CEG4311 **Digital Image Processing** **Dec. 10, 2003**

Final exam **Duration: 3 hours**

Professor: Eric Dubois

Closed-book exam: you may not use any books, notes or calculator. Answer all questions in the space provided and use the reverse side of the pages for rough work. Explain all assumption and calculations; I am more interested in the reasoning than in precise numerical answers. Unless otherwise specified, you may use the results provided on pages 19-20 without proof but with a clear explanation.

Vous pouvez répondre en anglais ou en français.

question	points	mark
1a	4	
1b	6	
2a	4	
2b	3	
2c	3	
3a	3	
3b	3	
3c	4	
4a	6	
4b	4	
	40	

Name:

Student number:

1(a) Determine the two-dimensional Fourier transform $P_{\mathcal{A}}(u,v)$ of the zero-one function $p_{\mathcal{A}}(x,y)$, where $\mathcal{A}$ is the plus-shaped region illustrated in Fig. 1. Note that $p_{\mathcal{A}}(x,y) = p_{\mathcal{A}}(-x,-y)$, so that $P_{\mathcal{A}}(u,v)$ is real. Express your solution in real form. (Hint: Express $p_{\mathcal{A}}(x,y)$ as the sum of functions whose Fourier transform can be computed using the tables and properties.)

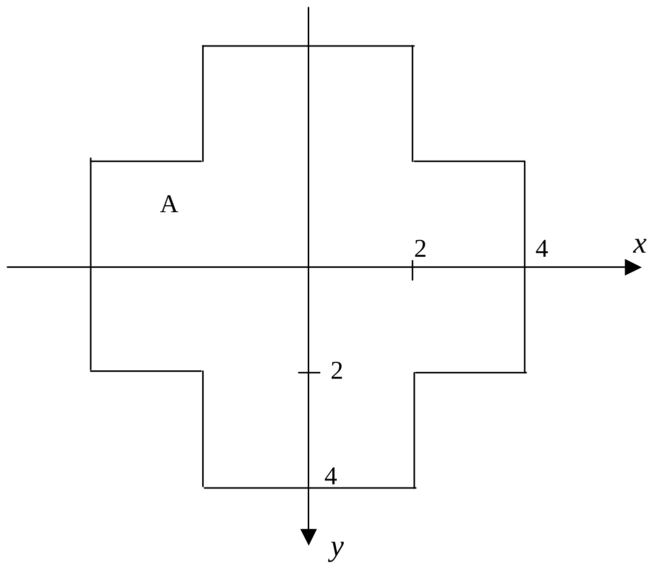

Figure 1: Region of support $\mathcal{A}$ of zero-one function $p_{\mathcal{A}}(x,y)$.

1(a) continued

1(b) A class of discrete-space still images is sampled on the sampling lattice Λ generated by the sampling matrix

$$V_\Lambda = \begin{bmatrix} 2X & X \\ 0 & 0.75X \end{bmatrix}.$$

These images are filtered by the linear shift-invariant FIR filter with unit-sample response $h[x, y] = 0.5\delta[x, y] + 0.25\delta[x - X, y + 0.75X] + 0.25\delta[x + X, y - 0.75X] + 0.125\delta[x - X, y - 0.75X] + 0.125\delta[x + X, y + 0.75X]$. Sketch, to scale, points of the lattice Λ for $-2X \leq x, y \leq 2X$, and illustrate $h[x, y]$ on the same axes. Compute the frequency response $H(u, v)$ of the filter and express it in real form.

1(b) continued

2(a) The cyan, magenta and yellow (CMY) primaries using for printing can be expressed as $[C] = [B] + [G]$, $[M] = [R] + [B]$ and $[YE] = [R] + [G]$. We assume here that $[R]$, $[G]$ and $[B]$ are the Rec. 709 RGB primaries. A color $[Q]$ can be expressed in terms of the RGB primaries as $[Q] = Q_R[R] + Q_G[G] + Q_B[B]$. Determine the representation of $[Q]$ in terms of the CMY primaries, using only the above information (i.e., don't quote an unproved result from the course notes). The following result *may* be of use:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

2(a) continued

2(b) Determine the tristimulus values of the color $[Q] = [\delta(\lambda - 480) + 0.5\delta(\lambda - 600)]$ with respect to the CMY primaries. You will need to use the color matching functions of the Rec. 709 RGB primaries, which are attached here as Table 1.

