CEG4311 **Digital Image Processing** **Dec. 21, 2001**

Final exam **Duration: 3 hours**

Professor: Eric Dubois

Closed-book exam: you may not use any books or notes. Answer all questions. You may use a simple calculator. However, explain all calculations; I am more interested in the reasoning than in precise numerical answers. Unless otherwise specified, you may use the results provided on pages 16-17 without proof but with a clear explanation.

Vous pouvez répondre en anglais ou en français.

question	points	mark
1a	7	
1b	8	
2a	15	
2b	10	
3a	10	
3b	10	
4a	8	
4b	6	
4c	8	
5a	10	
5b	8	
	100	

1. A photograph is 10 cm high and 15 cm wide. It is sampled with a rectangular sampling lattice Λ with 60 samples/cm in the horizontal direction and 40 samples/cm in the vertical direction.

(a) Determine a sampling matrix $\mathbf{V}$ for the sampling lattice and express it in both units of ph and cm. Determine the *sampling density* in both samples/ph^2 and samples/cm^2.

(b) Assume that each sample value is obtained by integrating the light reflected from a rectangular unit cell of Λ centered on the corresponding lattice point. This is equivalent to prefiltering the continuous space image with a 2D continuous space filter with impulse response $h_a(x, y)$ followed by ideal sampling on Λ. Write an expression for $h_a(x, y)$ in terms of the rect function (pg. 17). Determine the frequency response $H_a(u, v)$ of the prefilter, where u and v are in units of c/ph. Determine the frequency response if u and v are in units of c/cm.

2. The images in a certain imaging system are defined on a hexagonal lattice Λ determined by the sampling matrix

$$\mathbf{V} = \begin{bmatrix} 2X & X \\ 0 & X \end{bmatrix}$$

where X is in units of ph. An image is filtered by a linear shift-invariant system with unit sample response defined on Λ

$$h[x,y] = \begin{cases} \frac{1}{2} & (x,y) = (0,0) \\ -\frac{3}{8} & (x,y) = (X,-X) \\ -\frac{3}{8} & (x,y) = (-X,X) \\ \frac{1}{8} & (x,y) = (X,X) \\ \frac{1}{8} & (x,y) = (-X,-X) \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the frequency response $H(u,v)$ where u and v are in c/ph. Simplify your expression as much as possible. Sketch the reciprocal lattice Λ^* in the frequency domain, and indicate a unit cell of Λ^*. Carefully explain the significance of the reciprocal lattice and its unit cell with regard to the frequency response.

2(a) continued.

(b) Determine the output image $g[x, y]$ when the following test image is applied as input to the filter h of part (a).

$$f[x, y] = 0.5\left[1 + \cos\left(\frac{2\pi x}{4X} - \frac{2\pi y}{8X}\right)\right], \qquad (x, y) \in \Lambda$$

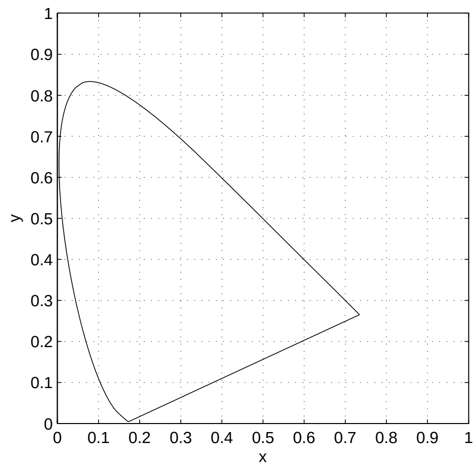
3. The recommendation 709 RGB primaries can be expressed in terms of the CIE XYZ primaries by

$$\begin{bmatrix} [R] \\ [G] \\ [B] \end{bmatrix} = \begin{bmatrix} 0.41 & 0.21 & 0.02 \\ 0.36 & 0.72 & 0.12 \\ 0.18 & 0.07 & 0.95 \end{bmatrix} \begin{bmatrix} [X] \\ [Y] \\ [Z] \end{bmatrix} .$$

Consider the cyan, magenta and yellow (CMY) primaries used in printing. These are given by $[C] = [B] + [G]$, $[M] = [R] + [B]$ and $[YE] = [R] + [G]$.

(a) Determine the *tristimulus values* of $[C]$, $[M]$ and $[YE]$ with respect to the XYZ primaries. Compute the XYZ *chromaticities* of $[C]$, $[M]$ and $[YE]$ and plot them on the xy chromaticity diagram on the next page. Comment on the suitability of cyan, magenta and yellow as primaries for an additive color display device like a cathode ray tube (CRT).

3(a) Continued.



