) FCA 18 (Space-Time) (b) FCA 184 (Space-Time)

Figure 1. The first 300 iteration of two Fuzzy Cellular Automata displayed with the space-time diagram.

an homogeneous configuration. There is another aspect of space-time diagrams that could mislead the observer: colours can show nice patterns, which allow the observer to differentiate between an extremely simple dynamics and a more complex one: they however do not give insights into the way a converging rule behaves towards its attractor.

In this paper we propose a new way of visualizing continuous CAs (*Radial Representation*) and we show that by observing their dynamics using this representation we can detect properties of their behavior that were hidden in the space-time diagram. In particular, we observe circular elementary FCA; the radial representation allows us to discover very quickly spatial correlations between neighbouring cells, which suggest the main features of the asymptotic behavior of the rule. Based on this new visualization point of view, we classify their observable behavior. We find that all FCA appear to converge towards asymptotic periodic behaviors: the only lengths of the periods that we have observed are one, two, four, and n.

The paper is organized as follows: in Section 2 we give definitions and introduce some terminology, Section 3 briefly outlines some of the existing classifications of cellular automata. In Section 4 we introduce a new visualization method; in Section 5 we propose a classification based on the lengths of the periods displayed by the various rules.

2. Notations and Preliminaries

2.1. Definitions

A one dimensional bi-infinite Boolean cellular automaton is a collection of cells arranged in a linear bi-infinite lattice. Cells have a Boolean values and they synchronously update their values according to a local rule applied to their neighborhood. A configuration $X = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots)$ is a description of all cell values at a given time. In the case of *elementary cellular automata* the neighborhood of a cell consists of the cell itself and its left and right neighbours, thus the local rule has the form: $g : \{0,1\}^3 \to \{0,1\}$. The global dynamics of an elementary one dimensional cellular automata is then defined by:

$$B : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}} \quad s.t. \quad \forall X \in \{0,1\}^{\mathbb{Z}}, \forall i \in \mathbb{Z}, B(X)_i = g(x_{i-1}, x_i, x_{i+1})$$

A cellular automata is *circular* if the lattice is finite and the last cell is considered to be neighbour of the first. In this case the global function has the following form, where the size of the lattice is n and the operations on the indices are modulo n:

$$B : \{0,1\}^n \to \{0,1\}^n \quad s.t. \quad \forall X \in \{0,1\}^n, \forall i \in \{0,\ldots,n-1\}, B(X)_i = g(x_{i-1}, x_i, x_{i+1})$$

In the following we concentrate only on elementary, circular cellular automata. The local rule $g : \{0,1\}^3 \to \{0,1\}$ of an elementary Boolean CA can be also given in tabular form by listing the 8 binary tuples corresponding to the 8 possible local configurations a cell can detect in its direct neighborhood, and mapping each tuple to a binary value q_i ($0 \le i \le 7$):

$$(000, 001, 010, 011, 100, 101, 110, 111) \to (q_0, \cdots, q_7).$$

The binary representation $(q_0, \cdots, q_7)$ is often converted into the decimal representation $\sum_{i=1:8} 2^{i-1} q_i$, and this value is used as the name of the rule. The local rule can also be canonically expressed as a *disjunctive normal form*:

$$g(v_1, v_2, v_3) = \bigvee_{i|q_i=1} \bigwedge_{j=1:3} v_j^{d_{ij}}$$

where d_{ij} is the j-th digit, from left to right, of the binary expression of i, and v_j^0 (resp. v_j^1) stands for $\neg v_j$ (resp. v_j). For example, the canonical expression of rule $18 = 2 + 16$, which is expressed in tabular form as: $(000, 001, 010, 011, 100, 101, 110, 111) \to (0, 1, 0, 0, 1, 0, 0, 0)$, is: $g(v_1, v_2, v_3) = (\neg v_1 \wedge \neg v_2 \wedge v_3) \vee (v_1 \wedge \neg v_2 \wedge \neg v_3)$.

In continuous cellular automaton (also known as Coupled Map Lattices [12]) the states assumed by the cells are real values in $[0, 1]$ and the local rule is any real function. In the case of one-dimensional, circular elementary continuous CA the local rule has the form $f : [0,1]^3 \to [0,1]$, and the corresponding global function is as follows (where the operations on the indices are modulo n):

$$F : [0,1]^n \to [0,1]^n \quad s.t. \quad \forall X \in [0,1]^n, \forall i \in \{0,\ldots,n-1\}, F(X)_i = f(x_{i-1}, x_i, x_{i+1})$$

A Fuzzy cellular automaton is a particular continuous cellular automata where the local rule is obtained by "fuzzification" of the local rule of a classical Boolean CA. The "fuzzification" consists of a fuzzy extension of the boolean operators AND, OR, and NOT. Depending on which fuzzy operators are used different types of Fuzzy cellular automata can be defined. Among the various possible choices, we consider the following: $(a \vee b)$ is replaced by $(a + b)$; $(a \wedge b)$ by (ab), and $(\neg a)$ by $(1 - a)$. The resulting local rule becomes the following real function that generalizes the original function on $\{0,1\}^3$, where $l(a, 0) = 1 - a$ and $l(a, 1) = a$:

$$f : [0,1]^3 \to [0,1] \;\; s.t., \;\; f(v_1, v_2, v_3) = \sum_{i=0\ldots7} q_i \prod_{j=1:3} l(v_j, d_{i,j})$$

The usual fuzzification of the expression $(a \vee b)$ would be $max\{1, (a + b)\}$ so as to ensure that the result is not larger than 1. Note however, that taking $(a + b)$ for the CA fuzzification does not lead to values greater than 1 since the maximum possible sum is 1 and occurs for rule 255 which contains the sum of all the expressions $(000, 001, 010, 011, 100, 101, 110, 111) \to (1, 1, 1, 1, 1, 1, 1, 1)$; any other CA rule is a partial sum and must be bounded by 1.

Example 1: *Fuzzification.*
Consider, for example, rule 18 whose canonical Boolean expression is: $g(x, y, z) = (\neg x \wedge \neg y \wedge z) \vee (x \wedge \neg y \wedge \neg z)$. The fuzzification process for the corresponding FCA yields: $f(x, y, z) = [(1-x) \cdot (1-y) \cdot z] + [x \cdot (1-y) \cdot (1-z)] = (1-y) \cdot (x + z - 2yz)$.

