Name: Solution
Student number:

CEG4311 Image Processing
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Closed book

Question

Consider the following three one-dimensional vectors in R^3:

$$\mathbf{b}_0 = [1 \;\; 1 \;\; 1] \quad \mathbf{b}_1 = [2 \;\; -1 \;\; -1] \quad \mathbf{b}_2 = [0 \;\; 1 \;\; -1]$$

(a) Show fully that these three vectors form an orthogonal set, and thus form a basis for R^3.

(b) Show explicitly that they do not form an orthonormal set.

(c) Normalize each vector to form a new set of vectors $\mathbf{c}_0$, $\mathbf{c}_1$, $\mathbf{c}_2$ that form an orthonormal set. Show them in the same format as used above for $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{b}_2$.

(d) A separable basis for $R^{3\times 3}$ is formed from the three orthonormal vectors $\mathbf{c}_0$, $\mathbf{c}_1$, $\mathbf{c}_2$ using

$$c_{mn}[i,j] = c_m[i]c_n[j], \quad m,n = 0,1,2; \, i,j = 0,1,2$$

Show as matrices (with i indicating the row and j the column) the two-dimensional basis vectors $\mathbf{c}_{00}$ and $\mathbf{c}_{12}$. Show explicitly that $\mathbf{c}_{00}$ is orthogonal to $\mathbf{c}_{12}$ and that they both have unit norm.

(e) A 3 by 3 data block is given in matrix form (with the same orientation as the basis vectors in (d)) by

$$\mathbf{f} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Determine the coefficient $f_c[1,2]$ in the expansion of $\mathbf{f}$ with respect to the two-dimensional orthonormal basis of (d).

(a) $\langle b_0 | b_1 \rangle = (1)(2) + (1)(-1) + (1)(-1) = 0 \quad \langle b_0 | b_2 \rangle = (1)(0) + (1)(1) + (1)(-1) = 0$

$\langle b_1 | b_2 \rangle = (2)(0) + (-1)(1) + (-1)(-1) = 0$

Thus, the three vectors form an orthogonal set, and thus a basis for $\mathbb{R}^3$.

(b) $\|b_0\|^2 = 3 \quad \|b_1\|^2 = 6 \quad \|b_2\|^2 = 2 \quad \neq 1$ Thus not an orthonormal set.

(c) $c_i = b_i / \|b_i\| \quad c_0 = \left[\frac{1}{\sqrt{3}} \; \frac{1}{\sqrt{3}} \; \frac{1}{\sqrt{3}}\right] \quad c_1 = \left[\frac{2}{\sqrt{6}} \; -\frac{1}{\sqrt{6}} \; -\frac{1}{\sqrt{6}}\right] \quad c_2 = \left[0 \; \frac{1}{\sqrt{2}} \; -\frac{1}{\sqrt{2}}\right]$

(d) $c_{00} = c_0^T c_0 = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \quad c_{12} = c_1^T c_2 = \begin{bmatrix} 0 & \frac{2}{\sqrt{12}} & -2/\sqrt{12} \\ 0 & -\frac{1}{\sqrt{12}} & 1/\sqrt{12} \\ 0 & -1/\sqrt{12} & 1/\sqrt{12} \end{bmatrix}.$

$\langle c_{00} | c_{12} \rangle = 3 \times (1/3)(0) + 1/3\left(\frac{2}{\sqrt{12}} - \frac{2}{\sqrt{12}}\right) + \frac{1}{3}\left(-1/\sqrt{12} + 1/\sqrt{12}\right) + 1/3\left(-\frac{1}{\sqrt{12}} + 1/\sqrt{12}\right) = 0$

$\|c_{00}\|^2 = 9\left(\frac{1}{3}\right)^2 = 1 \quad \|c_{12}\|^2 = \frac{1}{12}(4+4+1+1+1+1) = 1$

(e) $f_c[1,2] = \langle f | c_{12} \rangle = (1)(0) + 1\left(\frac{2}{\sqrt{12}}\right) + 0\left(-\frac{2}{\sqrt{12}}\right) + (1)(0) + (0)\left(-\frac{1}{\sqrt{12}}\right) + 0\left(1/\sqrt{12}\right)$
$+ 1(0) + 1\left(-1/\sqrt{12}\right) + 1\left(1/\sqrt{12}\right) = \frac{2}{\sqrt{12}} = 1/\sqrt{3}$