CEG 4311 Midterm Fall 2005
Solution.

1 (a) $V = \begin{bmatrix} X & X/2 \\ 0 & Y \end{bmatrix}$ $\quad 480Y = 1\,ph$

$\therefore Y = \frac{1}{480}\,ph$

$X = \frac{2Y}{\sqrt{3}} = \frac{1}{240\sqrt{3}}\,ph$

$\approx \frac{1}{415.7}\,ph$

$\therefore V = \begin{bmatrix} \frac{1}{240\sqrt{3}} & \frac{1}{480\sqrt{3}} \\ 0 & \frac{1}{480} \end{bmatrix} ph$

(b) A sensor element is one unit cell of Λ.
$\therefore$ The area of a sensor element is $d(\Lambda)$.

$$d(\Lambda) = XY = \frac{1}{(480)(240)\sqrt{3}} = \frac{1}{199532.25}\,ph^2$$

The sampling density is $\frac{1}{d(\Lambda)} = 199532.25 \ \underline{\underline{\frac{\text{samples}}{ph^2}}}$

(c) $ar = \frac{pw}{ph} = \frac{MX}{NY} = \frac{740 \cdot \frac{2}{\sqrt{3}}Y}{480 \quad Y} = \frac{1480}{480\sqrt{3}} = 1.7802$

$\frac{4}{3} = 1.\dot{3}\dot{3} \qquad \frac{16}{9} = 1.\dot{7}\dot{7} \approx 1.78$

The aspect ratio is approximately 16/9

CEG 4311 Midterm Fall 2005
Solution

1\. (d) Let the hexagonal unit cell of Λ as shown in Fig. 1 be denoted $\mathcal{P}_\Lambda$.

Then $f[\underline{x}] = c \int_{\mathcal{P}_\Lambda} f_c(\underline{x}+\underline{s})\, d\underline{s}$ $\quad , \underline{x} \in \Lambda$

$$= c \int_{\mathbb{R}_2} f_c(\underline{x}+\underline{s})\, a(\underline{s})\, d\underline{s}$$

where $a(\underline{s}) = \begin{cases} 1 & \underline{s} \in \mathcal{P}_\Lambda \\ 0 & \text{otherwise} \end{cases}$

Letting $\underline{s} = -\underline{s}$, and noting that $a(-\underline{s}) = a(\underline{s})$

$$f[\underline{x}] = c \int_{\mathbb{R}_2} f_c(\underline{x}-\underline{s})\, a(\underline{s})\, d\underline{s}$$

$$\therefore\ h_a(x,y) = \begin{cases} c & (x,y) \in \mathcal{P}_\Lambda \\ 0 & \text{otherwise} \end{cases}$$

The DC gain is $H_a(\underline{0}) = \int_{\mathbb{R}^2} h_a(\underline{s})\, d\underline{s} = cXY$

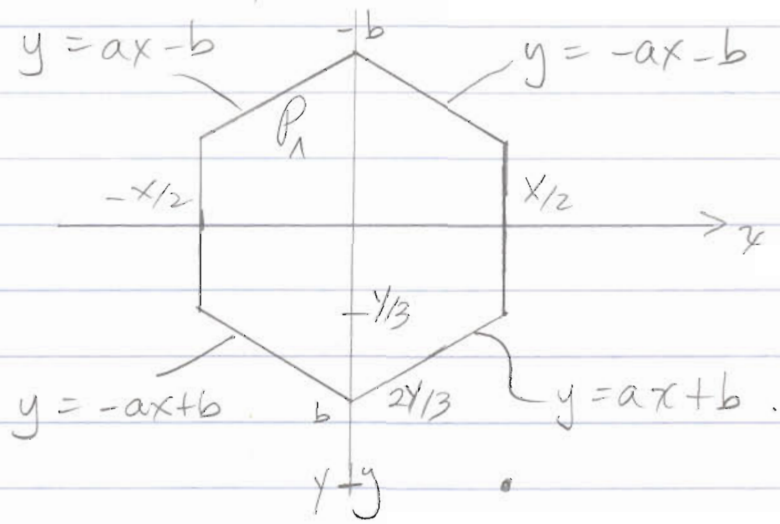
(from (b)) $\therefore\ c = \dfrac{1}{XY} = 199532.25$

