CEG4311 **Digital Image Processing** **Dec. 21, 2004**

Final exam **Duration: 3 hours**

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Closed-book exam: you may not use any books, notes or calculator. Answer all questions in the space provided and use the reverse side of the pages for rough work. *Explain all assumptions and calculations*; I am more interested in the reasoning than in precise numerical answers. Unless otherwise specified, you may use the results provided on pages 22–23 without proof but with a clear explanation. Vous pouvez répondre en anglais ou en français.

question	points	mark
1a	1	
1b	3	
1c	3	
2a	2	
2b	3	
2c	3	
2d	3	
3a	3	
3b	3	
3c	3	
4a	2	
4b	2	
4c	3	
4d	3	
4e	3	
	40	

Name:

Student number:

1. Consider the crosshatch image shown in Fig. 1 below.

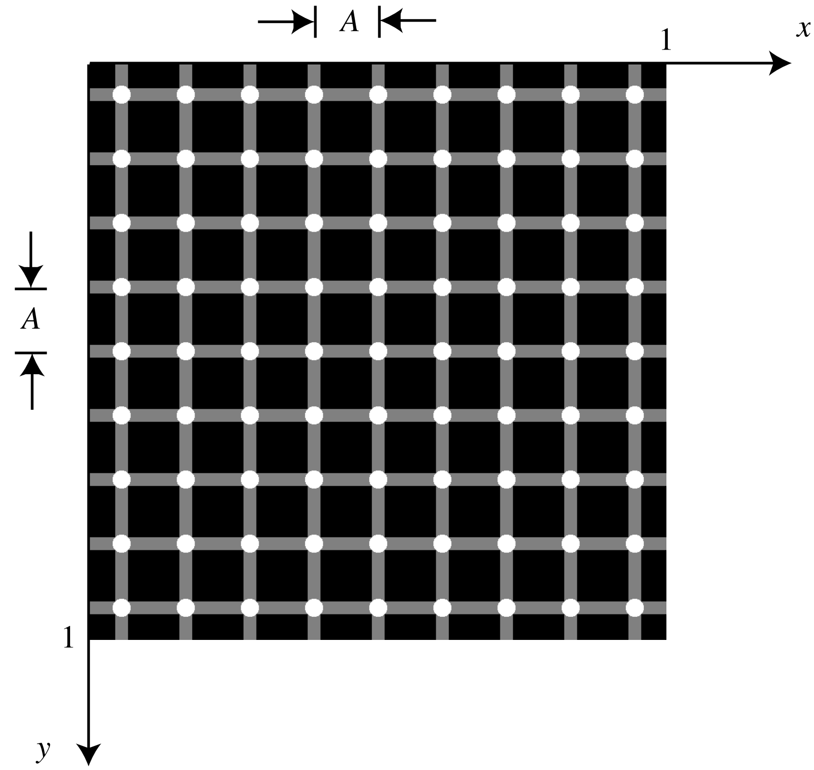


Figure 1: Crosshatch image.

1(a) Is this a still image or a time-varying image? Gaze generally at the image. What do you notice? Stare at one of the white dots. What do you notice now? Comment on your observations. Don't look at the image longer than you have to and hide it if it bothers you.

1(b) This image, considered as a continuous-space image, can be generated by repeating a basic function $f_A(x, y)$ on the points of a rectangular lattice with horizontal and vertical spacing A. The function $f_A(x, y)$ is illustrated in Fig. 2.

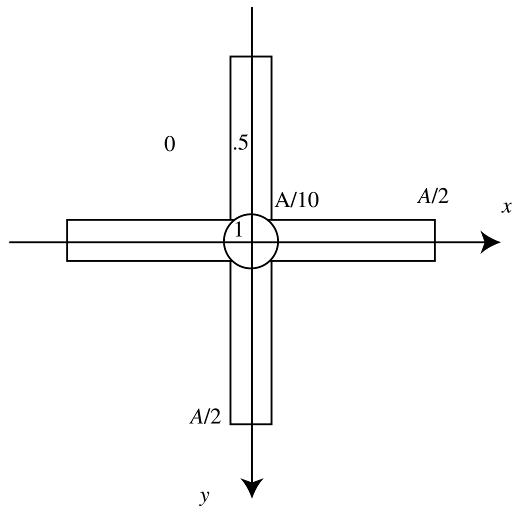


Figure 2: Basic function $f_A(x, y)$.