A *configuration* of a fuzzy cellular automata at a given time consists of the values assumed by the cells at that time: we shall indicate with $X^0 = (x_0^0, x_1^0, \ldots, x_{n-1}^0)$ an initial configuration, and with X^t a configuration at time t.

Definition 2.1. A *fixed point* $P \in [0,1]^N$ for a FCA with global rule $F : [0,1]^n \to [0,1]^n$ is a configuration P such that $F(P) = P$.

Definition 2.2. A rule has a *(right) shift* behavior if there is a time T such that $\forall t > T$, $F(X^{t+1}) = F(x_0, x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{n-1}, x_0)$.

Definition 2.3. A rule is *Temporally Periodic* with period p if $\forall \epsilon > 0$, $\exists T$ such that $\forall t > T$:

$$F(X^t) = F(X^{t+p})$$

Definition 2.4. A rule is *Spatially Periodic* with period p if $\forall \epsilon$, $\exists T$ such that $\forall t > T$, $\forall i$:

$$x_i^t = x_{i+p}^t$$

Notice that a rule with shifting behavior is temporally periodic with period n; moreover, a spatially periodic rule with period p with shifting behavior is also temporally periodic with the same period p.

Definition 2.5. A rule is *Asymptotically Periodic in time* (or Asymptotically Temporally Periodic) with period p if $\forall \epsilon > 0$ $\exists T$ such that $\forall t > T$ and $\forall i$

$$|x_i^t - x_i^{t+p}| < \epsilon$$

Definition 2.6. A rule is *Asymptotically Periodic in space* (or Asymptotically Spatially Periodic) with period q if $\forall \epsilon$ $\exists T$ such that $\forall t > T$ and $\forall i$

$$|x_i^t - x_{i+q}^t| < \epsilon$$

A rule which is asymptotically periodic in space with period 1 will be called *asymptotically homogeneous*. A rule which is asymptotically periodic in time with period 1 will be said to be *convergent to a fixed point*.

In the following, when a configuration at time t is spatially periodic with smallest period $\frac{n}{m}$, we shall indicate it as $X^t = (\alpha)^m$, where $\alpha = a_1, \ldots, a_{\frac{n}{m}}$ is the sequence of values corresponding to the smallest period. For example, a configuration $X = (a, b, c, d, a, b, c, d)$ of a circular CA of size $n = 8$ is spatially periodic of minimum period four, and can be indicated as $X = (a, b, c, d)^2$.

3. Classifications of CA behaviors

The classification of cellular automata has always been considered a fundamental problem. The first attempt to classify cellular automata has been done by Wolfram in [24] where cellular automata are grouped according to the observed behavior of their space-time diagram as follows:
- Class 1: automata that evolve to a unique, homogeneous state, after a finite transient,
- Class 2: automata whose evolution leads to a set of separated simple stable or periodic structures (space-time patterns),
- Class 3: automata whose evolution leads to aperiodic ("chaotic") space-time patterns,
- Class 4: automata that evolve to complex patterns with propagative localized structures, sometimes long-lived.

Although not formally precise, this classification captures important distinctions among cellular automata. Several other criteria for grouping CA have followed: some based on observable behaviors, some on intrinsic properties of CA rules (e.g., see [6, 7, 11, 21]). Since class membership is undecidable, the observation of the evolution of a CA starting from (possibly all) initial configurations becomes crucial to understand its dynamics.

Fuzzy cellular automata have been observed using the classical space-time diagram, where colours represent intervals of real values. Fuzzy cellular automata in boolean backgrounds have been grouped according to their observed behavior in [3, 4] and, in particular, according to the level of "spread" of fuzziness. Essentially the possible observed behaviors are three: 1) Boolean values are destroyed by Fuzzy values; 2) Fuzzy values are destroyed by Boolean values; 3) Fuzzy and Boolean values co-exist forming various patterns in the space-time diagram. Circular Fuzzy cellular automata have never been classified; from the observation of their space-time diagram it is clear that some evolve to a fixed-point, some present a shifting behavior, while some display "nice" patterns (for some examples of patterns see Figures 1).

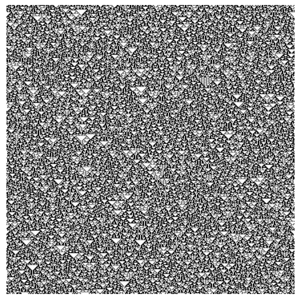
Interestingly, the behavior of Fuzzy cellular automata does not follow the behavior of their boolean counter-part: in fact, rules belonging to the same Wolfram-class can have, in some cases, an observable dynamics very different in the Fuzzy version (see Figure 2). In [8] the reasons for such differences are analytically explained for the case of rule 90.

Because of the complex patterns generated by some Fuzzy cellular automata in the space-time diagram, some rules were thought to be more complex than others, possibly having chaotic asymptotic dynamics. The only analytical study of Fuzzy cellular automata has been done by Mingarelli, who conducted a comprehensive analysis of FCA in quiescent backgrounds showing that none of them has a chaotic dynamics ([17, 18, 19]). No studies have been done for the circular case.

4. Radial Representations

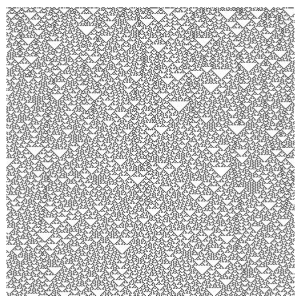
In the following we propose a new way to visualize circular continuous cellular automata.

Let k be the number of cells. Consider a unitary circle with center C divided in k equal sectors corresponding to radii $r_0, \ldots r_{k-1}$. Each cell of the CA corresponds to a radius r_i, its value x_i corresponds to distance x_i from C in r_i and is represented by a dot in the corresponding position. For example, cells with value 1 will have a dot in correspondence of the circumference, cells with value 0 will have a dot in the center, and any value between 0 and 1 will have its representational dot plotted inside the circle. A

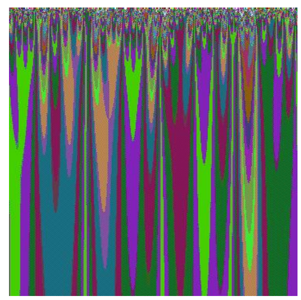


(a) Rule 90 Boolean

(b) Rule 90 Fuzzy



(c) Rule 18 Boolean



(d) Rule 18 Fuzzy

Figure 2. Two rules that display similar behavior in the Boolean setting and very different in the Fuzzy setting.

configuration of the CA then corresponds to a plotting on the circle of the various distances. When we connect the dot on radius r_i to the ones on the two neighbouring radii $r_{i\pm1}$ (the operations on indices are modulo n) we say that the Radial representation is *open*, it is *closed* otherwise. Finally, we also consider a *linear* version of the radial representation by arranging the values x_i on a line (see Figure 3 for an example of the three variants).