CE64311 Midterm Fall 2005
solution

(e) $H_a(u,v) = \mathcal{F}\{h_a(x,y)\}$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_a(x,y)\exp(-j2\pi(ux+vy))\,dx\,dy$$

$$= c\iint_{P_1} \exp(-j2\pi(ux+vy))\,dx\,dy$$



This could be evaluated by direct integration

$$H_a(u,v) = c\int_{-\frac{X}{2}}^{0}\int_{ax-b}^{-ax+b} \exp(-j2\pi(ux+vy))\,dx\,dy$$

$$+\, c\int_{0}^{X/2}\int_{-ax-b}^{ax+b} \exp(-j2\pi(ux+vy))\,dx\,dy$$

(f) $$F(\underline{u}) = \sum_{\underline{r} \in \Lambda^*} H_a(\underline{u}+\underline{r})\, F_c(\underline{u}+\underline{r})$$

$$\Lambda^* = LAT(\underline{V}^{-T})$$

$$V^{-T} = \begin{bmatrix} X & 0 \\ X/2 & Y \end{bmatrix}^{-1} = \frac{1}{XY}\begin{bmatrix} Y & 0 \\ -\frac{X}{2} & X \end{bmatrix} = \begin{bmatrix} \frac{1}{X} & 0 \\ -\frac{1}{2Y} & \frac{1}{Y} \end{bmatrix}$$

$$= \begin{bmatrix} 240\sqrt{3} & 0 \\ -240 & 480 \end{bmatrix} \text{c/ph}$$

$$\therefore F(u,v) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} H_a(u+240\sqrt{3}k_1,\ v-240k_1+480k_2) \cdot F_c(u+240\sqrt{3}k_1,\ v-240k_1+480k_2)$$

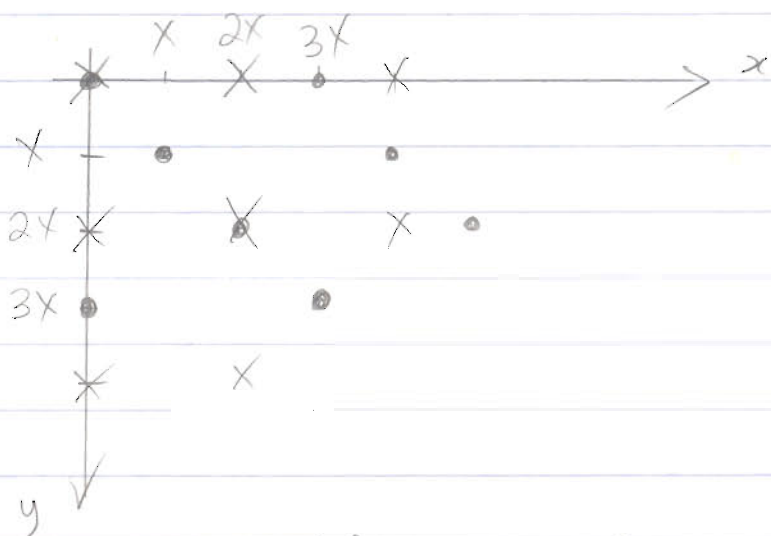
CEG4311 Midterm Fall 2005
Solution

2. $\underline{V}_1 = \begin{bmatrix} 3X & X \\ 0 & X \end{bmatrix}$ $\quad\quad$ $\underline{V}_2 = \begin{bmatrix} 2X & 0 \\ 0 & 2X \end{bmatrix}$