Note that the circle is completely inscribed within the plus-shaped figure and touches it at the four points $(\pm A/10, \pm A/10)$. Express this function $f_A(x, y)$ in terms of the standard rect function and circ function. There are several ways to do this, but a good way will give a solution of the form

$$f_A(x, y) = a_1 \operatorname{rect}(c_{11}x, c_{12}y) + a_2 \operatorname{rect}(c_{21}x, c_{22}y) - a_3 \operatorname{rect}(c_{31}x, c_{32}y) + a_4 \operatorname{circ}(c_{41}x, c_{42}y)$$

for some choice of these parameters a_i and c_{ij}. Clearly explain your reasoning. Express your solution in terms of the parameter A. From Fig. 1, what is the value of A?

1(b) continued

1(c) Determine the Fourier transform $F_A(u, v)$ of the basic function $f_A(x, y)$, using the attached tables. Explain clearly all steps and properties that you have used.

2. A class of discrete space images is defined on the lattice

$$\Lambda = \mathrm{LAT}\left(\begin{bmatrix} 4X & 2X \\ 0 & 1.5X \end{bmatrix}\right)$$

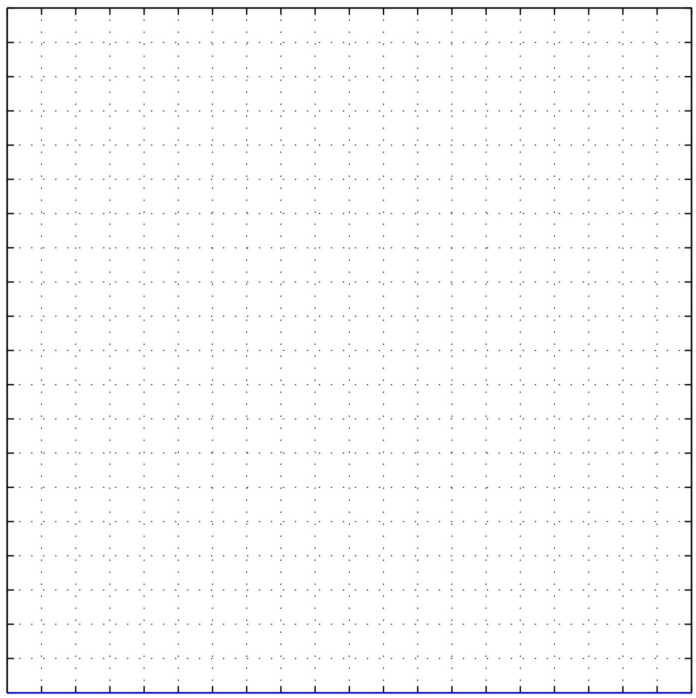
where $X = \frac{1}{100}$ ph and the aspect ratio is $ar = \frac{4}{3}$.

2(a) What is the sampling density and the approximate number of samples in one image?

