CEG4311 Image Processing
Mid-term exam

Date: Oct. 25, 2001
Time: 16:00-17:20
Professor: E. Dubois

Closed-book exam: you may not use any books or notes. You may use a pocket (nonprogrammable) calculator. However, explain all calculations; I am more interested in the reasoning than in precise numerical answers. Unless otherwise specified, you may use the results provided on pages 3 and 4 without proof.
Vous pouvez répondre en anglais ou en français.

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1. A two-dimensional continuous-space linear shift-invariant system has impulse response

$$h(x,y) = \begin{cases} \frac{1}{2\pi R_1 R_2}, & \left(\frac{x}{R_1}\right)^2 + \left(\frac{y}{R_2}\right)^2 \leq 1 \\ 0, & \text{otherwise,} \end{cases}$$

where $R_1 = 1/1000$ ph and $R_2 = 1/500$ ph.

5 (a) Sketch the region of support of the impulse response in the XY-plane, following the conventions used in the course for the labeling of axes. Express $h(x,y)$ in terms of the circ function (see pg. 4).

5 (b) Find the frequency response $H(u,v)$ of this system, where u and v are in c/ph.

5 (c) The image $f(x,y) = \mathrm{rect}(5(x-.5), 2(y-.5))$ is filtered with this system to produce the output $g(x,y) = f(x,y) * h(x,y)$. Determine the Fourier transform of the output, $G(u,v)$.

2. A two-dimensional continuous-space signal $f(x,y)$ has a bandlimited Fourier transform restricted to the region of the frequency domain shown in Fig. 1.

5 (a) Find a sampling matrix for the *least dense* sampling lattice Λ_1 that permits the sampling of this signal without aliasing (it will be hexagonal). Find a sampling matrix for the least dense *orthogonal* sampling lattice Λ_2 that allows the sampling of this signal without aliasing. In both cases, express the sampling matrix in units of ph and clearly show the units. State the sampling density in each case, giving units. (Note that you will want to find a sampling matrix for the reciprocal lattice first, then find a sampling matrix for the sampling lattice itself.)

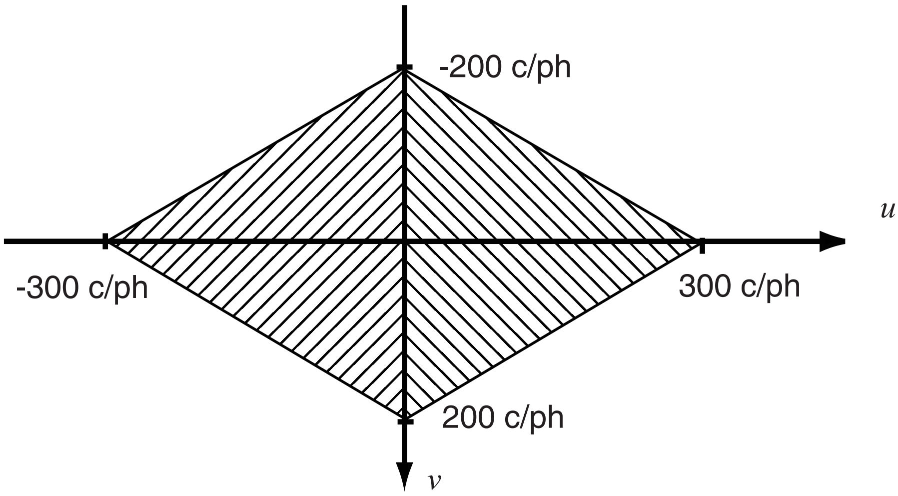

Figure 1: Region of support of Fourier transform of $f(x,y)$.

5 (b) Sketch the region of support of the Fourier transform of the signal sampled on lattice Λ_1 over the range -300 c/ph $< u <$ 300 c/ph, -400 c/ph $< v <$ 400 c/ph. You may reproduce the hatching in the Figure to make your drawing clearer. Label your axes.

5 (c) What is the approximate number of samples per image in each case (Λ_1 and Λ_2) if the image has an aspect ratio of 4/3?

3. An operator that estimates the horizontal derivative of a discrete space image is given by

$$g[n_1X, n_2X] = \frac{1}{2X}\left(f[(n_1+1)X, n_2X] - f[(n_1-1)X, n_2X]\right)$$

where $f[x,y]$ is defined on a rectangular lattice with equal horizontal and vertical spacing X.

5 (a) Explain why this operator defines a linear, shift-invariant two-dimensional system.

5 (b) What is the unit sample response $h[n_1X, n_2X]$ of this system?