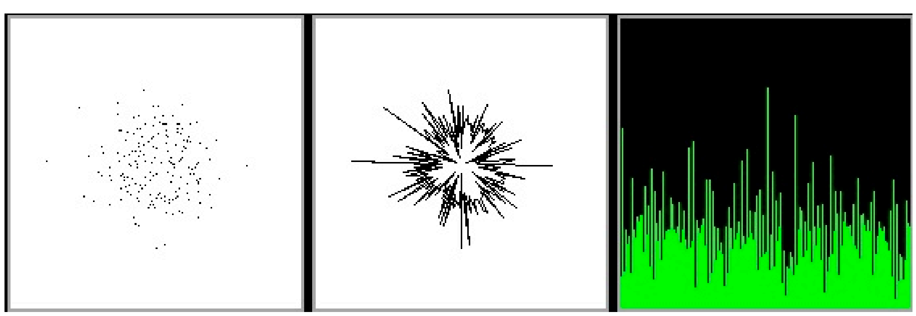
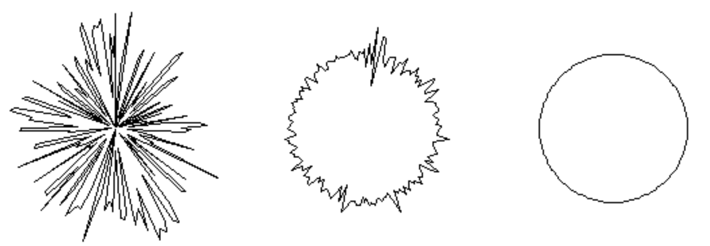


Figure 3. Radial representations (*open*, *close*, and *linear*) of a random initial configuration.

As opposed to the space-time diagram, with a radial representation we cannot observe the "life" of a CA in a 2-dimensional picture. In fact, the evolution of a CA corresponds to the time sequence of the plottings of the configurations. In other words, a radial representation provides a dynamic diagram where the observer can see the evolution through time in frames where each frame corresponds to a global configuration in an instant.

Example 3: *Radial Representations of FCA.*

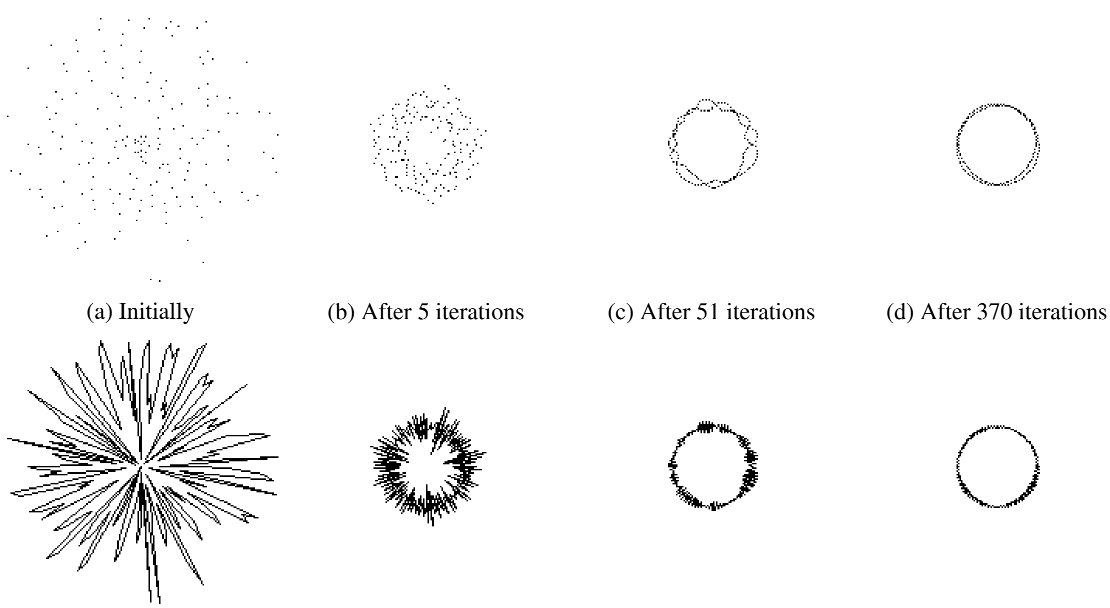
Consider FCA rule R_{110}. An example of closed radial representation is given in Figure 4, where a sequence of configurations of Rule R_{110} are displayed. In this case the rule reaches quickly to an homogeneous configuration and already after 20 iterations it is clear that all the dots corresponding to the FCA's states are placed on a circle.



(a) Initially (b) After 3 iterations (c) After 20 iterations

Figure 4. Some configurations in the evolution of rules R_{110} with the closed radial representation

Another, perhaps more interesting, example of radial representations is shown in Figure 5 where FCA rule R_{18} is displayed with both the open and the closed radial representations at different times in the evolution. It is interesting to observe a very quick formation of two curves during the evolution revealing a spatial correlation between the cells that becomes observable with the space-time diagram only after a very large number of iterations.



(a) Initially (b) After 5 iterations (c) After 51 iterations (d) After 370 iterations

Figure 5. Some configurations in the evolution of rules R_{18} with the open and closed radial representations

5. Experimental Classification of FCA using Radial Representations

In this Section we group elementary circular Fuzzy cellular automata according to their behavior when observed with the radial representations starting from random initial configuraitons. We observed all 256 elementary CAs; the results below refer to the 90 that are not equivalent under simple transformations (conjugation, reflection, and combination of conjugation and reflection). We executed each FCA starting from 50 different random initial configuration for lattices of size 45,90,180. In each initial configuration $X^0 = x_0^0, x_1^0, \ldots, x_{n-1}^0$, the states x_i^0 are randomly selected in the interval [0,1].

The observation of FCA on a computer obviously involves a discretization of the real values, and the values that we can observe are only approximations of the actual ones. Moreover, the calculations are all performed up to a certain decimal digit thus involving rounding and precision errors. In particular, we work with java doubles and truncate our observations at 16 decimal digits. Note that, due to limitations of precision, in any simulation rules that are asymptotically periodic (in time or in space) with some period p will appear to be truly periodic with period p after some initial time.