2(b) A moving average filter defined on this lattice is given by

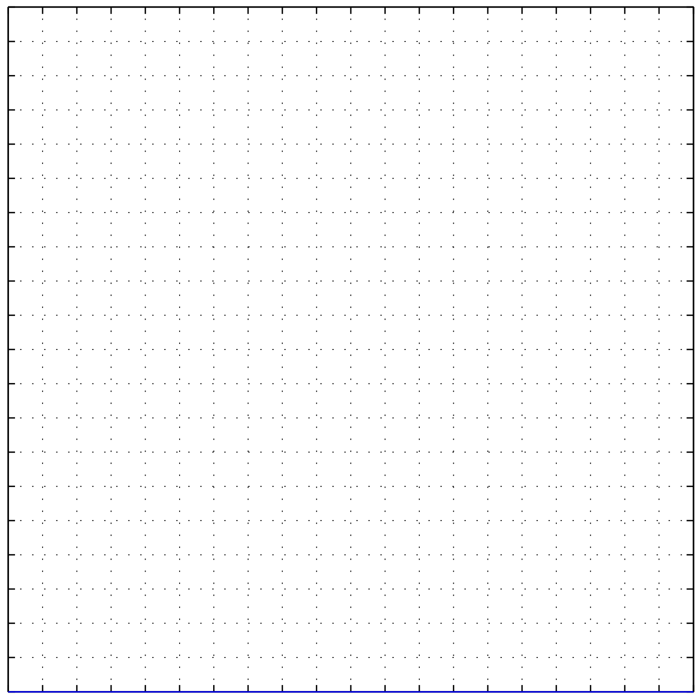
$$h[x,y] = \begin{cases} \frac{1}{5} & (x,y) = (0,0) \quad \text{or} \quad (x,y) = (\pm 2X, \pm 1.5X) \\ 0 & \text{otherwise.} \end{cases}$$

There are five non-zero coefficients. Sketch the points of Λ for $-5X \leq x \leq 5X$, $-5X \leq y \leq 5X$. Indicate $h[x,y]$ on your figure in the usual way.



2(c) Compute the frequency response $H(u, v)$ of the filter and express it in real form. What is the DC gain of this filter?

2(d) Sketch the points of the reciprocal lattice Λ^* for $-\frac{1}{X} \leq u, v \leq \frac{1}{X}$ and show the region corresponding to one period of $H(u,v)$ given by the Voronoi unit cell of Λ^*.



3. The primaries for the 1960 CIE Uniform Chromaticity Scale (UCS) color space are defined in terms of the standard CIE XYZ primaries $[X]$, $[Y]$ and $[Z]$ as follows:

$$[U] = 1.5[X] + 1.5[Z] \tag{1}$$

$$[V] = [Y] - 3[Z] \tag{2}$$

$$[W] = 2[Z] \tag{3}$$

An arbitrary color $[Q]$ is expressed in the XYZ space and the UVW space respectively as

$$[Q] = Q_X[X] + Q_Y[Y] + Q_Z[Z] \tag{4}$$

$$= Q_U[U] + Q_V[V] + Q_W[W] \tag{5}$$

3(a) The tristimulus values Q_U, Q_V and Q_W can be obtained from Q_X, Q_Y and Q_Z using the matrix equation

$$\begin{bmatrix} Q_U \\ Q_V \\ Q_W \end{bmatrix} = \mathbf{A} \begin{bmatrix} Q_X \\ Q_Y \\ Q_Z \end{bmatrix}$$

Using the information provide above, determine the matrix $\mathbf{A}$ from first principles. Do not quote any unproved (by you) results from the notes.

Formula which may be useful:

$$\begin{bmatrix} a & 0 & 0 \\ d & b & 0 \\ f & e & c \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ -\frac{d}{ab} & \frac{1}{b} & 0 \\ \frac{de-bf}{abc} & -\frac{e}{bc} & \frac{1}{c} \end{bmatrix}$$

3(a) continued

3(b) Determine the XYZ chromaticities of the four colors $[U]$, $[V]$, $[W]$, and $[U]+[V]+[W]$ and plot them on the xy chromaticity diagram below. Which, if any, of the four colors are physically realizable? Explain.

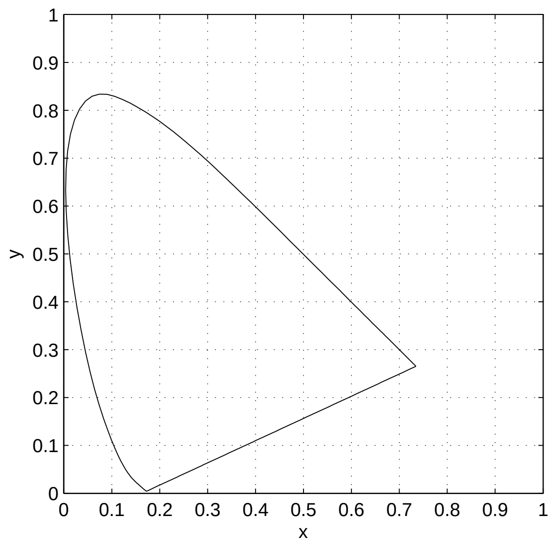


Figure 3: CIE XYZ chromaticity diagram.

