CEG 4311 Image Processing
Midterm Exam, Oct 23 2006
Solution

1 (i) It is best to work from the outside in:

$$f(x,y) = 0.8\, \mathrm{rect}\left(\frac{x}{2A}, \frac{y}{2A}\right) - 0.6\, \mathrm{rect}\left(\frac{x}{A}, \frac{y}{A}\right)$$

$$+ 0.4\, \mathrm{circ}\left(\frac{4x}{A}, \frac{4y}{A}\right)$$

(ii) We use properties (i) and (v) from the first table with Fourier transforms of rect and circ from the second table

1st term $\begin{bmatrix} x/2A \\ y/2A \end{bmatrix} = \underbrace{\begin{bmatrix} 1/2A & 0 \\ 0 & 1/2A \end{bmatrix}}_{\underline{M}_1} \begin{bmatrix} x \\ y \end{bmatrix}$ $\det \underline{M}_1 = \frac{1}{4A^2}$ $\underline{M}_1^{-T} = \begin{bmatrix} 2A & 0 \\ 0 & 2A \end{bmatrix}$

2ND term $\begin{bmatrix} x/A \\ y/A \end{bmatrix} = \underbrace{\begin{bmatrix} 1/A & 0 \\ 0 & 1/A \end{bmatrix}}_{\underline{M}_2} \begin{bmatrix} x \\ y \end{bmatrix}$ $\det \underline{M}_2 = \frac{1}{A^2}$ $\underline{M}_2^{-T} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$

3RD term $\begin{bmatrix} 4x/A \\ 4y/A \end{bmatrix} = \underbrace{\begin{bmatrix} 4/A & 0 \\ 0 & 4/A \end{bmatrix}}_{\underline{M}_3} \begin{bmatrix} x \\ y \end{bmatrix}$ $\det \underline{M}_3 = \frac{16}{A^2}$ $\underline{M}_3^{-T} = \begin{bmatrix} A/4 & 0 \\ 0 & A/4 \end{bmatrix}$

Thus

$$F(u,v) = 0.8 \cdot 4A^2 \cdot \frac{\sin \pi 2Au}{\pi 2Au} \frac{\sin \pi 2Av}{\pi 2Av} - 0.6 \cdot A^2 \frac{\sin \pi Au}{\pi Au} \frac{\sin \pi Av}{\pi Av}$$

$$+ 0.4 \cdot \frac{A^2}{16} \cdot \frac{1}{\sqrt{\left(\frac{Au}{4}\right)^2 + \left(\frac{Av}{4}\right)^2}} J_1\left(2\pi \sqrt{\left(\frac{Au}{4}\right)^2 + \left(\frac{Av}{4}\right)^2}\right)$$

...

1 (ii) cont

$$F(u,v) = 0.8 \frac{\sin(2\pi A u)\sin(2\pi A v)}{\pi^2 uv} - 0.6 \frac{\sin(\pi A u)\sin(\pi A v)}{\pi^2 uv}$$

$$+ 0.1 A \frac{1}{\sqrt{u^2+v^2}} J_1\left(\frac{\pi A}{2}\sqrt{u^2+v^2}\right)$$

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2. Picture width is $3840X$ and picture height is $2160Y$. Thus, the aspect ratio is

$$ar = \frac{3840X}{2160Y} = \frac{16X}{9Y}$$

Since the aspect ratio is given to be $\frac{16}{9}$, thus $Y = X$

Now $Y = \frac{1}{2160} ph$ so $X = \frac{1}{2160} ph$

Thus, the rectangular lattice has sampling matrix

$$V = \begin{bmatrix} \frac{1}{2160} & 0 \\ 0 & 1/2160 \end{bmatrix} \qquad d(\Lambda) = |\det V| = \frac{1}{(2160)^2} ph^2$$

The sampling density is $\frac{1}{d(\Lambda)} = (2160)^2 = 4{,}665{,}600 \frac{\text{samples}}{ph^2}$

Picture area is $\frac{16}{9} ph^2 \Rightarrow$ no of samples is $\frac{16}{9} \times 4{,}665{,}600$
$= 8{,}294{,}400$ samples/image. $(= 3840 \times 2160)$

3. $V_\Lambda = \begin{bmatrix} 2X & X \\ 0 & X \end{bmatrix}$ $\quad V_\Gamma = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}$

$d(\Lambda) = 2X^2 \qquad d(\Gamma) = X^2$

(i) $\begin{bmatrix} 2X \\ 0 \end{bmatrix} = 2\begin{bmatrix} X \\ 0 \end{bmatrix} \in \Gamma \qquad \begin{bmatrix} X \\ X \end{bmatrix} = \begin{bmatrix} X \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ X \end{bmatrix} \in \Gamma$

Both basis vectors of Λ belong to Γ, so all elements of Λ belong to Γ and $\Lambda \subset \Gamma$

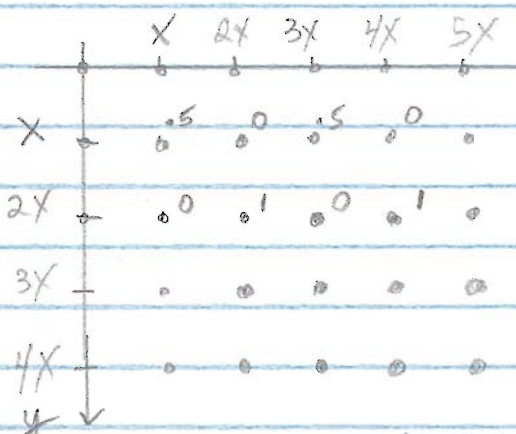
Alternatively, using Theorem 2 of the notes

$$(V_\Gamma)^{-1} V_\Lambda = \begin{bmatrix} 1/X & 0 \\ 0 & 1/X \end{bmatrix}\begin{bmatrix} 2X & X \\ 0 & X \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$ an integer matrix

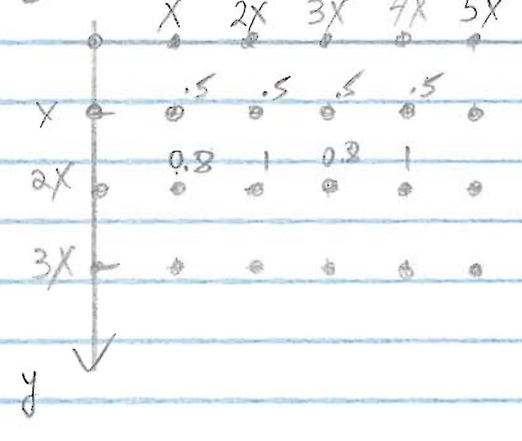
(ii)

$$f[x,y] \longrightarrow \boxed{\Lambda \uparrow \Gamma} \xrightarrow{g[x,y]} \boxed{H(u,v)} \longrightarrow g[x,y]$$

(iii)



$g[x,y] \quad X \le x \le 4X$
$X \le y \le 2X$



$g[x,y] \quad X \le x \le 4X$
$X \le y \le 2X$

(iv) $$H(0,0) = \sum_{(x,y)\in\Gamma} h[x,y] = 1 + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} = 2$$

We expect the DC gain to be equal to the upsampling factor $d(\Lambda)/d(\Gamma) = 2$, so it is as expected.