λ	$\bar{r}$	$\bar{g}$	$\bar{b}$
400	0.01	-0.01	0.07
410	0.04	-0.03	0.22
420	0.11	-0.10	0.69
430	0.21	-0.20	1.48
440	0.22	-0.22	1.86
450	0.15	-0.18	1.88
460	0.02	-0.10	1.77
470	-0.15	0.03	1.35
480	-0.31	0.20	0.84
490	-0.45	0.38	0.45
500	-0.62	0.61	0.22
510	-0.82	0.94	0.07
520	-0.93	1.27	-0.06
530	-0.81	1.46	-0.12
540	-0.54	1.51	-0.16
550	-0.13	1.45	-0.17
560	0.40	1.29	-0.17
570	1.01	1.05	-0.15
580	1.63	0.74	-0.12
590	2.16	0.43	-0.10
600	2.47	0.15	-0.07
610	2.48	-0.03	-0.05
620	2.18	-0.11	-0.03
630	1.67	-0.13	-0.02
640	1.18	-0.11	-0.01
650	0.75	-0.07	-0.01
660	0.44	-0.05	-0.00
670	0.23	-0.02	-0.00
680	0.13	-0.01	-0.00
690	0.06	-0.01	-0.00
700	0.03	-0.00	-0.00

Table 1. Color matching functions of Rec. 709 RGB primaries.

2(c) Fig. 2 shows a portion of the chromaticity diagram in the Rec. 709 RGB space. Illustrate on this diagram the set of chromaticities of all colors that can be synthesized as a linear of combination of $[R]$, $[G]$ and $[B]$ with non-negative tristimulus values. On the same diagram, show the set of chromaticities of all colors that can be synthesized as a linear of combination of $[C]$, $[M]$ and $[YE]$ with non-negative tristimulus values. Clearly distinguish the two sets on your diagram and explain how you got them. Comment on the suitability of the CMY primaries as opposed to the RGB primaries for additive reproduction of colors.

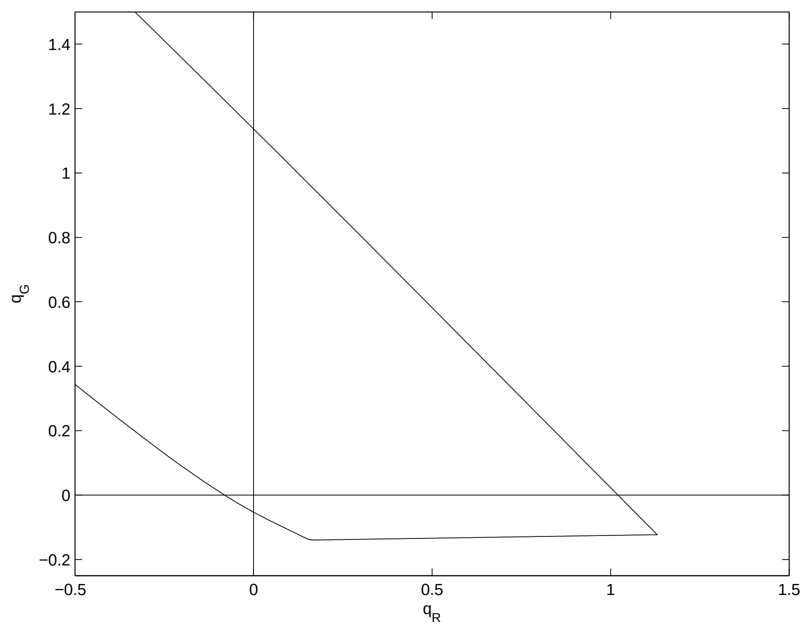

Figure 2: Portion of Rec. 709 RGB rg chromaticity diagram.