3(c) A light $[Q]$consisting consisting of the superposition of two monochromatic (spectral) lights has the power density spectrum approximated by

$$Q(\lambda) = \delta(\lambda - 600) + 0.5\delta(\lambda - 520)$$

where λ is measured in nanometers (nm). Estimate the XYZ tristimulus values of $[Q(\lambda)]$ using the plotted XYZ color matching functions. What are the corresponding UVW tristimulus values. Give these values in equations with correct notation. Your calculations do not have to be very precise, and you may give answers as arithmetic expressions. Do explain what you are doing.

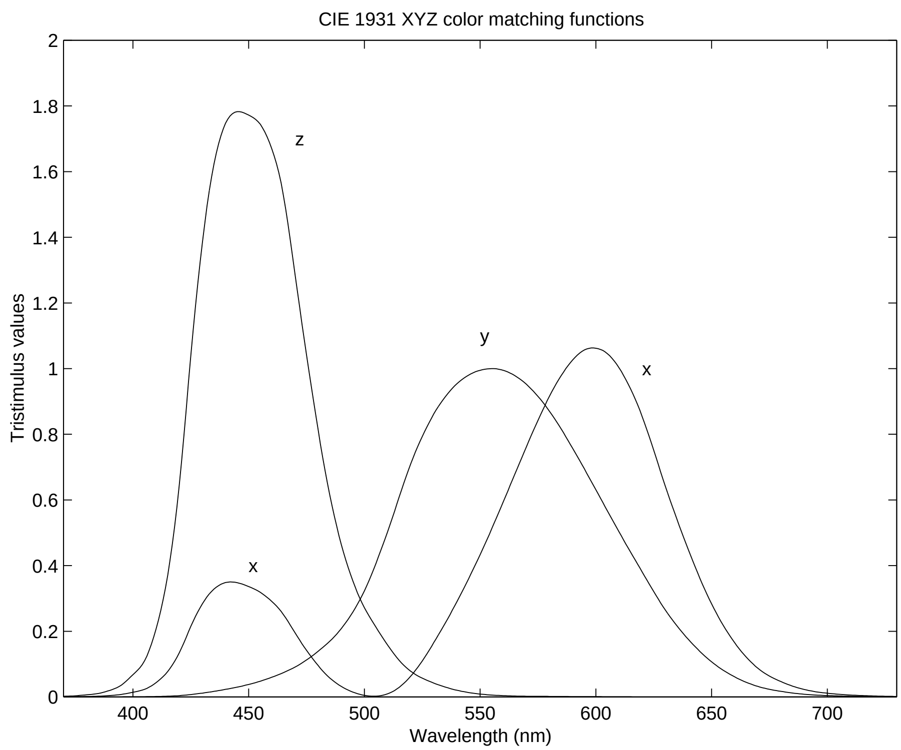


Figure 4: CIE XYZ color matching functions.

3(c) continued

4. Consider the following four one-dimensional vectors in $\mathbb{R}^4$:

$$\vec{\boldsymbol{b}}_{h0} = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}$$
$$\vec{\boldsymbol{b}}_{h1} = \begin{bmatrix}1 & 1 & -1 & -1\end{bmatrix}$$
$$\vec{\boldsymbol{b}}_{h2} = \begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix}$$
$$\vec{\boldsymbol{b}}_{h3} = \begin{bmatrix}1 & -1 & 1 & -1\end{bmatrix}$$

4(a) Show that $\vec{\boldsymbol{b}}_{hi}, i = 0, 1, 2, 3$ form an *orthogonal* basis for $\mathbb{R}^4$. Normalize the basis vectors to obtain an *orthonormal* basis $\vec{\boldsymbol{c}}_{hi}, i = 0, 1, 2, 3$.