$d(V_1) = 3X^2$ $\quad\quad$ $d(V_2) = 4X^2$

(a) We have a reduction of sampling density, but not by an integer factor. Thus, neither $\Lambda_1 \subset \Lambda_2$ nor $\Lambda_2 \subset \Lambda_1$

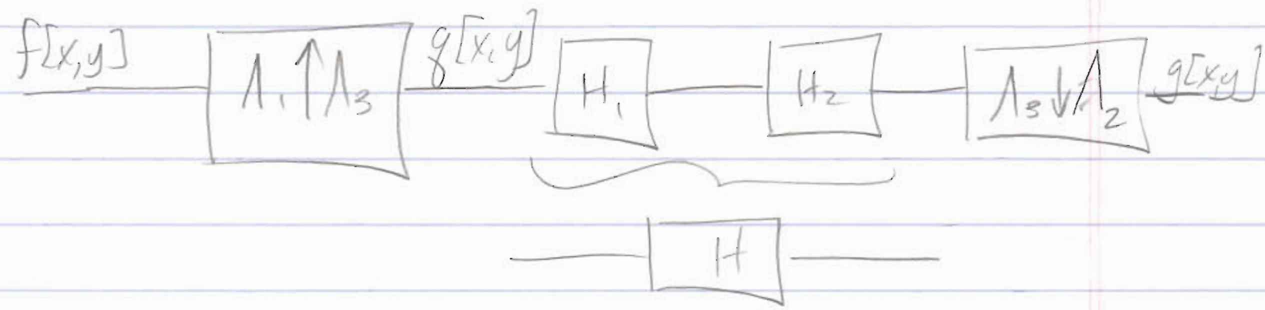
(b) Sketch the lattices



By inspection $\begin{bmatrix} X \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ X \end{bmatrix}$ must be contained in any lattice that contains both Λ_1 & Λ_2
Also, we see that $\Lambda_3 = LAT\left(\begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}\right)$ contains both Λ_1 & Λ_2. Thus Λ_3 is the least dense lattice containing both Λ_1 and Λ_2
$d(\Lambda_3) = X^2$.

Solution

Conversion system



(i) Upsample from Λ_1 to Λ_3 by zero insertion

$$g[x,y] = \begin{cases} f[x,y] & (x,y) \in \Lambda_1 \\ 0 & (x,y) \in \Lambda_3 \setminus \Lambda_1 \end{cases}$$

(ii) interpolate with a lowpass filter H_1

$$H_1(u,v) = \begin{cases} 3 & (u,v) \in P_{\Lambda_1^*} \\ 0 & (u,v) \in P_{\Lambda_3^*} \setminus P_{\Lambda_1^*} \end{cases}$$

$$\Lambda_1^* = LAT\left(V_1^{-T}\right) = LAT\left(\begin{bmatrix} 3X & 0 \\ X & X \end{bmatrix}^{-1}\right) = LAT\left(\frac{1}{3X^2}\begin{bmatrix} X & 0 \\ -X & 3X \end{bmatrix}\right)$$

$$= LAT\left(\begin{bmatrix} \frac{1}{3X} & 0 \\ -\frac{1}{3X} & \frac{1}{X} \end{bmatrix}\right)$$

$$\Lambda_3^* = LAT\left(\begin{bmatrix} \frac{1}{X} & 1 \\ 0 & \frac{1}{X} \end{bmatrix}\right)$$