(b) Suppose that $[C]$, $[M]$ and $[YE]$ as in part (a) are taken as primaries in a color system. Determine the tristimulus values of a monochromatic light $\delta(\lambda - 650nm)$ with respect to these primaries. You will need to use the XYZ color matching functions tabulated on the next page. Carefully explain all steps. Can the given light be *physically* synthesized as a sum of a positive quantity of the $[C]$, $[M]$ and $[YE]$ primaries? You may use the expression

$$\begin{bmatrix} [X] \\ [Y] \\ [Z] \end{bmatrix} = \begin{bmatrix} -2.08 & 2.13 & 1.12 \\ 1.60 & -1.80 & 0.26 \\ 0.80 & 0.26 & -0.76 \end{bmatrix} \begin{bmatrix} [C] \\ [M] \\ [YE] \end{bmatrix}$$

but explain how you obtain this equation from the results in part (a) of this question.

$\bar{x}$	$\bar{y}$	$\bar{z}$	λ
0.0143	0.0004	0.0679	400 nm
0.0435	0.0012	0.2074	
0.1344	0.0040	0.6456	
0.2839	0.0116	1.3856	
0.3483	0.0230	1.7471	
0.3362	0.0380	1.7721	450 nm
0.2908	0.0600	1.6692	
0.1954	0.0910	1.2876	
0.0956	0.1390	0.8130	
0.0320	0.2080	0.4652	
0.0049	0.3230	0.2720	500 nm
0.0093	0.5030	0.1582	
0.0633	0.7100	0.0782	
0.1655	0.8620	0.0422	
0.2904	0.9540	0.0203	
0.4334	0.9950	0.0087	550 nm
0.5945	0.9950	0.0039	
0.7621	0.9520	0.0021	
0.9163	0.8700	0.0017	
1.0263	0.7570	0.0011	
1.0622	0.6310	0.0008	600 nm
1.0026	0.5030	0.0003	
0.8544	0.3810	0.0002	
0.6424	0.2650	0.0000	
0.4479	0.1750	0.0000	
0.2835	0.1070	0.0000	650 nm
0.1649	0.0610	0.0000	
0.0874	0.0320	0.0000	
0.0468	0.0170	0.0000	
0.0227	0.0082	0.0000	
0.0114	0.0041	0.0000	700 nm

Color matching functions of XYZ primaries.

4. Consider the following four one-dimensional vectors in $\mathbb{R}^4$:

$$\vec{\boldsymbol{b}}_{h0} = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}$$
$$\vec{\boldsymbol{b}}_{h1} = \begin{bmatrix}1 & 1 & -1 & -1\end{bmatrix}$$
$$\vec{\boldsymbol{b}}_{h2} = \begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix}$$
$$\vec{\boldsymbol{b}}_{h3} = \begin{bmatrix}1 & -1 & 1 & -1\end{bmatrix}$$

(a) Show that $\vec{\boldsymbol{b}}_{hi}, i = 0, 1, 2, 3$ form an *orthogonal* basis for $\mathbb{R}^4$. Normalize the basis vectors to obtain an *orthonormal* basis $\vec{\boldsymbol{c}}_{hi}, i = 0, 1, 2, 3$.

(b) A two-dimensional separable basis for the vector space of 4 by 4 real blocks, $\vec{\boldsymbol{c}}_{ij}, i, j = 0, 1, 2, 3$ is obtained from the orthonormal vectors of (a) by

$$c_{ij}[m,n] = c_{hi}[m]c_{hj}[n], \qquad m,n = 0,1,2,3$$

Determine and sketch the basis vector $\vec{\boldsymbol{c}}_{23}$.

(c) Suppose that an image block is represented as a 4×4 matrix $\mathbf{f}$ where $[\mathbf{f}]_{ij} = f[i-1, j-1]$. Show how to obtain the 4×4 matrix of transform coefficients $\mathbf{f}_c$ for the separable basis of (b), and compute the transform coefficients for the image block given by

$$\mathbf{f} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

5. A cartoon image is formed using six colors: white [W], red [R], yellow [Y], green [G], blue [B] and black [K]. These occur in the cartoon show with the following relative frequencies (or probabilities)

[W]	[R]	[Y]	[G]	[B]	[K]
0.1	0.2	0.05	0.05	0.5	0.1

(a) Based on these probabilities, construct a Huffman code that minimizes the average codeword length. Compute the average codeword length that you obtain.

(b) A series of consecutive image values along a scan line are ... [B] [B] [B] [R] [R] [K] [B] [B] ... Generate the sequence of code bits produced by your code for this data and calculate the average number of bits per image sample. Explain why you are able to uniquely separate the sequence of bits into code words and thus regenerate the image values.