4(b) A two-dimensional separable basis for the vector space of 4 by 4 real blocks, $\vec{\boldsymbol{c}}_{ij}, i, j = 0, 1, 2, 3$ is obtained from the orthonormal vectors of (a) by

$$c_{ij}[m, n] = c_{hi}[m]c_{hj}[n], \qquad m, n = 0, 1, 2, 3$$

Determine and illustrate the basis vector $\vec{\boldsymbol{c}}_{12}$ as a matrix with the first index indicating the row and the second index indicating the column.

4(c) Consider an image block illustrated as a matrix with the same indexing convention as in 4(b):

$$\mathbf{f} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Determine the 4×4 matrix of transform coefficients $\mathbf{f}_b$ using the orthonormal separable basis vectors of 4(b). (Hint: factor any constant terms out of the matrices to simplify computation.)

4(d) A *different* image block is transformed with respect to the given basis to obtain the transform matrix

$$\mathbf{f}_b = \begin{bmatrix} 2.3 & .6 & -.5 & .2 \\ -.7 & .2 & .3 & -.1 \\ .1 & -.5 & .2 & 0 \\ -.2 & 0 & .3 & 0 \end{bmatrix}$$

This matrix is quantized using the *same* uniform quantizer with step size $\Delta = 0.2$ for all coefficients. Determine the output of the quantizer encoders and the quantizer decoders, displayed as matrices.

4(d) continued

4(e) Yet another *different* image block has been transformed and quantized. The output of the quantizer encoders is shown below in matrix form:

$$\begin{bmatrix} 12 & -2 & 5 & 2 \\ -4 & 3 & 6 & 0 \\ 2 & 0 & -4 & 0 \\ 0 & 3 & -2 & 0 \end{bmatrix}.$$

These indexes are scanned in a zigzag fashion to form a one dimensional length-16 vector. These are losslessly encoded using the Golomb-Rice code with $m = 4$. Convert the data above to a bitstream using this Golomb-Rice code. Show how to deal with negative values. Explain how to decode your bitstream unambiguously to recover the 4×4 matrix of quantizer indices. Recall that a Golomb codeword for an integer n has the form [q-zeros 1 k-bit representation of r], where $n = mq + r$ and $m = 2^k$.

(4(d) continued

	$f(\boldsymbol{x}) = \int_{\mathbb{R}^D} F(\boldsymbol{u}) \exp(j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{u}$	$F(\boldsymbol{u}) = \int_{\mathbb{R}^D} f(\boldsymbol{x}) \exp(-j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{x}$
(i)	$af_1(\boldsymbol{x}) + bf_2(\boldsymbol{x})$	$aF_1(\boldsymbol{u}) + bF_2(\boldsymbol{u})$
(ii)	$f(\boldsymbol{x} - \boldsymbol{x}_0)$	$F(\boldsymbol{u}) \exp(-j2\pi \boldsymbol{u} \cdot \boldsymbol{x}_0)$
(iii)	$f(\boldsymbol{x}) \exp(j2\pi \boldsymbol{u}_0 \cdot \boldsymbol{x})$	$F(\boldsymbol{u} - \boldsymbol{u}_0)$
(iv)	$f^*(\boldsymbol{x})$	$F^*(-\boldsymbol{u})$
(v)	$f(\mathbf{A}\boldsymbol{x})$	$\frac{1}{\lvert \det \mathbf{A} \rvert} F(\mathbf{A}^{-T} \boldsymbol{u})$
(vi)	$f_1(\boldsymbol{x}) * f_2(\boldsymbol{x})$	$F_1(\boldsymbol{u}) F_2(\boldsymbol{u})$
(vii)	$f_1(\boldsymbol{x}) f_2(\boldsymbol{x})$	$F_1(\boldsymbol{u}) * F_2(\boldsymbol{u})$
(viii)	$f_1(x) f_2(y)$	$F_1(u) F_2(v)$
(ix)	$\int_{\mathbb{R}^D} \lvert f(\boldsymbol{x}) \rvert^2\, d\boldsymbol{x} = \int_{\mathbb{R}^D} \lvert F(\boldsymbol{u}) \rvert^2\, d\boldsymbol{u}$	

Multidimensional Fourier transform properties.

