

1. Consider a two-dimensional sinusoidal signal $f(x, y) = A \cos(2\pi(ux + vy) + \phi)$ where x and y are in ph and u and v are in c/ph. Form the one-dimensional signal $g(z)$ by tracing $f(x, y)$ along the line $y = cx$, where c is some real constant, as a function of distance along the line, $z = \sqrt{x^2 + y^2}$.
 - (a) Show that $g(z)$ is a sinusoidal signal $g(z) = A \cos(2\pi wz + \phi)$ and determine the spatial frequency w in c/ph, as a function of u , v and c .
 - (b) Explain what happens when $c = 0$ and when $c \rightarrow \infty$.
 - (c) Show that the spatial frequency w is greatest along the line $y = (v/u)x$, if $u \neq 0$. What is the value of this maximum spatial frequency? What happens if $u = 0$?
2. Show that for each of the following functions $\delta_N(x, y)$,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta_N(x, y) dx dy = 1$$

and

$$\lim_{N \rightarrow \infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta_N(x, y) f(x, y) dx dy = f(0, 0)$$

for any function $f(x, y)$ that is continuous at $(x, y) = (0, 0)$.

- (a) $\delta_N(x, y) = N^2 \text{rect}(Nx, Ny)$;
- (b) $\delta_N(x, y) = N^2 \exp(-N^2 \pi(x^2 + y^2))$;
- (c) $\delta_N(x, y) = \frac{N^2}{\pi} \text{circ}(Nx, Ny)$.

3. Show that

$$\delta(ax, by) = \frac{1}{|ab|} \delta(x, y)$$

where $a, b \neq 0$.

4. Prove the Fourier transform properties shown in Table 2.1.
5. Let $f(x, y) = 0.5 \text{rect}(4(x-0.5), 2(y-0.25))$ and $h(x, y) = \text{rect}(10x, 10y)$, where x and y are in ph.

- (a) Sketch the region of support of $f(x, y)$ and $h(x, y)$ in the XY-plane (i.e., the area where these two signals are nonzero).
- (b) Compute the two-dimensional convolution $f(x, y) * h(x, y)$ from the definition using integration in the spatial domain.
- (c) Suppose that $f(x, y)$ is the input to a two-dimensional system, and the output of this system is computed as in (b). What can we say about this system?
- (d) Determine the continuous-space Fourier transforms $F(u, v)$, $H(u, v)$ and $G(u, v)$ of the above three signals. Make liberal use of Fourier transform properties. What are the units of u and v ?
- (e) Continuing with question (c), what is the interpretation of $H(u, v)$?
6. A two-dimensional continuous-space linear shift-invariant system has impulse response

$$h(x, y) = \begin{cases} \frac{1}{2\pi R_1 R_2}, & \left(\frac{x}{R_1}\right)^2 + \left(\frac{y}{R_2}\right)^2 \leq 1 \\ 0, & \text{otherwise,} \end{cases}$$

where $R_1 = 1/1000$ ph and $R_2 = 1/500$ ph.

- (a) Sketch the region of support of the impulse response in the XY-plane, following the conventions used in the course for the labelling of axes. Express $h(x, y)$ in terms of the circ function.
- (b) Find the frequency response $H(u, v)$ of this system, where u and v are in c/ph.
- (c) The image $f(x, y) = \text{rect}(5(x - .5), 2(y - .5))$ is filtered with this system to produce the output $g(x, y) = f(x, y) * h(x, y)$. Determine the Fourier transform of the output, $G(u, v)$.
7. Compute the two-dimensional continuous-space Fourier transform of the following signals:
- (a) The separable signal $f(x, y) = h_X^{(1)}(x)h_Y^{(1)}(y)$ where

$$h_T^{(1)}(t) = \begin{cases} 1 - \frac{|t|}{T} & |t| \leq T, \\ 0 & \text{otherwise.} \end{cases}$$