Period 1 (Quiescent)	R_0, R_8, R_{32}, R_{40}, R_{72}, R_{104}, R_{128}, R_{136} R_{160}, R_{168}, R_{24}, R_{36}, R_{152}, R_{164} , R_{44}, R_{56} R_{74}, R_{200}
Period 1 (Homogeneous)	R_6, R_9, R_{22}, R_{25}, R_{26}, R_{30}, R_{33}, R_{35}, R_{37}, R_{38}, R_{41}, R_{45}, R_{54}, R_{57} R_{60}, R_{61}, R_{62}, R_{73}, R_{90}, R_{105}, R_{106}, R_{110}, R_{122}, R_{126} R_{134}, R_{150}, R_{154} R_{172}.
Period 1 (Heterogeneous)	R_4, R_{12}, R_{13}, R_{76}, R_{77}, R_{132}, R_{140}, R_{204}, R_{232} R_{28} R_{108}, R_{156}
Period 1 for n even (Heterogeneous)	R_{78}, R_{94}
Period 2 for all n	$R_1, R_5, R_{19}, R_{23}, R_{27}, R_{50}, R_{51}R_{178}$
Period 2 for n even	R_{18}, R_{29}, R_{58}, R_{146}, R_{184}
Period 4 for n multiple of 4	R_{46}
Period n (Shifts)	R_2, R_{10}, R_{15}, R_{34}, R_{42}, R_{130}, R_{138}, R_{162},R_{170} $R_3, R_7, R_{11}, R_{14}, R_{43}, R_{142}$

Table 1. Observed dynamics of Circular Elementary Cellular Automata

In our observations, all rules eventually display periodic behaviors with periods smaller than n, which leads us to speculate that they all have an asymtotically periodic behavior. Interestingly, the lengths of the periods that we have observed are only: $p = 1$, $p = 2$, $p = 4$ and $p = n$. In the rest of the paper we group the elementary FCAs on the basis of their observed period. As mentioned before, an asymptotic behavior will necessarily be observed as an exact behavior after a certain time due to discretization, while the reverse is not necessarily true. We however conjecture that when we observe periodicity in a FCA, its asymptotic behavior is indeed periodic. In the following, whenever we say that a rule “appears to converge” or “seems to be asymtotically periodic” our statements are only based on empirical observations.

5.1. Periods of Length 1: Fixed Points

5.1.1. Homogeneous Configurations

In the majority of Fuzzy cellular automata rules the observed configurations become homogeneous in time, suggesting convergence to an homogeneous fixed point. In all cases, it can be easily shown that the homogeneous configuration reached by each rule is indeed its homogeneous fixed point, i.e., the solution of $F(P) = P$ when starting from an homogeneous configuration $X^0 = (x, x, \ldots, x)$. Fixed points for all rules starting from homogeneous configurations can be easily calculated (a table containing all of them can be found in [17]).

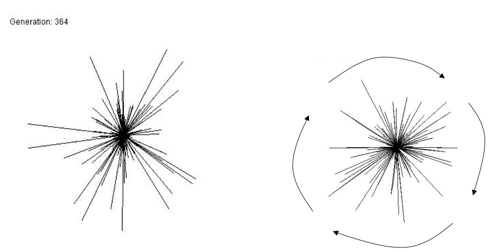
Among the rules that reach a quiescent configuration we observe that for rules R_0, R_8, R_{32}, R_{40}, R_{72}, R_{104}, R_{128}, R_{136}, R_{160}, R_{168}, R_{200} this is evident after a very small small number of iterations (less then 100 when $n = 180$), while for rules R_{24}, R_{36}, R_{152}, R_{164} it takes considerable more time for the rule to stabilize; finally, for rules R_{44} R_{56} and R_{74} the quiescent state is reached extremely slowly.

In our experiments, for several rules the dots quickly arrange in a circle; i.e., the values reach an homogeneous configuration different from zero (see an example in Figure 4). This is the case of Rules

R_6, R_9, R_{22}, R_{25}, R_{26}, R_{30}, R_{33}, R_{35}, R_{37}, R_{38}, R_{41}, R_{45}, R_{54}, R_{57}, R_{60}, R_{61}, R_{62}, R_{73}, R_{90}, R_{105}, R_{106}, R_{110}, R_{122}, R_{126} R_{134}, R_{150}, R_{154} R_{172}.

5.1.2. Non-homogeneous Configurations

In the case of rule R_{204} ($f(x, y, z) = y$), we have that any inital configuration is a fixed point ($\forall X \in [0, 1]^n$, $F(X) = X$). Other rules that appear to have an asymtotic convergence to a non-homogeneous fixed point after a few iterations are: R_4, R_{12}, R_{13}, R_{76}, R_{77}, R_{132}, R_{140}, R_{232}, R_{28} R_{108}, R_{156}.



(a) FCA 4 (Radial) (b) FCA 10 (Radial)

Figure 6. (a) a fixed point for rule R_4 observed after 364 steps (b) a shifting (i.e. rotating) configuration for rule R_{10}, observed after 364 steps.

Finally, Rules R_{94} and R_{78} merit special attention and are separately described below. Both rules appear to converge to a fixed point, which is homogeneous when the size is odd, while it consists of a periodic configuration, when the size is even. During the evolution, the radial representations show very quickly a spatial correlation between cells through the formation of two co-existing curves with consecutive values belonging to different curves, eventually stabilizing in two concentric circles (see, for Example, two moments in the evolution of Rule R_{78} in Figure 8).

	Period 1 (Fixed Points)
$f_{94}(x,y,z)$	$x+y-xy+z-2xz-yz+xyz$
n even	$(a,b)^{\frac{n}{2}}$ with $2a+2b-ab=2$
n odd	$(2-\sqrt{2})^n$
$f_{78}(x,y,z)$	$y+z-xz-yz$
n even	$(a,b)^{\frac{n}{2}}$ with $a+b=1$
n odd	$(\frac{1}{2})^n$

The Case of Rule R_{94}. Our simulations show that rule R_{94} reaches fixed points of the type $(a,b)^{\frac{n}{2}}$, with $2a+2b-ab=2$ and $a \neq b$, suggesting that it is asymptotically periodic both in time and space. The following Property shows that an exact configuration of this type (which is possibly never reached by the rule) is indeed a fixed point.

