CEG 4311 Image Processing
Problem Set 1, Fall 2007
Solutions

1. $h_1(x,y) = c \exp\left(-(x^2+y^2)/2r^2\right)$

(a) $H_1(u,v) = c\, 2\pi r^2 \exp\left(-2\pi^2 (u^2+v^2) r^2\right)$
(table 2.2 & prop. (i) table 2.1)

(b) DC gain $= H_1(0,0) = c\, 2\pi r^2 = 1 \Rightarrow c = \dfrac{1}{2\pi r^2}$

If $f(x,y) = K$, then the output $g(x,y) = K\, H_1(0,0)$
$= f(x,y)$ if $H_1(0,0) = 1$.

(c) By linearity and shift invariance

$$g(x,y) = h_1(x-r, y-0.5r) = c \exp\left(-\frac{(x-r)^2+(y-.5r)^2}{2r^2}\right)$$

(d) $f(x,y) = 2\cos\left(2\pi\left(\frac{x}{4r}+\frac{y}{2r}\right)\right)$

$$= \exp\left(j2\pi\left(\frac{x}{4r}+\frac{y}{2r}\right)\right) + \exp\left(-j2\pi\left(\frac{x}{4r}+\frac{y}{2r}\right)\right)$$

$$g(x,y) = H\left(\frac{1}{4r},\frac{1}{2r}\right)\exp\left(j2\pi\left(\frac{x}{4r}+y/2r\right)\right) + H\left(-\frac{1}{4r},-\frac{1}{2r}\right)\exp\left(-j2\pi\left(\frac{x}{4r}+\frac{y}{2r}\right)\right)$$

$H(u,v) = H(-u,-v)$

$$\therefore g(x,y) = 2H\left(\frac{1}{4r},\frac{1}{2r}\right)\cos\left(2\pi\left(\frac{x}{4r}+y/2r\right)\right)$$

$$= c\, 4\pi r^2 \exp\left(-2\pi^2 \cdot \frac{5}{16}\right)\cos\left(2\pi\left(\frac{x}{4r}+y/2r\right)\right)$$

(e) $h_2(x,y) = c\exp\left(-\left(\frac{x^2}{4r^2} + \frac{y^2}{r^2}\right)\right) = c\exp\left(-\frac{(x/\sqrt{2})^2 + (\sqrt{2}y)^2}{2r^2}\right)$

Apply property (v) with $A = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$ $\det A = 1$

$A^{-T} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}$

$H_2(u,v) = H_1(\sqrt{2}u, v/\sqrt{2}) = c\, 2\pi r^2 \left(-2\pi^2\left(2u^2 + \frac{v^2}{2}\right)r^2\right)$

etc.

2. $F(u,v) = 2 + 4\cos(2\pi u X) + j2\sin(2\pi v Y) - 8e^{-j2\pi u X}e^{-j2\pi v Y}$

$$= 2 + 2\exp(j2\pi u X) + 2\exp(-j2\pi u X)$$

$$+ \exp(j2\pi v Y) - \exp(-j2\pi v Y)$$

$$- 8\exp(-j2\pi(uX + vY)))$$

It follows that (by inspection)

$$f[x,y] = 2\delta[x,y] + 2\delta[x-X, y] + 2\delta[x+X, y]$$

$$- \delta[x, y-Y] + \delta[x, y+Y]$$

$$- 8\delta[x-X, y-Y]$$

(a)

$$V_{\Lambda_1} = \begin{bmatrix} X & 0 \\ X & 2X \end{bmatrix} \qquad V_{\Lambda_2} = \begin{bmatrix} 2X & 2X \\ 0 & 2X \end{bmatrix}$$

$$d(\Lambda_1) = 2X^2 \qquad d(\Lambda_2) = 4X^2$$

Thus, it is possible that $\Lambda_2 \subset \Lambda_1$. To check, using Theorem 2

$$(V_{\Lambda_1})^{-1} V_{\Lambda_2} = \begin{bmatrix} \frac{1}{X} & 0 \\ -\frac{1}{2X} & \frac{1}{2X} \end{bmatrix} \begin{bmatrix} 2X & 2X \\ 0 & 2X \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}$$

which is an integer matrix. Thus $\Lambda_2 \subset \Lambda_1$.

