CEG4311 Image Processing

Mid-term make-up exam

Date: Nov. 12, 2002

Time: 11:30-12:50

Professor: E. Dubois

Closed-book exam: you may not use any books or notes. You may use a pocket (nonprogrammable) calculator. However, explain all calculations; I am more interested in the reasoning than in precise numerical answers. Unless otherwise specified, you may use the results provided on pages 4 and 5 without proof.

Vous pouvez répondre en anglais ou en français.

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1. An image sensor consists of sensor elements laid out in a pattern as shown in Fig. 1. This figure is intentionally *not* drawn to scale. Note that the dimensions of the sensor elements are smaller than the intersample spacing to allow room for circuitry. Specifically, the width of each sensor element is $0.9X$ and the height of each sensor element is $0.9Y$. The image acquired consists of 1280 samples per row of sensors, and there are 720 rows of sensors. The aspect ratio $ar = W/H$ of the image is 16/9.

2 (a) Using the information provided above, determine the numerical values of X and Y in units of picture heights (ph).

3 (b) The sampling lattice Λ consists of the centres of the sensor elements (with extension over all space), as shown in Fig. 1. Determine a sampling matrix $\mathbf{V}$ for Λ numerically in units of ph (i.e., use the numerical values of X and Y found in (a)). Compute the determinant of Λ and the sampling density, with correct units.

2 (c) Determine a sampling matrix for the reciprocal lattice Λ^*, again with the correct units.

3 (d) The values of the image samples are obtained by integrating the irradiance of the light falling on each sensor element, scaling by a suitable gain K, and assigning the value to the lattice point at the centre of the sensor element. The gain K is chosen so that a constant continuous-space input image $f_c(x, y) = 1$ produces a constant sampled image $f[x, y] = 1, \quad (x, y) \in \Lambda$. As seen in class, this sampling operation can be modelled as the continuous-space filtering of the continuous-space irradiance $f_c(x, y)$ with a continuous-space, linear, shift-invariant filter with impulse response