3(a) Consider the following three vectors in $\mathbb{R}^3$: $\mathbf{b}_{h0} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, $\mathbf{b}_{h1} = \begin{bmatrix} -1 & 2 & -1 \end{bmatrix}$, $\mathbf{b}_{h2} = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$. Verify that these form an orthogonal, but not orthonormal, basis for $\mathbb{R}^3$. Determine an orthonormal basis $\mathbf{c}_{hi}$, $i = 0, 1, 2$ by normalizing these basis vectors.

3(b) These one-dimensional basis vectors are used to form a separable basis for the nine-dimensional space of 3×3 image blocks, $c_{ij}[m,n] = c_{hi}[m]c_{hj}[n]$, where $0 \leq i,j,m,n \leq 2$. Illustrate the following three basis vectors: $\mathbf{c}_{00}$, $\mathbf{c}_{02}$, $\mathbf{c}_{21}$ Make sure to indicate the horizontal and vertical directions in your illustrations. Check that these three basis vectors satisfy the orthonormality condition in $\mathbb{R}^{3\times 3}$.

3(b) continued

3(c) The following 3×3 image block is analysed in terms of the separable basis of 3(b).

$$\mathbf{f} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

The block is shown in matrix orientation, with the horizontal axis oriented downward and the vertical axis oriented to the right. The block is to be reconstructed using only the four basis vectors $\mathbf{c}_{ij}$, $i, j = 0, 1$. Show the transformed image block and the reconstructed image block. Calculate the total squared error between the original block and the reconstructed block in the space domain and in the transform domain, and verify that they are equal.

3(c) continued

4(a) A computer-generated image clip is formed using six colors: white [W], red [R], yellow [Y], green [G], blue [B] and black [K]. These occur in the clip with the following relative frequencies (or probabilities)

[W]	[R]	[Y]	[G]	[B]	[K]
0.5	0.1	0.05	0.05	0.2	0.1

Based on these probabilities, construct a Huffman code that minimizes the average codeword length. Compute the average codeword length that you obtain. Construct a second Huffman code with a different set of codeword lengths, and verify that the average codeword length is the same.

4(a) continued

4(b) A series of consecutive image values along a scan line are ... [W] [W] [W] [R] [R] [K] [B] [B] [Y] [W]... Generate the sequence of code bits produced by your first Huffman code for this data and calculate the average number of bits per image sample. Explain why you are able to uniquely separate the sequence of bits into code words and thus regenerate the image values without error.

	$f(\boldsymbol{x}) = \int_{\mathbb{R}^D} F(\boldsymbol{u}) \exp(j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{u}$	$F(\boldsymbol{u}) = \int_{\mathbb{R}^D} f(\boldsymbol{x}) \exp(-j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{x}$
(i)	$af_1(\boldsymbol{x}) + bf_2(\boldsymbol{x})$	$aF_1(\boldsymbol{u}) + bF_2(\boldsymbol{u})$
(ii)	$f(\boldsymbol{x} - \boldsymbol{x}_0)$	$F(\boldsymbol{u}) \exp(-j2\pi \boldsymbol{u} \cdot \boldsymbol{x}_0)$
(iii)	$f(\boldsymbol{x}) \exp(j2\pi \boldsymbol{u}_0 \cdot \boldsymbol{x})$	$F(\boldsymbol{u} - \boldsymbol{u}_0)$
(iv)	$f^*(\boldsymbol{x})$	$F^*(-\boldsymbol{u})$
(v)	$f(\mathbf{A}\boldsymbol{x})$	$\frac{1}{\lvert\det \mathbf{A}\rvert} F(\mathbf{A}^{-T}\boldsymbol{u})$
(vi)	$f_1(\boldsymbol{x}) * f_2(\boldsymbol{x})$	$F_1(\boldsymbol{u}) F_2(\boldsymbol{u})$
(vii)	$f_1(\boldsymbol{x}) f_2(\boldsymbol{x})$	$F_1(\boldsymbol{u}) * F_2(\boldsymbol{u})$
(viii)	$f_1(x) f_2(y)$	$F_1(u) F_2(v)$
(ix)	$\int_{\mathbb{R}^D} \lvert f(\boldsymbol{x})\rvert^2\, d\boldsymbol{x} = \int_{\mathbb{R}^D} \lvert F(\boldsymbol{u})\rvert^2\, d\boldsymbol{u}$	

Multidimensional Fourier transform properties.

