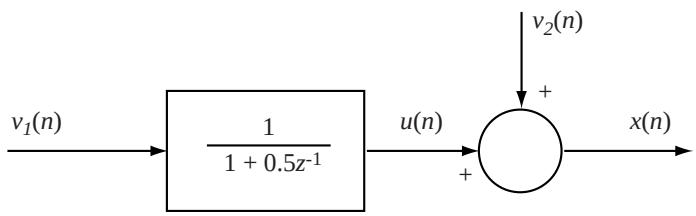
ELG5377 Adaptive Signal Processing
Mid-term exam

Date: Mar. 1, 1999
Time: 11:30-12:50
Professor: E. Dubois

Closed-book exam: you may not use any books or notes. Explain all calculations; I am more interested in the reasoning than in precise numerical answers. Unless otherwise specified, you may use the results provided on page 3 without proof if you wish.
Vous pouvez répondre en anglais ou en français.

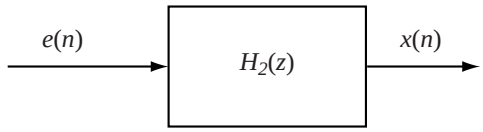
---

1. A real-valued, zero-mean, wide-sense-stationary white noise process $v_1(n)$ with variance σ_1^2 is passed through a linear shift-invariant filter with transfer function $H_1(z) = 1/(1 + 0.5z^{-1})$, producing the process $u(n)$. A second zero-mean white-noise process $v_2(n)$, uncorrelated with $v_1(n)$, of variance σ_2^2 is added to $u(n)$ to produce $x(n)$. The situation is illustrated below. We suppose that $\sigma_1^2 = 2.5$ and $\sigma_2^2 = 6.0$.



4 (a) Determine the autocorrelation functions $r_{\cdot}(k)$, the complex spectral density functions $S_{\cdot}(z)$ and the power spectral density $S_{\cdot}(e^{j\omega})$ for the four processes $v_1(n)$, $v_2(n)$, $u(n)$ and $x(n)$.

4 (b) Show that $x(n)$ can be considered to be an ARMA(1,1) process generated as shown below, where $e(n)$ is a zero-mean white noise process with variance $\sigma_e^2 = 1.0$ and $H_2(z)$ is a minimum-phase filter. Specifically, show that $H_2(z) = 3(1 + \frac{1}{3}z^{-1})/(1 + \frac{1}{2}z^{-1})$. What is the difference equation relating $x(n)$ and $e(n)$.



5 2. Let $\{U(n)\}$ be a zero-mean, wide-sense-stationary, complex-valued discrete-time random process. Use the Cauchy-Schwartz inequality to prove that $|r_u(k)| \leq r_u(0)$. Carefully justify all steps.

3. Two sensors measure two related complex signals $u_1(n)$ and $u_2(n)$. The two signals are realizations of two jointly wide-sense-stationary, zero-mean random processes with autocorrelation functions $r_{u_1}(k)$ and $r_{u_2}(k)$ and cross-correlation function $r_{u_1 u_2}(k)$. We wish to estimate $u_1(n)$ by $\widehat{u}_1(n) = w u_2(n)$, where the complex parameter w is chosen to minimize $\mathrm{E}[|U_1(n) - \widehat{U}_1(n)|^2]$.

3 (a) Show how the problem can be formulated as the projection of a vector on a subspace of a Hilbert space. Clearly specify the definitions of the Hilbert space, the subspace, the inner product and the norm.

4 (b) Apply the projection theorem to obtain the optimal coefficient w_0. What is the minimum mean-squared error? Express your result in terms of the correlation functions $r_\cdot(k)$. Justify all steps completely.

ELG5377 Adaptive Signal Processing
Results

Cauchy-Schwartz inequality: $|\langle \boldsymbol{f} \mid \boldsymbol{g} \rangle|^2 \leq \langle \boldsymbol{f} \mid \boldsymbol{f} \rangle \langle \boldsymbol{g} \mid \boldsymbol{g} \rangle$ with equality iff $\boldsymbol{f} = \alpha \boldsymbol{g}$ for some $\alpha \in \mathbb{C}$.

Projection: If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_M$ are linearly independent, then the projection of $\boldsymbol{x}$ on $\mathrm{span}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_M)$ is $\widehat{\boldsymbol{x}} = \sum_{i=1}^{M} \alpha_i \boldsymbol{u}_i$, where $\mathcal{U}\boldsymbol{\alpha}^* = \boldsymbol{p}$, $[\mathcal{U}]_{ij} = \langle \boldsymbol{u}_i \mid \boldsymbol{u}_j \rangle$ and $[\boldsymbol{p}]_i = \langle \boldsymbol{u}_i \mid \boldsymbol{x} \rangle$. $\|\boldsymbol{x} - \widehat{\boldsymbol{x}}\|^2 = \|\boldsymbol{x}\|^2 - \boldsymbol{\alpha}^T \boldsymbol{p}$.

Linear shift-invariant filtering of a wide-sense stationary random process u:
$S_y(z) = H(z)H^*(1/z^*)S_u(z)$, and $S_y(e^{j\omega}) = |H(e^{j\omega})|^2 S_u(e^{j\omega})$.

Inverse Z-transform by residues:

$$\begin{aligned} x(n) &= \frac{1}{2\pi j} \oint_{\mathcal{C}} X(z) z^{n-1}\, dz \\ &= \sum [\text{residues of } X(z)z^{n-1} \text{ at the poles inside } \mathcal{C}]. \end{aligned}$$

For a simple pole at $z = p$ inside $\mathcal{C}$, the residue of $X(z)z^{n-1}$ at p is $X(z)z^{(n-1)}(z-p)|_{z=p}$.