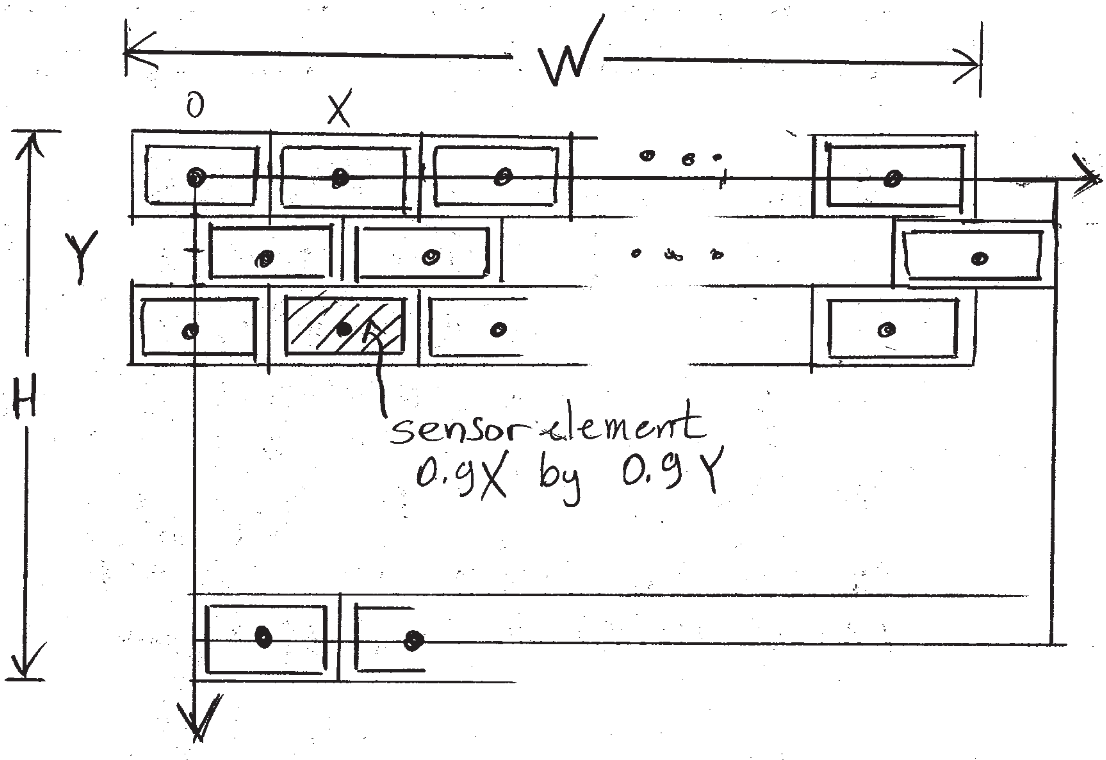

Figure 1: Image sensor.

$h_a(x, y)$, followed by ideal sampling on the lattice Λ. The impulse response $h_a(x, y)$ incorporates the gain K. Using the information provided above, determine $h_a(x, y)$ and the corresponding frequency response $H_a(u, v)$. Use the numerical values of X and Y found in (a) in your expressions.

3 (e) Suppose that $f_c(x, y) = .5(1 + \sin(600\pi x))$. The magnitude of the Fourier transform of this signal is illustrated schematically in Fig. 2, where the $\times$ symbol represents a Dirac delta function in the frequency domain. Using this symbol, sketch $F(u, v)$, the Fourier transform of the sampled signal $f[x, y]$, $\quad (x, y) \in \Lambda$, over frequency range $-1000\text{c/ph} \leq u, v \leq 1000\text{c/ph}$. Draw your diagram to scale and label your axes.

2. A signal $f[x, y]$ is defined on the lattice $\Lambda = \text{LAT}(\mathbf{V}_\Lambda)$, where

$$\mathbf{V}_\Lambda = \begin{bmatrix} 2X & X \\ 0 & X \end{bmatrix}.$$

We want to up-sample $f[x, y]$ to the rectangular lattice Γ with sampling matrix $\mathbf{V}_\Gamma$, where

$$\mathbf{V}_\Gamma = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}.$$

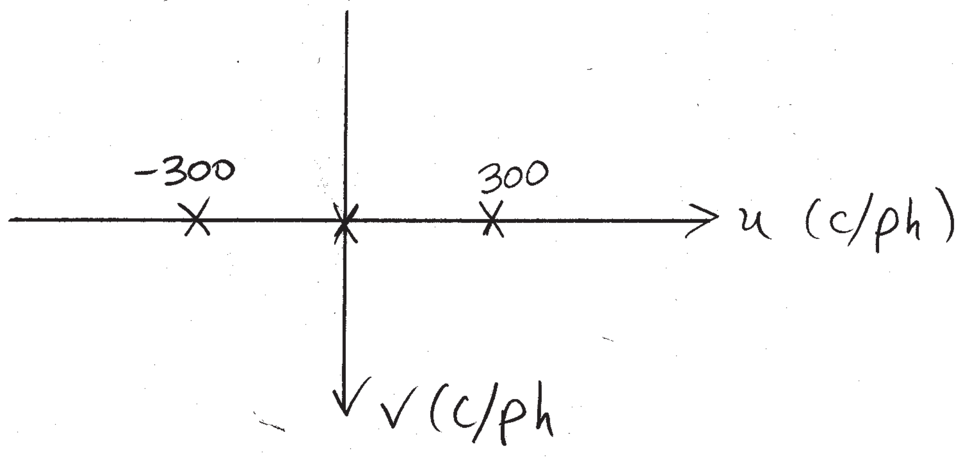

Figure 2: Fourier transform of continuous-space sinusoidal signal.

This is accomplished using the system shown in Fig. 3, where the linear shift-invariant filter $\mathcal{H}$ has unit sample response

$$h[x,y] = \begin{cases} 1 & (x,y) = (0,0) \\ 0.3 & (x,y) = (\pm X, 0) \\ 0.2 & (x,y) = (0, \pm X) \\ 0 & \text{elsewhere in } \Gamma \end{cases}.$$

There are five non-zero values.