The sampling structure conversion system has the form

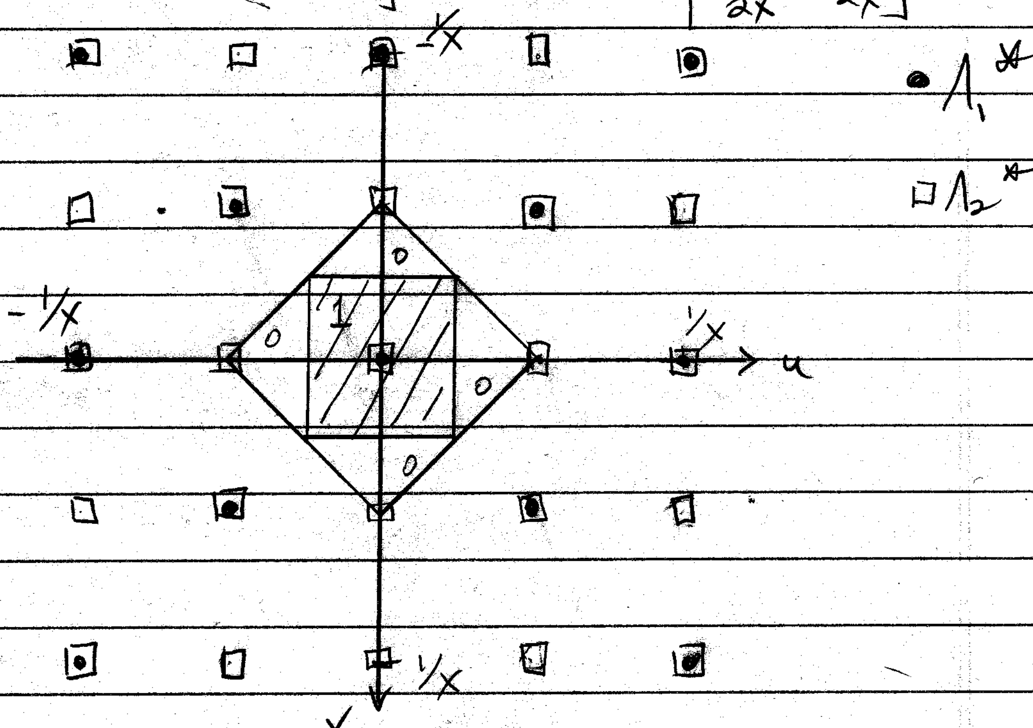
$$f \longrightarrow \boxed{\mathcal{H}} \longrightarrow \boxed{\Lambda_1 \downarrow \Lambda_2} \longrightarrow g$$

$\mathcal{H}$ is a lowpass filter, defined on Λ_1, with passband equal to a unit cell of Λ_2

$$H(\underline{u}) = \begin{cases} 1 & \underline{u} \in P_{\Lambda_2^*} \\ 0 & \underline{u} \in P_{\Lambda_1^*} \setminus P_{\Lambda_2^*} \end{cases}$$

The reciprocal lattices are defined by the matrices $(V_{\Lambda_1})^{-T}$, $(V_{\Lambda_2})^{-T}$

$$(V_{\Lambda_1})^{-T} = \begin{bmatrix} \frac{1}{X} & -\frac{1}{2X} \\ 0 & \frac{1}{2X} \end{bmatrix} \qquad (V_{\Lambda_2})^{-T} = \begin{bmatrix} \frac{1}{2X} & 0 \\ -\frac{1}{2X} & \frac{1}{2X} \end{bmatrix}$$



One period of the frequency response $H(u,v)$ is shown. The frequency response has period Λ_1^*.

Note that Λ_2 and Λ_2^* are rectangular. They could equivalently be represented by the sampling matrices

$$V'_{\Lambda_2} = \begin{bmatrix} 2X & 0 \\ 0 & 2X \end{bmatrix} \qquad V'_{\Lambda_2^*} = \begin{bmatrix} \frac{1}{2X} & 0 \\ 0 & \frac{1}{2X} \end{bmatrix}$$

3 (ii) $V_{\Lambda_3} = \begin{bmatrix} .5X & 0 \\ 0 & .5X \end{bmatrix}$

$$\boxed{\Lambda_1 \uparrow \Lambda_3} \longrightarrow \boxed{H(u,v)} \longrightarrow \boxed{\Lambda_3 \downarrow \Lambda_2}$$