- (b) A Gaussian function $f(x, y) = \frac{1}{2\pi r_0^2} e^{-(x^2+y^2)/2r_0^2}$.
- (c) A real zoneplate, $f(x, y) = \cos(\pi(x^2 + y^2)/r_0^2)$. (Hint: Find the Fourier transform of the complex zoneplate $\exp(j\pi(x^2 + y^2)/r_0^2)$ and use linearity. You can use $\int_{-\infty}^{\infty} e^{jy^2} dy = \sqrt{\pi} e^{j\pi/4}$.)
- (d) Diamond-shaped pulse

$$f(x, y) = \begin{cases} 1 & |x| + |y| \leq 1, \\ 0 & |x| + |y| > 1. \end{cases}$$

(Hint: obtain this function from a rect function using an affine transformation.)

- (e) Gabor function

$$f(x, y) = \cos(2\pi(u_0x + v_0y)) \exp\left(-\frac{(x - x_0)^2 + (y - y_0)^2}{2r_0^2}\right)$$

- (f) The two-dimensional zero-one function $p_{\mathcal{A}}(x, y)$ where \mathcal{A} is an elliptical region, with semi-minor axis X and semi-major axis $2X$, oriented at 45° as shown in the figure

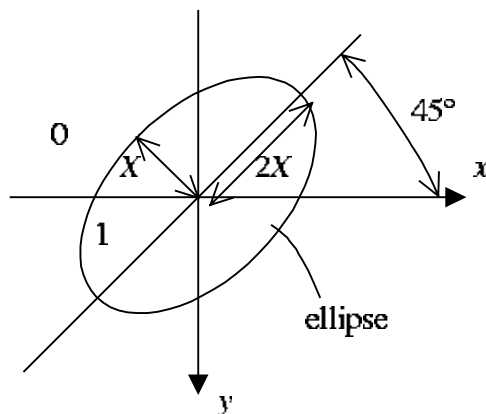


Figure 1: Elliptical region of support of two-dimensional zero-one function.

8. If \mathbf{A} is a $D \times D$ matrix and \mathbf{x} , \mathbf{y} are $D \times 1$ column matrices, show that $\mathbf{x} \cdot (\mathbf{A}\mathbf{y}) = (\mathbf{A}^T \mathbf{x}) \cdot \mathbf{y}$.

9. A two-dimensional continuous signal $f_c(x, y)$ has Fourier transform

$$F_c(u, v) = \begin{cases} ce^{-\alpha(|u|+|v|)} & u^2 + v^2 < W^2 \\ 0 & u^2 + v^2 \geq W^2 \end{cases}$$

for some real number W . The signal is sampled on a hexagonal lattice Λ with sampling matrix

$$\mathbf{V} = \begin{bmatrix} X & X/2 \\ 0 & \sqrt{3}X/2 \end{bmatrix}$$

to give the sampled signal $f[x, y]$, $(x, y) \in \Lambda$, with Fourier transform $F(u, v)$.

- (a) What is the expression for $F(u, v)$ in terms of $F_c(u, v)$?
- (b) Find the largest possible value of X such that there is no aliasing? Sketch the region of support of the Fourier transform of the sampled signal in this case (including all replicas), and also indicate a unit cell of the reciprocal lattice Λ^* .
10. For each of the following pairs of lattices Λ_1 and Λ_2 , state whether $\Lambda_1 \subset \Lambda_2$, $\Lambda_2 \subset \Lambda_1$ or neither. If neither, find (by inspection) the least dense lattice Λ_3 such that $\Lambda_1 \subset \Lambda_3$ and $\Lambda_2 \subset \Lambda_3$. For each lattice Λ_1 , Λ_2 and Λ_3 (if required), determine and sketch the reciprocal lattice and a unit cell of the reciprocal lattice.

