ELG5377 Adaptive Signal Processing

Mid-term exam answers

Date: Oct. 29, 2012

Time: 10:00-11:20

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Closed-book exam: you may not use any books or notes. Explain all calculations; I am more interested in the reasoning than in precise numerical answers. Unless otherwise specified, you may use the results provided on page 3 without proof but with an explanation.

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10 1. Let $\mathcal{S} = \mathbb{R}^N$ be the vector space of real N-tuples that we consider here to be $N \times 1$ column matrices $\mathbf{u}$, with the usual inner product $\langle \mathbf{u}_i \mid \mathbf{u}_j \rangle = \sum_{n=1}^{N} u_i(n) u_j(n) = \mathbf{u}_i^T \mathbf{u}_j$. Assume that $N = 4M$ for some integer M. Consider the subspace $\mathcal{M} = \text{span}(\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_M)$, where

$$
\begin{aligned}
\mathbf{s}_1^T &= \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \\
\mathbf{s}_2^T &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \\
&\vdots \\
\mathbf{s}_M^T &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

or explicitly

$$
s_i(n) = \begin{cases} 1 & n = 4(i-1)+1, \ldots, 4i \\ 0 & \text{otherwise} \end{cases}
$$

We want to approximate an arbitrary $\mathbf{u} \in \mathbb{R}^N$ with the closest element of $\mathcal{M}$, where the distance is defined by the norm associated with the inner product given above:

$$
\widehat{\mathbf{u}} = \sum_{j=1}^{M} a_j \mathbf{s}_j \qquad \text{where } \|\mathbf{u} - \widehat{\mathbf{u}}\| \text{ is minimized}
$$

(a) Use the projection theorem to find an expression for a_j for $j = 1, 2, \ldots, M$.

Ans: $\langle \mathbf{s}_i \mid \mathbf{s}_j \rangle = 4\delta_{ij}$, or $\mathbf{U} = 4\mathbf{I}_M$. $p_j = \langle \mathbf{s}_j \mid \mathbf{u} \rangle = u(4j-3) + u(4j-2) + u(4j-1) + u(4j)$. Thus $\mathbf{a} = \mathbf{U}^{-1}\mathbf{p} = \frac{1}{4}\mathbf{I}_M \mathbf{p} = \frac{1}{4}\mathbf{p}$, and $a_j = \frac{1}{4}(u(4j-3) + u(4j-2) + u(4j-1) + u(4j))$.

(b) What is the resulting norm squared of the error?

Ans: $\|\mathbf{u} - \widehat{\mathbf{u}}\|^2 = \|\mathbf{u}\|^2 - \sum_{j=1}^{M} p_j a_j = \|\mathbf{u}\|^2 - \frac{1}{4}\sum_{j=1}^{M} p_j^2 = \|\mathbf{u}\|^2 - \frac{1}{4}\sum_{j=1}^{M}(u(4j-3) + u(4j-2) + u(4j-1) + u(4j))^2$.

(c) Let $\mathbf{S}$ be the $N \times M$ matrix $\begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \cdots & \mathbf{s}_M \end{bmatrix}$. Show that $\mathbf{a} = (\mathbf{S}^T\mathbf{S})^{-1}\mathbf{S}^T\mathbf{u}$.

Ans:

$$\mathbf{S}^T\mathbf{S} = \begin{bmatrix} \mathbf{s}_1^T \\ \mathbf{s}_2^T \\ \vdots \\ \mathbf{s}_M^T \end{bmatrix} \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \cdots & \mathbf{s}_M \end{bmatrix}$$

$$[\mathbf{S}^T\mathbf{S}]_{ij} = \mathbf{s}_i^T\mathbf{s}_j = \langle \mathbf{s}_i \mid \mathbf{s}_j \rangle = U_{ij}$$

$$\mathbf{S}^T\mathbf{u} = \begin{bmatrix} \mathbf{s}_1^T \\ \mathbf{s}_2^T \\ \vdots \\ \mathbf{s}_M^T \end{bmatrix} \mathbf{u}$$

$$[\mathbf{S}^T\mathbf{u}]_j = \mathbf{s}_j^T\mathbf{u} = \langle \mathbf{s}_j \mid \mathbf{u} \rangle = p_j \quad \mathbf{S}^T\mathbf{u} = \mathbf{p}$$

$$\mathbf{a} = \mathbf{U}^{-1}\mathbf{p} = (\mathbf{S}^T\mathbf{S})^{-1}\mathbf{S}^T\mathbf{u}$$

(d) Can we be sure that $\mathbf{S}^T\mathbf{S}$ is non-singular? Explain why.