$f(\boldsymbol{x}) = \int_{\mathbb{R}^D} F(\boldsymbol{u}) \exp(j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{u}$	$F(\boldsymbol{u}) = \int_{\mathbb{R}^D} f(\boldsymbol{x}) \exp(-j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{x}$
$\mathrm{rect}(x, y)$	$\frac{\sin \pi u}{\pi u} \frac{\sin \pi v}{\pi v}$
$\mathrm{circ}(x, y)$	$\frac{1}{\sqrt{u^2+v^2}} J_1(2\pi \sqrt{u^2 + v^2})$
$\exp(-(x^2 + y^2)/2r_0^2)$	$2\pi r_0^2 \exp(-2\pi^2 (u^2 + v^2) r_0^2)$
$\cos(\pi (x^2 + y^2)/r_0^2)$	$r_0^2 \sin(\pi (u^2 + v^2) r_0^2)$
$\exp(j\pi (x^2 + y^2)/r_0^2)$	$jr_0^2 \exp(-j\pi (u^2 + v^2) r_0^2)$
$\delta(\boldsymbol{x})$	1

Multidimensional Fourier transform of selected functions.

Formulas

$$\mathrm{rect}(x,y) = \begin{cases} 1, & \text{if } |x| \le 0.5 \text{ and } |y| \le 0.5; \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathrm{circ}(x,y) = \begin{cases} 1, & \text{if } x^2+y^2 \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

If $\Lambda = \mathrm{LAT}(\mathbf{V})$, then $d(\Lambda) = |\det(\mathbf{V})|$, and $\Lambda^* = \mathrm{LAT}(\mathbf{V}^{-T})$.

The Fourier transform pair for a signal $f[\boldsymbol{x}]$ defined on the lattice Λ is given by

$$F(\boldsymbol{u}) = \sum_{\boldsymbol{x}\in\Lambda} f[\boldsymbol{x}] \exp(-j2\pi \boldsymbol{u}\cdot\boldsymbol{x})$$

$$f[\boldsymbol{x}] = d(\Lambda) \int_{\mathcal{P}^*} F(\boldsymbol{u}) \exp(j2\pi \boldsymbol{u}\cdot\boldsymbol{x})\, d\boldsymbol{u}$$

Sampling

If $f[\boldsymbol{x}] = f_c(\boldsymbol{x}), \quad \boldsymbol{x} \in \Lambda$ then

$$F(\boldsymbol{u}) = \frac{1}{d(\Lambda)} \sum_{r\in\Lambda^*} F_c(\boldsymbol{u}+\boldsymbol{r})$$

Tristimulus values

$$C_i = \int_{\lambda_{\min}}^{\lambda_{\max}} C(\lambda)\bar{p}_i(\lambda)\, d\lambda.$$

where $\bar{p}_i(\lambda)$ are the color matching functions of the primaries $[P_i]$, $i = 1,2,3$.

Chromaticities

$$c_i = \frac{C_i}{C_1+C_2+C_3}$$

Luminance

$$C_L = C_1P_{1L} + C_2P_{2L} + C_3P_{3L}$$

Obtaining tristimulus values from luminance and chromaticities

$$C_i = \frac{C_L c_i}{c_1P_{1L} + c_2P_{2L} + c_3P_{3L}}$$

Representation by orthogonal basis vectors

$$\vec{\boldsymbol{f}} = \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} f_b[i,j]\vec{\boldsymbol{b}}_{ij}$$

$$f_b[k,l] = \frac{\langle \vec{\boldsymbol{f}} \mid \vec{\boldsymbol{b}}_{kl}\rangle}{\|\vec{\boldsymbol{b}}_{kl}\|^2}.$$