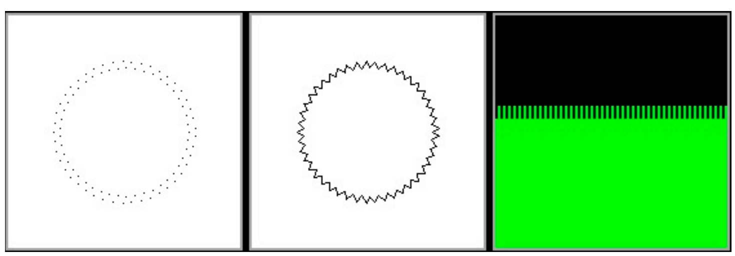


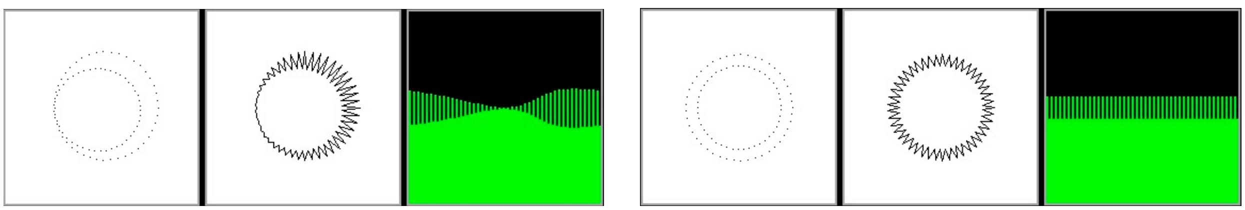
Figure 7. Rule R_{94} after 5000 iteration with 90 cells.

Property 5.1. Consider rule R_{94} on a configuration of even size n. Let $X^t = (a,b)^{\frac{n}{2}}$. Configuration X^t is a fixed point if and only if $2a+2b-ab=2$.

Proof:
Rule R_{94} has the following analytical form: $f(x,y,z) = x+y-xy+z-2xz-yz+xyz$. Let a,b,a be three consecutive values in configuration X^t; the only way to obtain a fixed point dynamic with configuration X^t is when the local function f satisfies the condition $f(a,b,a)=b$ (and $f(b,a,b)=a$). We have that $f(a,b,a) = a+b-ab+a-2a^2-ab+a^2b = 2a+b-2ab-2a^2+a^2b$, thus $f(a,b,a)=b$ when either $a=0$ or $2a+2b-ab=2$. When $2a+2b-ab=2$, we also have that $f(b,a,b)=b$ and thus this condition indeed guarantees that $(a,b)^{\frac{n}{2}}$ is a fixed point. When $a=0$ it must be $f(b,0,b)=0$, which is verified only when $b \in \{0,1\}$. It is easy to see that the homogeneous configuration $(0)^n$ is repelling, the remaining spatially 2-periodic fixed point $(0,1)^{\frac{n}{2}}$ satisfies also condition $2a+2b-ab=2$.
$\square$

On the other hand, if the size of the CA is odd, the two curves merge into a single circle suggesting asymptotic convergence to an homogeneous fixed point $(2-\sqrt{2})^n$, which is the only attracting homogeneous fixed point of rule R_{94}.

The Case of Rule R_{78}. During the evolution the behavior of the CA is shifting (i.e., the two curves rotate); however, the CA eventually reaches a spatially periodic fixed point of the form $(a,b)^{\frac{n}{2}}$, with $(a+b)=1$ and $a \neq b$, suggesting an asymptotically periodic behavior both in time and space. We can easily verify that $(a,b)^{\frac{n}{2}}$ is indeed a fixed point for rule R_{78} under the condition $(a+b)=1$.



(a) Rule 78 during the evolution

(b) Rule 78 at the convergence point

Figure 8. Rule R_{78} with 90 cells observed with Radial representations (a) after 2500 iterations (b) after 14,000 iterations.

Property 5.2. Consider rule R_{78} on a configuration of even size n. Let $X^t = (a,b)^{\frac{n}{2}}$ with $a \neq b$. Configuration X^t is a fixed point if and only if $a+b=1$.

Proof:
The proof is analogous to the one of property 5.1. Rule R_{78} has the following analytical form: $f(x,y,z) = y+z-xz-yz$. Let a,b,a be three consecutive values in configuration X^t. We have that $f(a,b,a)=b$ for $b+a-a^2-ab=b$, which implies that either $a+b=1$ or $a=0$. In the first case we also have $f(b,a,b)=a$, in the second case, we must have that $b \in \{0,1\}$. Configuration $(0)^n$ is repelling, the remaining spatially 2-periodic fixed point $(0,1)^{\frac{n}{2}}$ satisfies condition $a+b=1$ as well. □

As for the case of rule R_{94}, rule R_{78} reaches an homogeneous configuration when starting our simulations from configurations of odd size. The reached configuration $(\frac{1}{2})^n$ is the only attracting homogeneous fixed point of the rule. Thus, also in this case, the observed behavior suggests an asymptotic convergence to an homogeneous fixed point.

5.2. Periods of Length Two

Some rules show a periodic behavior of length two for any value of n (the size of the initial configuration), while some other rules display this periodicity only when n is even while they reach an homogeneous configuration when n is odd. This is the case of rules where the configuration become spatially periodic in time and the temporal periodicity is actually given by a shifting behavior.

5.2.1. Periodicity for all sizes of the initial configuration.

The simplest rule with obvious periodic behavior is the complement rule R_{51} ($f(x,y,z) = 1-y$). Other rules that quickly show a periodic behavior of length two are $R_1, R_5, R_{19}, R_{23}, R_{50}, R_{178}$. Rule R_{27} also become periodic of length two, but it takes a longer time. For this rule, each configuration in the period is homogeneous and is the complement of the previous (i.e., $X^t = (a)^n, X^{t+1} = (1-a)^n, \ldots$). This happens regardless of the parity of the initial configuration. It is indeed very simple to see that if $X^t = (a)^n$, the next configuration is its complement. In fact, it comes form the observation that, for rule R_{27}, whose analytical expression is $f(x,y,z) = 1 - y - xz + yz$ we have that $f(a,a,a) = (1-a)$.