Ans. The $\mathbf{s}_i$ are clearly linearly independent; in fact they are orthogonal. Thus, $\mathbf{U} = \mathbf{S}^T\mathbf{S}$ is positive definite and thus non-singular. In this case it is $4\mathbf{I}_M$ which is clearly non-singular.

10 2. A desired signal $d(n)$ that we wish to observe can be considered to be a first-order real autoregressive process (AR(1)), where the filter $H_G(z) = 1/(1 - az^{-1})$. However, we in fact observe $u(n) = d(n) + v(n)$, where v is zero-mean white noise, uncorrelated with d, with variance σ_v^2. Assume that d is zero mean, with variance σ_d^2.

(a) Determine $r_d(k)$, $r_v(k)$, $r_u(k)$ and $r_{du}(k)$ in terms of a, σ_d^2 and σ_v^2.

Ans. We have seen in class that the autocorrelation of an AR(1) process is $r_d(k) = \sigma_d^2 a^{|k|}$. This was shown using the Yule-Walker equation. Since v is white noise, $r_v(k) = \sigma_v^2 \delta(k)$. $r_u(k) = r_d(k) + r_v(k) = \sigma_d^2 a^{|k|} + \sigma_v^2 \delta(k)$. Since d and v are uncorrelated, $r_{du}(k) = r_d(k)$.

(b) We wish to estimate d by filtering u with a second-order causal filter $H_w(z) = w_0 + w_1 z^{-1}$, to obtain $\hat{d}(n)$. Set up the linear equations that must be solved to find the column matrix $\mathbf{w} = \begin{bmatrix} w_0 & w_1 \end{bmatrix}^T$ for the filter that minimizes the MSE (the Wiener filter), again in terms of the parameters a, σ_d^2 and σ_v^2.

Ans. Using the projection theorem, $\mathbf{w} = \mathbf{U}^{-1}\mathbf{p}$, where

$$\mathbf{U} = \begin{bmatrix} r_u(0) & r_u(1) \\ r_u(1) & r_u(0) \end{bmatrix} = \begin{bmatrix} \sigma_d^2 + \sigma_v^2 & \sigma_d^2 a \\ \sigma_d^2 a & \sigma_d^2 + \sigma_v^2 \end{bmatrix}$$

$$\mathbf{p} = \begin{bmatrix} r_{ud}(0) \\ r_{ud}(-1) \end{bmatrix} = \begin{bmatrix} r_d(0) \\ r_d(1) \end{bmatrix} = \begin{bmatrix} \sigma_d^2 \\ \sigma_d^2 a \end{bmatrix}$$

(c) Give an equation for the MSE between $\hat{d}$ and d. What is the improvement in SNR due to Wiener filtering?

Ans. The MSE after Wiener filtering is $\sigma_d^2 - \mathbf{p}^T\mathbf{w}$. The MSE before Wiener filtering is σ_v^2. Thus, the SNR improvement is

$$10 \log_{10} \frac{\sigma_d^2}{\sigma_d^2 - \mathbf{p}^T\mathbf{w}} - 10 \log_{10} \frac{\sigma_d^2}{\sigma_v^2} = 10 \log_{10} \frac{\sigma_v^2}{\sigma_d^2 - \mathbf{p}^T\mathbf{w}}$$

(d) In particular, what is the SNR improvement when $a = .9$, $\sigma_d^2 = 1$ and $\sigma_v^2 = 1$?

Ans. With these values, $\mathbf{w} = \begin{bmatrix} 0.3730 & 0.2821 \end{bmatrix}^T$ and the SNR improvement is $10 \log_{10} \frac{1}{1 - 0.6269} = 4.28$ dB.