	$f(\boldsymbol{x}) = \int_{\mathbb{R}^D} F(\boldsymbol{u}) \exp(j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{u}$	$F(\boldsymbol{u}) = \int_{\mathbb{R}^D} f(\boldsymbol{x}) \exp(-j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{x}$
(i)	$af_1(\boldsymbol{x}) + bf_2(\boldsymbol{x})$	$aF_1(\boldsymbol{u}) + bF_2(\boldsymbol{u})$
(ii)	$f(\boldsymbol{x} - \boldsymbol{x}_0)$	$F(\boldsymbol{u}) \exp(-j2\pi \boldsymbol{u} \cdot \boldsymbol{x}_0)$
(iii)	$f(\boldsymbol{x}) \exp(j2\pi \boldsymbol{u}_0 \cdot \boldsymbol{x})$	$F(\boldsymbol{u} - \boldsymbol{u}_0)$
(iv)	$f^*(\boldsymbol{x})$	$F^*(-\boldsymbol{u})$
(v)	$f(\mathbf{A}\boldsymbol{x})$	$\frac{1}{\lvert\det \mathbf{A}\rvert} F(\mathbf{A}^{-T}\boldsymbol{u})$
(vi)	$f_1(\boldsymbol{x}) * f_2(\boldsymbol{x})$	$F_1(\boldsymbol{u}) F_2(\boldsymbol{u})$
(vii)	$f_1(\boldsymbol{x}) f_2(\boldsymbol{x})$	$F_1(\boldsymbol{u}) * F_2(\boldsymbol{u})$
(viii)	$f_1(x) f_2(y)$	$F_1(u) F_2(v)$
(ix)	$\int_{\mathbb{R}^D} \lvert f(\boldsymbol{x})\rvert^2\, d\boldsymbol{x} = \int_{\mathbb{R}^D} \lvert F(\boldsymbol{u})\rvert^2\, d\boldsymbol{u}$	

Multidimensional Fourier transform properties.

$f(\boldsymbol{x}) = \int_{\mathbb{R}^D} F(\boldsymbol{u}) \exp(j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{u}$	$F(\boldsymbol{u}) = \int_{\mathbb{R}^D} f(\boldsymbol{x}) \exp(-j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{x}$
$\mathrm{rect}(x, y)$	$\frac{\sin \pi u}{\pi u} \frac{\sin \pi v}{\pi v}$
$\mathrm{circ}(x, y)$	$\frac{1}{\sqrt{u^2+v^2}} J_1(2\pi\sqrt{u^2 + v^2})$
$\exp(-(x^2 + y^2)/2r^2)$	$2\pi r^2 \exp(-2\pi^2(u^2 + v^2)r^2)$
$\cos(\pi(x^2 + y^2)/r^2)$	$r^2 \sin(\pi(u^2 + v^2)r^2)$
$\exp(j\pi(x^2 + y^2)/r^2)$	$jr^2 \exp(-j\pi(u^2 + v^2)r^2)$
$\delta(\boldsymbol{x})$	1

Multidimensional Fourier transform of selected functions.

Formulas

$$\text{rect}(x,y) = \begin{cases} 1, & \text{if } |x| \le 0.5 \text{ and } |y| \le 0.5; \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{circ}(x,y) = \begin{cases} 1, & \text{if } x^2 + y^2 \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

If $\Lambda = \text{LAT}(\mathbf{V})$, then $d(\Lambda) = |\det(\mathbf{V})|$, and $\Lambda^* = \text{LAT}(\mathbf{V}^{-T})$.

The Fourier transform pair for a signal $f[\boldsymbol{x}]$ defined on the lattice Λ is given by

$$F(\boldsymbol{u}) = \sum_{\boldsymbol{x}\in\Lambda} f[\boldsymbol{x}] \exp(-j2\pi \boldsymbol{u}\cdot\boldsymbol{x})$$

$$f[\boldsymbol{x}] = d(\Lambda) \int_{\mathcal{P}^*} F(\boldsymbol{u}) \exp(j2\pi \boldsymbol{u}\cdot\boldsymbol{x})\, d\boldsymbol{u}$$

Sampling

If $f[\boldsymbol{x}] = f_c(\boldsymbol{x}), \quad \boldsymbol{x} \in \Lambda$ then

$$F(\boldsymbol{u}) = \frac{1}{d(\Lambda)} \sum_{r\in\Lambda^*} F_c(\boldsymbol{u} + \boldsymbol{r})$$

Tristimulus values

$$C_i = \int_{\lambda_{\min}}^{\lambda_{\max}} C(\lambda) \bar{p}_i(\lambda)\, d\lambda.$$

where $\bar{p}_i(\lambda)$ are the color matching functions of the primaries $[P_i]$, $i = 1, 2, 3$.

Chromaticities

$$c_i = \frac{C_i}{C_1 + C_2 + C_3}$$

Luminance

$$C_L = C_1 P_{1L} + C_2 P_{2L} + C_3 P_{3L}$$

Obtaining tristimulus values from luminance and chromaticities

$$C_i = \frac{C_L c_i}{c_1 P_{1L} + c_2 P_{2L} + c_3 P_{3L}}$$

Representation by orthogonal basis vectors

$$\vec{\boldsymbol{f}} = \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} f_b[i,j] \vec{\boldsymbol{b}}_{ij}$$

$$f_b[k,l] = \frac{\langle \vec{\boldsymbol{f}} \mid \vec{\boldsymbol{b}}_{kl} \rangle}{\|\vec{\boldsymbol{b}}_{kl}\|^2}.$$