5 (c) Compute the frequency response $H(u,v)$ of the system.

5 (d) Sketch a contour plot of the magnitude response $|H(u,v)|$ over the frequency range $-(1/X) \leq u, v \leq (1/X)$.

	$f(\boldsymbol{x}) = \int_{\mathbb{R}^D} F(\boldsymbol{u}) \exp(j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{u}$	$F(\boldsymbol{u}) = \int_{\mathbb{R}^D} f(\boldsymbol{x}) \exp(-j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{x}$
(i)	$af_1(\boldsymbol{x}) + bf_2(\boldsymbol{x})$	$aF_1(\boldsymbol{u}) + bF_2(\boldsymbol{u})$
(ii)	$f(\boldsymbol{x} - \boldsymbol{x}_0)$	$F(\boldsymbol{u}) \exp(-j2\pi \boldsymbol{u} \cdot \boldsymbol{x}_0)$
(iii)	$f(\boldsymbol{x}) \exp(j2\pi \boldsymbol{u}_0 \cdot \boldsymbol{x})$	$F(\boldsymbol{u} - \boldsymbol{u}_0)$
(iv)	$f^*(\boldsymbol{x})$	$F^*(-\boldsymbol{u})$
(v)	$f(\mathbf{A}\boldsymbol{x})$	$\frac{1}{\lvert\det \mathbf{A}\rvert} F(\mathbf{A}^{-T}\boldsymbol{u})$
(vi)	$f_1(\boldsymbol{x}) * f_2(\boldsymbol{x})$	$F_1(\boldsymbol{u}) F_2(\boldsymbol{u})$
(vii)	$f_1(\boldsymbol{x}) f_2(\boldsymbol{x})$	$F_1(\boldsymbol{u}) * F_2(\boldsymbol{u})$
(viii)	$f_1(x) f_2(y)$	$F_1(u) F_2(v)$
(ix)	$\int_{\mathbb{R}^D} \lvert f(\boldsymbol{x})\rvert^2\, d\boldsymbol{x} = \int_{\mathbb{R}^D} \lvert F(\boldsymbol{u})\rvert^2\, d\boldsymbol{u}$	

Table 1: Multidimensional Fourier transform properties.

$f(\boldsymbol{x}) = \int_{\mathbb{R}^D} F(\boldsymbol{u}) \exp(j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{u}$	$F(\boldsymbol{u}) = \int_{\mathbb{R}^D} f(\boldsymbol{x}) \exp(-j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{x}$
$\mathrm{rect}(x, y)$	$\frac{\sin \pi u}{\pi u} \frac{\sin \pi v}{\pi v}$
$\mathrm{circ}(x, y)$	$\frac{1}{\sqrt{u^2+v^2}} J_1(2\pi\sqrt{u^2+v^2})$
$\exp(-(x^2+y^2)/2r^2)$	$2\pi r^2 \exp(-2\pi^2(u^2+v^2)r^2)$
$\cos(\pi(x^2+y^2)/r^2)$	$r^2 \sin(\pi(u^2+v^2)r^2)$
$\exp(j\pi(x^2+y^2)/r^2)$	$jr^2 \exp(-j\pi(u^2+v^2)r^2)$
$\delta(\boldsymbol{x})$	1

Table 2: Multidimensional Fourier transform of selected functions.

Definitions

$$\mathrm{rect}(x, y) = \begin{cases} 1, & \text{if } |x| \le 0.5 \text{ and } |y| \le 0.5; \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathrm{circ}(x, y) = \begin{cases} 1, & \text{if } x^2 + y^2 \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

If $\Lambda = \mathrm{LAT}(\mathbf{V})$, then $d(\Lambda) = |\det(\mathbf{V})|$, and $\Lambda^* = \mathrm{LAT}(\mathbf{V}^{-T})$.
The Fourier transform pair for a signal $f[\boldsymbol{x}]$ defined on the lattice Λ is given by

$$F(\boldsymbol{u}) = \sum_{\boldsymbol{x} \in \Lambda} f[\boldsymbol{x}] \exp(-j2\pi \boldsymbol{u} \cdot \boldsymbol{x})$$

$$f[\boldsymbol{x}] = d(\Lambda) \int_{\mathcal{P}^*} F(\boldsymbol{u}) \exp(j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{u}$$

If $f[\boldsymbol{x}] = f_c(\boldsymbol{x}), \quad \boldsymbol{x} \in \Lambda$ then

$$F(\boldsymbol{u}) = \frac{1}{d(\Lambda)} \sum_{r \in \Lambda^*} F_c(\boldsymbol{u} + \boldsymbol{r})$$