ELG 5377 Midterm 1999
Solutions

1. a) $r_{v_1}(k) = 2.5\,\delta(k) \qquad S_{v_1}(z) = S_{v_1}(e^{j\omega}) = 2.5$

$r_{v_2}(k) = 6.0\,\delta(k) \qquad S_{v_2}(z) = S_{v_2}(e^{j\omega}) = 6.0$

4 $S_u(z) = S_{v_1}(z)\,H_1(z)\,H_1^*(1/z^*) = \dfrac{2.5}{(1+.5z^{-1})(1+.5z)}$

$$S_u(e^{j\omega}) = \frac{2.5}{(1+.5e^{-j\omega})(1+.5e^{j\omega})} = \frac{5}{2.5+2\cos\omega}$$

For $k \geq 0$, $r_u(k) = \mathrm{Res}\left[S_u(z)z^{k-1},\ z=-.5\right]$

$$= \left.\frac{2.5z^k}{1+.5z}\right|_{z=-.5} = \frac{10}{3}(-.5)^k$$

$r_u(-k) = r_u(k)$, so $r_u(k) = \frac{10}{3}(-.5)^{|k|} \quad -\infty < k < \infty$.

Since v_2 is uncorrelated with v_1, it is uncorrelated with u.

$\therefore\ r_x(k) = r_u(k) + r_{v_2}(k) = \frac{10}{3}(-.5)^{|k|} + 6.0\,\delta(k)$

$$S_x(z) = S_u(z) + S_{v_2}(z) = \frac{5z}{z^2+2.5z+1} + 6 = \frac{6z^2+20z+6}{z^2+2.5z+1}$$

$$S_x(e^{j\omega}) = \frac{6e^{2j\omega}+20e^{j\omega}+6}{e^{2j\omega}+2.5e^{j\omega}+1} = \frac{20+12\cos\omega}{2.5+2\cos\omega}$$

(b) The poles of $S_x(z)$ are -0.5 and -2.0.

4 The zeros of $S_x(z)$ are the roots of $6z^2+20z+6=0$, i.e. $-1/3$ and -3.

We can write $6z^2+20z+6 = 6(z+1/3)(z+3)$

Thus

$$S_x(z) = \frac{6(z+\frac{1}{3})(z+3)}{(z+\frac{1}{2})(z+2)} = \frac{9(1+\frac{1}{3}z^{-1})(1+\frac{1}{3}z)}{(1+\frac{1}{2}z^{-1})(1+\frac{1}{2}z)}$$

$$= H_2(z)\,H_2^*(1/z^*)$$

$$= H_2(z)\,H_2(1/z) \quad \text{since signals and filters are real.}$$

Thus, $H_2(z) = \dfrac{3(1+\frac{1}{3}z^{-1})}{(1+\frac{1}{2}z^{-1})}$ pole & zero inside unit circle $\Rightarrow$ minimum phase.

Thus $x(n)$ is modeled as white noise passed through a filter with one pole and one zero, and so is ARMA(1,1).

$$x(n) + \tfrac{1}{2}x(n-1) = 3e(n) + e(n-1)$$

or $$x(n) = -\tfrac{1}{2}x(n-1) + 3e(n) + e(n-1)$$

5 2. Let S be the space of random variables on the sample space underlying the random process $\{U(n)\}$. The inner product is given by $\langle U_1 | U_2 \rangle = E[U_1 U_2^*]$.
By the Cauchy Schwartz inequality

$$|r_u(k)|^2 = |\langle U(n) | U(n-k)\rangle|^2 \leq \langle U(n) | U(n)\rangle \langle U(n-k) | U(n-k)\rangle$$

$$= r_u(0)\, r_u(0) = r_u^2(0) \in \mathbb{R}$$

$$\therefore |r_u(k)| \leq r_u(0)$$

3 3. (a) $U_1(n)$ & $U_2(n)$ are random variables defined on a sample space Ω. Let S be the set of all random variables on Ω of finite variance. Let $\mathcal{M} = \text{span}(U_2(n))$. If U & $V \in S$ $\langle U | V\rangle = E[UV^*]$.
Then $\hat{U}_1(n)$ is the projection of $U_1(n)$ on $\mathcal{M}$.

4 (b) $\hat{U}_1(n) = w_0 U_2(n)$

$$U_1(n) - w_0 U_2(n) \perp U_2(n)$$

$$\langle U_1(n) - w_0 U_2(n) | U_2(n)\rangle = 0$$

$$\langle U_1(n) | U_2(n)\rangle = w_0 \langle U_2(n) | U_2(n)\rangle$$

$$w_0 = \frac{r_{u_1 u_2}(0)}{r_{u_2}(0)}$$

$$MSE = \| U_1(n) - w_0 U_2(n)\|^2 = \langle U_1(n) | U_1(n) - w_0 U_2(n)\rangle$$

$$= \|U_1(n)\|^2 - w_0^* \langle U_1(n) | U_2(n)\rangle = r_{u_1}(0) - \frac{r^*_{u_1 u_2}(0)}{r_{u_2}(0)}\, r_{u_1 u_2}(0)$$

$$= r_{u_1}(0) - |r_{u_1 u_2}(0)|^2 / r_{u_2}(0)$$