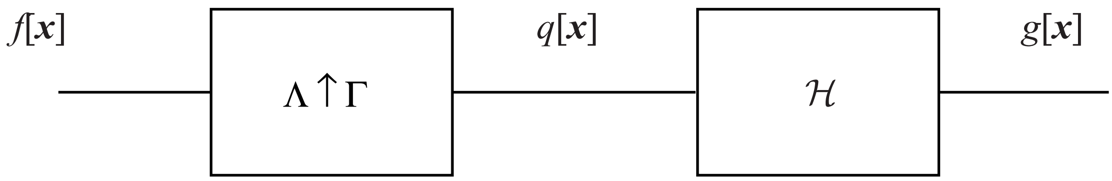

Figure 3: Upsampling system.

2 (a) What is the upsampling factor (or interpolation ratio)?

3 (b) Determine the frequency response $H(u,v)$ of the LSI filter $\mathcal{H}$. Express it as a real function of u and v.

3 (c) Suppose that the input signal is $f[x,y] = \delta[x,y] + \delta[x-X, y-X] + \delta[x+X, y-X]$, $(x,y) \in \Lambda$. Sketch this signal in the space domain. Determine the output signal $g[x,y]$ for all $(x,y) \in \Gamma$ and sketch this signal on a separate set of axes.

	$f(\boldsymbol{x}) = \int_{\mathbb{R}^D} F(\boldsymbol{u}) \exp(j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{u}$	$F(\boldsymbol{u}) = \int_{\mathbb{R}^D} f(\boldsymbol{x}) \exp(-j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{x}$
(i)	$af_1(\boldsymbol{x}) + bf_2(\boldsymbol{x})$	$aF_1(\boldsymbol{u}) + bF_2(\boldsymbol{u})$
(ii)	$f(\boldsymbol{x} - \boldsymbol{x}_0)$	$F(\boldsymbol{u}) \exp(-j2\pi \boldsymbol{u} \cdot \boldsymbol{x}_0)$
(iii)	$f(\boldsymbol{x}) \exp(j2\pi \boldsymbol{u}_0 \cdot \boldsymbol{x})$	$F(\boldsymbol{u} - \boldsymbol{u}_0)$
(iv)	$f^*(\boldsymbol{x})$	$F^*(-\boldsymbol{u})$
(v)	$f(\mathbf{A}\boldsymbol{x})$	$\frac{1}{\lvert\det \mathbf{A}\rvert} F(\mathbf{A}^{-T}\boldsymbol{u})$
(vi)	$f_1(\boldsymbol{x}) * f_2(\boldsymbol{x})$	$F_1(\boldsymbol{u}) F_2(\boldsymbol{u})$
(vii)	$f_1(\boldsymbol{x}) f_2(\boldsymbol{x})$	$F_1(\boldsymbol{u}) * F_2(\boldsymbol{u})$
(viii)	$f_1(x) f_2(y)$	$F_1(u) F_2(v)$
(ix)	$\int_{\mathbb{R}^D} \lvert f(\boldsymbol{x})\rvert^2\, d\boldsymbol{x} = \int_{\mathbb{R}^D} \lvert F(\boldsymbol{u})\rvert^2\, d\boldsymbol{u}$	

Multidimensional Fourier transform properties.