5.2.2. Period two for configurations of even size only

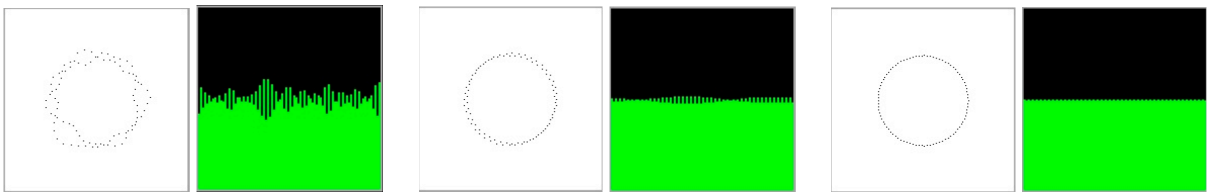
The observed periodic behavior of the rules described below (except for rule R_{29}) is actually a shifting behavior occurring on spatially periodic configurations with period two, which causes the behavior to be spatially and temporally 2-periodic. Due to this reason, this dynamics is visible only if the configuration is of even size. Interestingly, if the size is odd, all the rules reach an homogeneous fixed point. From a pictorial point of view, when the size of the configuration is even, after a few time steps they all display two co-existing curves with consecutive values belonging to different curves (see, for an example, Figure 10), and they eventually form two concentric rotating circles.

The case of rule R_{29} is similar in the sense that it also shows spatially periodic configurations of period two, when n is even, and a temporal behavior of period two; in this case, however, the shift-like behavior is slightly more complicated since a value x_i^t becomes the complement of the neighbour x_{i+1}^{t-1} at the previous time step.

	Spatially and Temporally periodic		Spatially and Temporally periodic
f_{18}	$x - xy + z - 2xz - yz + 2xyz$	f_{146}	$x - xy + z - 2xz - yz + 3xyz$
n even	$(a,b)^{\frac{n}{2}}$ with $2a + 2b - 2ab = 1$	n even	$(a,b)^{\frac{n}{2}}$ with $2a + 2b - 3ab = 1$
n odd	$(1 - \frac{\sqrt{2}}{2})$	n odd	$(\frac{1}{3})^n$
f_{58}	$x - xy + z - xz$	f_{184}	$x - xy + yz$
n even	$(a,b)^{\frac{n}{2}}$ with $a + b = 1$	n even	$(a,b)^{\frac{n}{2}}$, $\forall a,b$
n odd	$(\frac{1}{2})^n$	n odd	$(a)^n, \forall a$

	Shift of Complement
f_{29}	$1 - z + yz - xy$
n even	$(a,b)^{\frac{n}{2}}$ $\forall a,b$
n odd	$(a)^n$, $\forall a$

The Case of Rules R_{18}, R_{146}, R_{58}. When the configuration size of is even, these fuzzy cellular automata form two co-existing rotating curves that eventually form two concentric circles with consecutive values belonging to different curves. In other words, the observed configurations become of the form $(a,b)^{\frac{n}{2}}$ with $a \neq b$ (alternating with $(b,a)^{\frac{n}{2}}$), thus suggesting an asymptotic periodic behavior both in space and time (see, for example, three different moments in the evolution of Rule R_{58}). We can show that, for these rules, an exact configuration $X^t = (a,b)^{\frac{n}{2}}$ has indeed a shifting behavior (and thus a periodic behavior of length two) for a certain relationship between a and b, which depends on the rule.



(a) (b) (c) (d) (e) (f)

Figure 9. Three moments in the evolution of Rule R_{58} with the Radial and the Bar representations.

For example, in the case of rule R_{18} the observed configurations become of the form $(a,b)^{\frac{n}{2}}$ with $a+b-ab = \frac{1}{2}$; indeed the following property shows that for rule R_{18}, $X^t = (a,b)^{\frac{n}{2}}$ has a shifting behavior (and thus a periodic behavior of length two) only when $a+b-ab = \frac{1}{2}$ or $X^t = (0,1)^{\frac{n}{2}}$, analogous properties can be shown for the other rules of this class.

Property 5.3. Consider rule R_{18} on a configuration of even size n. Let $X^t = (a,b)^{\frac{n}{2}}$ with $a \neq b$. Configuration $X^{t+1} = (b,a)^{\frac{n}{2}}$ if and only if $a+b-ab = \frac{1}{2}$ or $a=0, b=1$.

Proof:
Rule R_{18} has the following analytical form: $f(x,y,z) = x - xy + z - 2xz - yz + 2xyz$. Let a,b,a be three consecutive values in configuration X^t; in order for the global dynamics to be shifting, the local rule f must verify the condition $f(a,b,a) = a$ (and $f(b,a,b) = b$). We have that $f(a,b,a) = 2a - 2ab - 2a^2 + 2a^2b$, which means that it must be either $a+b-ab = \frac{1}{2}$ or $a = 0$. Condition $a+b-ab = \frac{1}{2}$ verifies also $f(b,a,b) = b$. If $a = 0$, solving $f(b,0,b)$ we have that b must be either 0 or 1. Configuration $(0)^n$ is a repelling homogeneous fixed point, so the only two spatially 2-periodic shifting configurations are $(a,b)^{\frac{n}{2}}$ with $a+b-ab = \frac{1}{2}$, and $(0,1)^{\frac{n}{2}}$. □

Although configuration $(0,1)^{\frac{n}{2}}$ is also a fixed point for rule R_{18}, we never observe it in our experiments from random initial configurations, which always appear to converge to $(a,b)^{\frac{n}{2}}$ for some a and b such that $a+b-ab = \frac{1}{2}$.

Analogously to the case of rule R_{18}, it can be shown that a shifting configuration of the form $(a,b)^{\frac{n}{2}}$ must verify condition $2a+2b-3ab = 1$ for rule R_{146}, and condition $a+b = 1$ for rule R_{58}.

When the configuration size is odd, the radial diagram of these rules still show two co-existing curves with consecutive values belonging to different curves in all places except for one. The two curves eventually merge into one and the rule reaches an homogeneous configuration. Thus, our experiments starting from random configurations suggest asymptotic behavior that becomes homogeneous in space and converges to a fixed point in time: rule R_{18} appears to asymptotically converge to $(1-\frac{\sqrt{2}}{2})^n$, rule R_{146} to $(\frac{1}{3})^n$, and rule R_{58} to $(\frac{1}{2})^n$. It can be easily shown that these are the only attracting homogeneous fixed points for these rules.