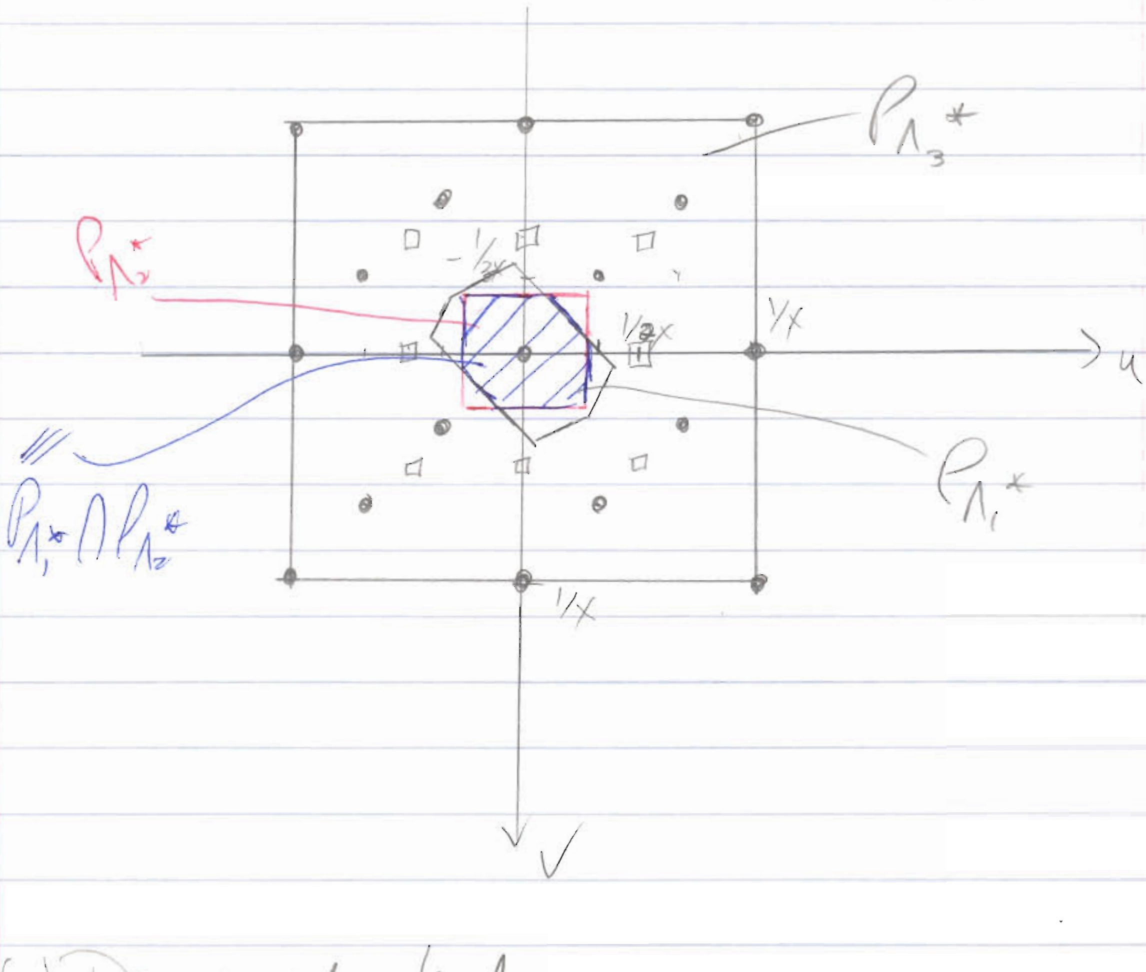
$$\Lambda_2^* = LAT\left(\begin{bmatrix} \frac{1}{2X} & 0 \\ 0 & \frac{1}{2X} \end{bmatrix}\right)$$