(a) $V_{\Lambda_1} = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}$	$V_{\Lambda_2} = \begin{bmatrix} 2X & 0 \\ 0 & 2X \end{bmatrix}$
(b) $V_{\Lambda_1} = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}$	$V_{\Lambda_2} = \begin{bmatrix} 3X & X \\ 0 & X \end{bmatrix}$
(c) $V_{\Lambda_1} = \begin{bmatrix} 2X & 0 \\ 0 & 2X \end{bmatrix}$	$V_{\Lambda_2} = \begin{bmatrix} X & X \\ X & -X \end{bmatrix}$
(d) $V_{\Lambda_1} = \begin{bmatrix} 1.5X & 0 \\ 0 & 1.5X \end{bmatrix}$	$V_{\Lambda_2} = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}$
(e) $V_{\Lambda_1} = \begin{bmatrix} X & X \\ X & -X \end{bmatrix}$	$V_{\Lambda_2} = \begin{bmatrix} 1.5X & 1.5X \\ 1.5X & -1.5X \end{bmatrix}$

11. Prove the properties of the multidimensional Fourier transform over a lattice Λ given in Table 2.3. Of course, property (vi) has already been done.
12. Consider an ideal discrete-space circularly symmetric lowpass filter defined on the rectangular lattice with horizontal and vertical sample spacing X and Y . The passband is $C_W = \{(u, v) \mid u^2 + v^2 \leq W^2\}$ and the unit cell of the reciprocal lattice is $P^* = \{(u, v) \mid -1/2X \leq u < 1/2X, -1/2Y \leq v < 1/2Y\}$. Assume that $W < \min(1/2X, 1/2Y)$.

$$H(u, v) = \begin{cases} 1 & (u, v) \in C_W \\ 0 & (u, v) \in P^* \setminus C_W \end{cases}$$

where of course $H(u + k/X, v + l/Y) = H(u, v)$ for all integers k, l . Show that the unit sample response of this filter is given by

$$h[mX, nY] = \frac{WXY}{\sqrt{X^2m^2 + Y^2n^2}} J_1(2\pi W \sqrt{X^2m^2 + Y^2n^2})$$

where $J_1(s)$ is the Bessel function of the first kind and first order. You may use the following identities:

$$J_0(s) = \frac{1}{2\pi} \int_0^{2\pi} \exp[js \cos(\theta + \phi)] d\theta, \quad \text{for any } \phi$$

$$\int s J_0(s) ds = s J_1(s)$$

Simplify the expression in the case $X = Y$.

13. The face-centered cubic lattice is the most efficient lattice for the packing of spheres in three dimensions. A sampling matrix for this lattice is given by

$$\mathbf{V} = K \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where K is some real constant. Suppose that a bandlimited three-dimensional signal $f(\mathbf{x})$ satisfying $F(\mathbf{u}) = 0$ for $|\mathbf{u}| > W$ is sampled on a lattice whose reciprocal lattice is face-centered cubic. Find the least dense lattice such that there is no aliasing. Compare the resulting sampling density with the best orthogonal sampling for which there is no aliasing.

14. A square image (pw = ph) is sampled on the hexagonal lattice Λ^* generated by the sampling matrix

$$\mathbf{V} = \begin{bmatrix} X & X/2 \\ 0 & \sqrt{3}X/2 \end{bmatrix}$$

where $X = 1/512$ ph. Design a Gaussian FIR filter with unit sample response $h[\mathbf{x}] = c \exp(-\|\mathbf{x}\|^2/2r^2)$ for $\mathbf{x} \in \mathcal{B}$ having a 3dB bandwidth of $0.2/X$ c/ph. The region of support of the FIR filter is $\mathcal{B} = \{\mathbf{x} \in \Lambda \mid \|\mathbf{x}\| \leq 3X\}$. Having determined the correct values of c and r , give the coefficients of the filter. Give an analytical approximation for the frequency response of the filter. Make a contour plot and a perspective plot of the frequency response of the filter over the frequency range $-2/X \leq u \leq 2/X$, $-4/(\sqrt{3}X) \leq v \leq 4/(\sqrt{3}X)$. Sketch by hand on the contour plot the points of the reciprocal lattice Λ^* and a Voronoi unit cell of Λ^* , and comment on the periodicity of the frequency response. Recall that the Voronoi unit cell consists of all points in Λ^* closer to the origin than to any other point of Λ^* .