The Case of Rule R_{184}. Rule R_{184} has the following analytical form: $f(x,y,z) = x - xy + yz$. This rule is special in many ways. First of all, when starting from homogeneous configurations, any configuration is a fixed point (this is due to the fact that $f(a,a,a) = a, \forall a$). Furthermore, any 2-periodic configuration gives rise to a periodic behavior (this is due to the fact that $f(a,b,a) = a, \forall a, b$).

As before, when the size of the CA is even, the radial diagram shows 2 co-existing curves with consecutive values belonging to different curves thus reaching a spatially and temporally 2-periodic configuration $(a,b)^{\frac{n}{2}}$ (i.e., to two rotating concentric circles, as shown in Figure 10). However, the values of a and b are not linked to each other. From the experiments it appears that the CA can be attracted, depending on the initial configuration to various shifting configuration of the type $(a,b)^{\frac{n}{2}}$.

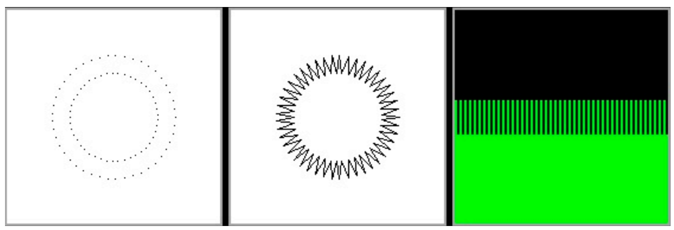


Figure 10. Rule R_{184} after 5500 iteration with a configuration of 90 cells.

When the size of the FCA is odd the rule reaches, in our simulations, an homogeneous configuration $(a)^n$, suggesting an asymptotic behavior that converges to a fixed point in time and becomes homogeneous in space. The value of a is variable and appears to depend on the initial configuration.

The Case of Rule R_{29}. Also this rule has a unique behavior. When the size of the FCA is even, the observed rule reaches a spatially 2-periodic configuration of the type $(a,b)^{\frac{n}{2}}$ and then alternates with the shift of its complement $(1-b, 1-a)^{\frac{n}{2}}$ still giving rise to a period of length 2. Rule R_{29} has the following analytical form: $f(x,y,z) = 1 - xy - z + yz$, which means that $f(a,b,a) = 1 - a$ for any values of a and b. As a consequence $(a,b)^{\frac{n}{2}}$ is clearly a spatial and temporal period-2 configuration.

When the size of theF CA is odd the rule reaches a periodic behavior of the type $X^t = (a)^n, X^{t+1} = (1-a)^n, \ldots$. The value of a varies and depends on the initial configuration.

5.3. Periods of Length 4: Rule R_{46}

There is only one rule in this class: rule R_{46}. During the evolution, we observe the formation of four co-existing rotating curves that eventually form four concentric circles (see Figure 11). Similarly to the cases of the rules of the previous subsection, the periodic behavior is due to the shift in time of spatially 4-periodic configurations of the type $(a, b, 1-a, 1-b)^{\frac{n}{4}}$, thus the requirement of configurations with size multiple of 4.

	Spatially periodic Shift
$f_{46}(x,y,z)$	$y - xy + z - yz$
n multiple of four	$(a,b,c,d)^{\frac{n}{4}}$ with $c = 1-a$, $d = 1-b$
n not multiple of four	$(\frac{1}{2})^n$

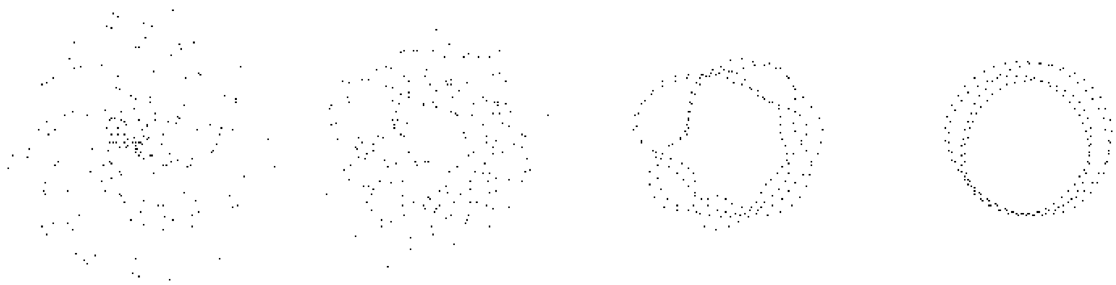


Figure 11. Some configurations in the evolution of rules R_{46} with the Radial representation.

In the following we show that, when the size of the configuration is multiple of four, the four-periodic configuration $(a, b, c, d)^{\frac{n}{4}}$ is a shifting configuration (and thus resulting in a periodic behavior of length four) only when $c = 1 - a$ and $d = 1 - b$).

Property 5.4. Consider rule R_{46} on a configuration of size $n = 4m$. Let $X^t = (a, b, c, d)^m$ with $a \neq b$, then $X^{t+4} = X^t$ iff $c = 1 - a$ and $d = 1 - b$.

Proof:
The analytical form of rule R_{46} is $f(x, y, z) = y - xy + z - yz$. In order for the configuration to be of the form $(a, b, c, d)^m$, we must have that: $f(a, b, c) = c$, $f(b, c, d) = d$ and $f(c, d, a) = a$. Solving the equations, we have that $b - ab + c - bc = c$ for $b = 0$ or $a + c = 1$; $c - bc + d - cd = d$ for $c = 0$ or $b + d - 1$; $d - cd + a - da = a$ for $d = 0$ or $c + a = 1$. Thus, the only spatially 4-periodic shifting configuration (with $a \neq b$) is $X^t = (a, b, 1 - a, 1 - b)^m$. □

When the configuration size is even but not a multiple of four or when it is odd, we observe that the CA reaches (very slowly) its only homogeneous fixed point $(1/2)^n$, suggesting an asymptotic convergent behavior (both in time and in space).

5.4. Periods of Length n: Shifts and Double Alternating Shifts

Rules R_{170} ($f(x, y, z) = z$) is the perfect shift and the radial representation shows a perfect rotation of the initial configuration. A shifting behavior after a short transient is displayed also by rules R_2, R_{10}, R_{34}, R_{42}, R_{130}, R_{138}, R_{162} (Stars). An example of shifting configuration for rule R_{10} is shown in Figure 6, where the configuration rotates as indicated by the arrows.