$f(\boldsymbol{x}) = \int_{\mathbb{R}^D} F(\boldsymbol{u}) \exp(j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{u}$	$F(\boldsymbol{u}) = \int_{\mathbb{R}^D} f(\boldsymbol{x}) \exp(-j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{x}$
$\mathrm{rect}(x, y)$	$\frac{\sin \pi u}{\pi u} \frac{\sin \pi v}{\pi v}$
$\mathrm{circ}(x, y)$	$\frac{1}{\sqrt{u^2+v^2}} J_1(2\pi\sqrt{u^2+v^2})$
$\exp(-(x^2+y^2)/2r_0^2)$	$2\pi r_0^2 \exp(-2\pi^2(u^2+v^2)r_0^2)$
$\cos(\pi(x^2+y^2)/r_0^2)$	$r_0^2 \sin(\pi(u^2+v^2)r_0^2)$
$\exp(j\pi(x^2+y^2)/r_0^2)$	$jr_0^2 \exp(-j\pi(u^2+v^2)r_0^2)$
$\delta(\boldsymbol{x})$	1

Multidimensional Fourier transform of selected functions.

Formulas

$$\mathrm{rect}(x,y)=\begin{cases}1, & \text{if } |x|\le 0.5 \text{ and } |y|\le 0.5;\\ 0, & \text{otherwise.}\end{cases}$$

$$\mathrm{circ}(x,y)=\begin{cases}1, & \text{if } x^2+y^2\le 1;\\ 0, & \text{otherwise.}\end{cases}$$

If $\Lambda=\mathrm{LAT}(\mathbf{V})$, then $d(\Lambda)=|\det(\mathbf{V})|$, and $\Lambda^*=\mathrm{LAT}(\mathbf{V}^{-T})$.

The Fourier transform pair for a signal $f[\boldsymbol{x}]$ defined on the lattice Λ is given by

$$F(\boldsymbol{u})=\sum_{\boldsymbol{x}\in\Lambda} f[\boldsymbol{x}]\exp(-j2\pi\boldsymbol{u}\cdot\boldsymbol{x})$$

$$f[\boldsymbol{x}]=d(\Lambda)\int_{\mathcal{P}^*} F(\boldsymbol{u})\exp(j2\pi\boldsymbol{u}\cdot\boldsymbol{x})\,d\boldsymbol{u}$$

Sampling

If $f[\boldsymbol{x}]=f_c(\boldsymbol{x}),\quad \boldsymbol{x}\in\Lambda$ then

$$F(\boldsymbol{u})=\frac{1}{d(\Lambda)}\sum_{r\in\Lambda^*} F_c(\boldsymbol{u}+\boldsymbol{r})$$

Tristimulus values

$$C_i=\int_{\lambda_{\min}}^{\lambda_{\max}} C(\lambda)\bar{p}_i(\lambda)\,d\lambda.$$

where $\bar{p}_i(\lambda)$ are the color matching functions of the primaries $[P_i]$, $i=1,2,3$.

Chromaticities

$$c_i=\frac{C_i}{C_1+C_2+C_3}$$

Luminance

$$C_L=C_1P_{1L}+C_2P_{2L}+C_3P_{3L}$$

Obtaining tristimulus values from luminance and chromaticities

$$C_i=\frac{C_Lc_i}{c_1P_{1L}+c_2P_{2L}+c_3P_{3L}}$$

Representation by orthogonal basis vectors

$$\vec{\boldsymbol{f}}=\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} f_b[i,j]\vec{\boldsymbol{b}}_{ij}$$

$$f_b[k,l]=\frac{\langle\vec{\boldsymbol{f}}\mid\vec{\boldsymbol{b}}_{kl}\rangle}{\|\vec{\boldsymbol{b}}_{kl}\|^2}.$$