$f(\boldsymbol{x}) = \int_{\mathbb{R}^D} F(\boldsymbol{u}) \exp(j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{u}$	$F(\boldsymbol{u}) = \int_{\mathbb{R}^D} f(\boldsymbol{x}) \exp(-j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{x}$
$\mathrm{rect}(x, y)$	$\frac{\sin \pi u}{\pi u} \frac{\sin \pi v}{\pi v}$
$\mathrm{circ}(x, y)$	$\frac{1}{\sqrt{u^2+v^2}} J_1(2\pi\sqrt{u^2 + v^2})$
$\exp(-(x^2 + y^2)/2r^2)$	$2\pi r^2 \exp(-2\pi^2(u^2 + v^2) r^2)$
$\cos(\pi(x^2 + y^2)/r^2)$	$r^2 \sin(\pi(u^2 + v^2) r^2)$
$\exp(j\pi(x^2 + y^2)/r^2)$	$jr^2 \exp(-j\pi(u^2 + v^2) r^2)$
$\delta(\boldsymbol{x})$	1

Multidimensional Fourier transform of selected functions.

	$f[\boldsymbol{x}] = d(\Lambda)\int_{\mathcal{P}^*} F(\boldsymbol{u})\exp(j2\pi\boldsymbol{u}\cdot\boldsymbol{x})\,d\boldsymbol{u}$	$F(\boldsymbol{u}) = \sum_{\boldsymbol{x}\in\Lambda} f[\boldsymbol{x}]\exp(-j2\pi\boldsymbol{u}\cdot\boldsymbol{x})$
(i)	$af_1[\boldsymbol{x}] + bf_2[\boldsymbol{x}]$	$aF_1(\boldsymbol{u}) + bF_2(\boldsymbol{u})$
(ii)	$f[\boldsymbol{x} - \boldsymbol{x}_0]$	$F(\boldsymbol{u})\exp(-j2\pi\boldsymbol{u}\cdot\boldsymbol{x}_0)$
(iii)	$f[\boldsymbol{x}]\exp(j2\pi\boldsymbol{u}_0\cdot\boldsymbol{x})$	$F(\boldsymbol{u} - \boldsymbol{u}_0)$
(iv)	$f^*[\boldsymbol{x}]$	$F^*(-\boldsymbol{u})$
(v)	$f_1[\boldsymbol{x}] * f_2[\boldsymbol{x}]$	$F_1(\boldsymbol{u})F_2(\boldsymbol{u})$
(vi)	$f_1[\boldsymbol{x}]f_2[\boldsymbol{x}]$	$d(\Lambda)\int_{\mathcal{P}^*} F_1(\boldsymbol{r})F_2(\boldsymbol{u}-\boldsymbol{r})\,d\boldsymbol{r}$
(vii)	$\sum_{\boldsymbol{x}\in\Lambda} \lvert f[\boldsymbol{x}]\rvert^2 = d(\Lambda)\int_{\mathcal{P}^*} \lvert F(\boldsymbol{u})\rvert^2\,d\boldsymbol{u}$	

Properties of the multidimensional Fourier transform over a lattice Λ.

Formulas

$$\mathrm{rect}(x,y) = \begin{cases} 1, & \text{if } |x| \le 0.5 \text{ and } |y| \le 0.5; \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathrm{circ}(x,y) = \begin{cases} 1, & \text{if } x^2 + y^2 \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

If $\Lambda = \mathrm{LAT}(\mathbf{V})$, then $d(\Lambda) = |\det(\mathbf{V})|$, and $\Lambda^* = \mathrm{LAT}(\mathbf{V}^{-T})$.

The Fourier transform pair for a signal $f[\boldsymbol{x}]$ defined on the lattice Λ is given by

$$F(\boldsymbol{u}) = \sum_{\boldsymbol{x}\in\Lambda} f[\boldsymbol{x}]\exp(-j2\pi\boldsymbol{u}\cdot\boldsymbol{x})$$

$$f[\boldsymbol{x}] = d(\Lambda)\int_{\mathcal{P}^*} F(\boldsymbol{u})\exp(j2\pi\boldsymbol{u}\cdot\boldsymbol{x})\,d\boldsymbol{u}$$

Sampling

If $f[\boldsymbol{x}] = f_c(\boldsymbol{x}), \quad \boldsymbol{x}\in\Lambda$ then

$$F(\boldsymbol{u}) = \frac{1}{d(\Lambda)}\sum_{r\in\Lambda^*} F_c(\boldsymbol{u}+\boldsymbol{r})$$