Rule R_{15} ($f(x, y, z) = (1 - x)$) is the complement of a perfect shift: since every configuration is the complement of the shift of the previous, we observe what we call an *Double alternating Shift* or a shifting periodic behavior: every other configuration is shifted by two positions[2]. The radial representation clearly shows both the periodic and the shifting nature of the rule. Similar but more complex behavior have rules $R_3, R_7, R_{11}, R_{14}, R_{43}, R_{142}$. In these cases, depending on the initial configuration, and on its size, the dynamics could differ. Consider, for example, the case of rule R_{43}. For large values of n the

[2]We have seen another double alternating shift, Rule R_{29}, in subsection 5.2.2.

rule appears to converge always to a periodic behavior of period n from a random initial configuration; however, for smaller values of n, under certain circumstances, it could actually become totally Boolean stabilizing in a Boolean shifting spatially periodic configuration $(1100)^{\frac{n}{4}}$, thus resulting in a temporal periodic behavior of period four. The detailed analysis of this class of rules is left for subsequent research.

6. Concluding Remarks

In this paper we have described an experimental classification of circular fuzzy cellular automata based on a new visualization method. The different visualization method has allowed us to observe dynamics that were not easily detectable with the classical space-time diagram. In particular, it has allowed us to see the quick emergence of spatial correlation between cells that was not evident from the space time diagram (compare, for example, snapshots of configurations during the evolution of some rules observed with the classical space-time diagram, and with the Radial representations in Figure 12).

We have observed that all circular elementary FCA from random initial configurations of size n display at some point a periodic dynamics with period $p \leq n$, and we have grouped them on the basis of the length of their periods. Interestingly, the only observed periods lengths are one, two, four, and n. In the most interesting cases we have also analytically verified that the configurations reached by the FCA in our experiments are indeed periodic points. The empirical observations lead us to conjecture that all circular elementary rules have an asymptotic periodic behavior of the observed length.

Several problems are now under investigation. First of all we are now conducting a detail study of each rule to analytically show that the periodic behavior that we observe are indeed asymptotic periodic behaviors and not simply due to discretization. We are also investigating the reasons for FCA to have these particular period lengths. The classifications of infinite configurations and configurations in zero backgrounds are also under investigation. Finally, radial representation has been employed in this paper to observe FCA only; it would be interesting to see if it can be useful in the observation of other types of continuous cellular automata

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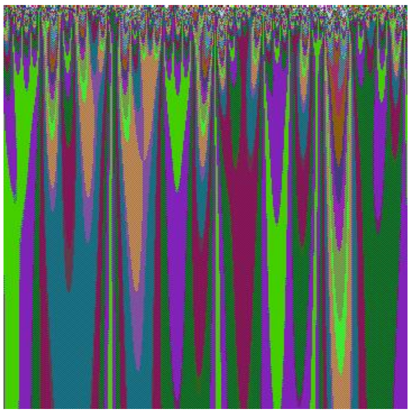
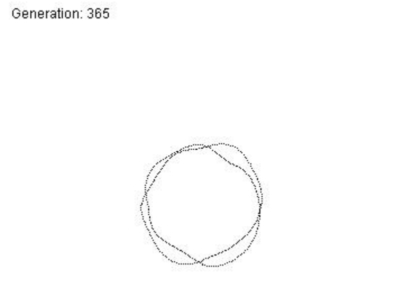
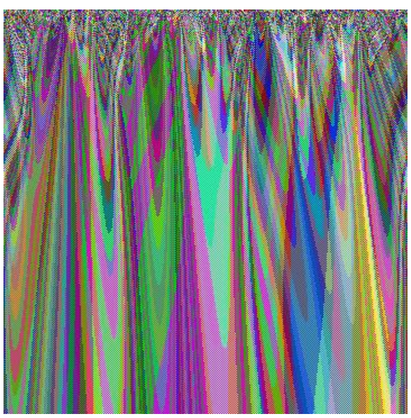


(a) Rule 18 Space-Time

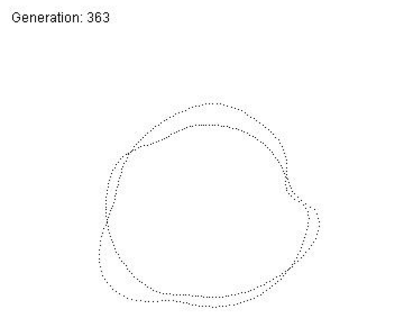


(b) Rule 18 Open Radial

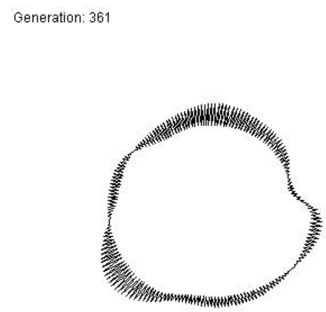
(c) Rule 18 Closed Radial



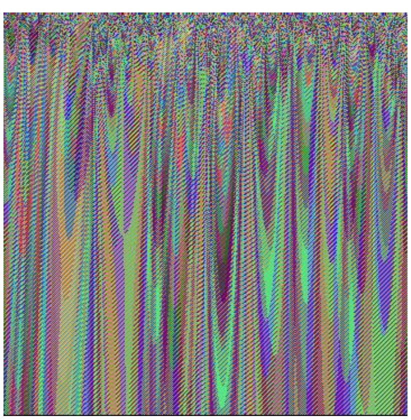
(d) Rule 184 Space-Time



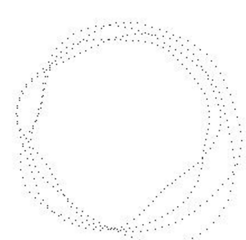
(e) Rule 184 Open Radial



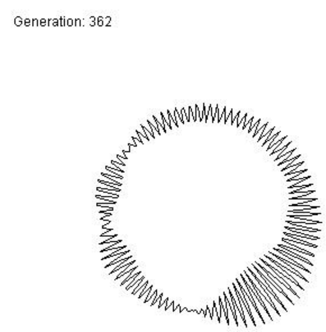
(f) Rule 184 Closed Radial



(g) Rule 46 Space-Time



(h) Rule 46 Open Radial



(i) Rule 46 Closed Radial

Figure 12. Rules R_{18}, R_{184}, and R_{46} visualized with the classical space-time diagram in (a) (d) (g), and a configurations in their evolution visualized with the Radial representations. In all cases, the configuration has 180 cells.