iii) Filter with a Lowpass filter $H_2(u,v)$

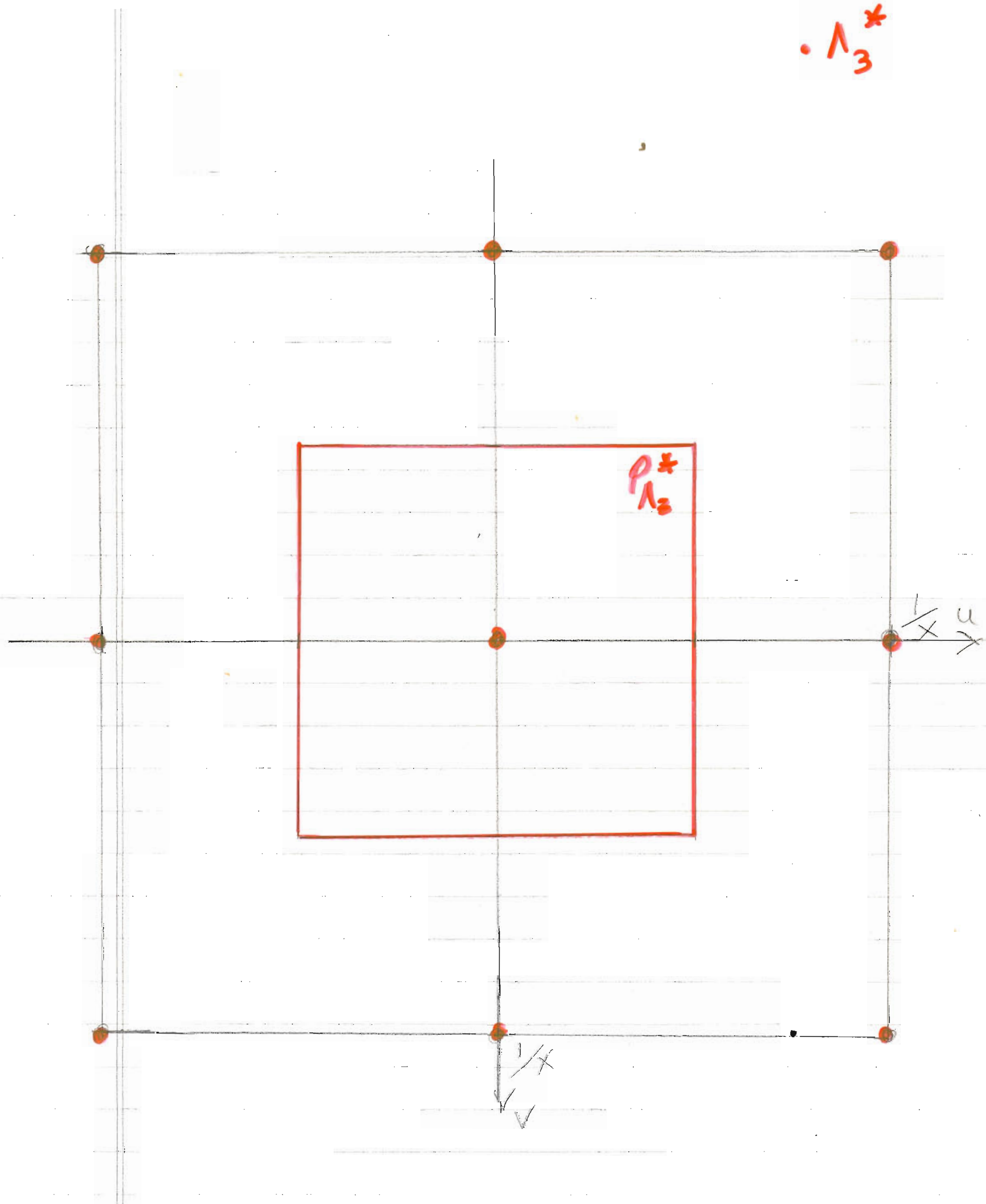
$$H_2(u,v) = \begin{cases} 1 & (u,v) \in P_{\Lambda_2^*} \\ 0 & (u,v) \in P_{\Lambda_3^*} \setminus P_{\Lambda_2^*} \end{cases}$$

Combining

$$H(u,v) = \begin{cases} 3 & (u,v) \in P_{\Lambda_1^*} \cap P_{\Lambda_2^*} \\ 0 & \text{elsewhere in } P_{\Lambda_3^*} \end{cases}$$

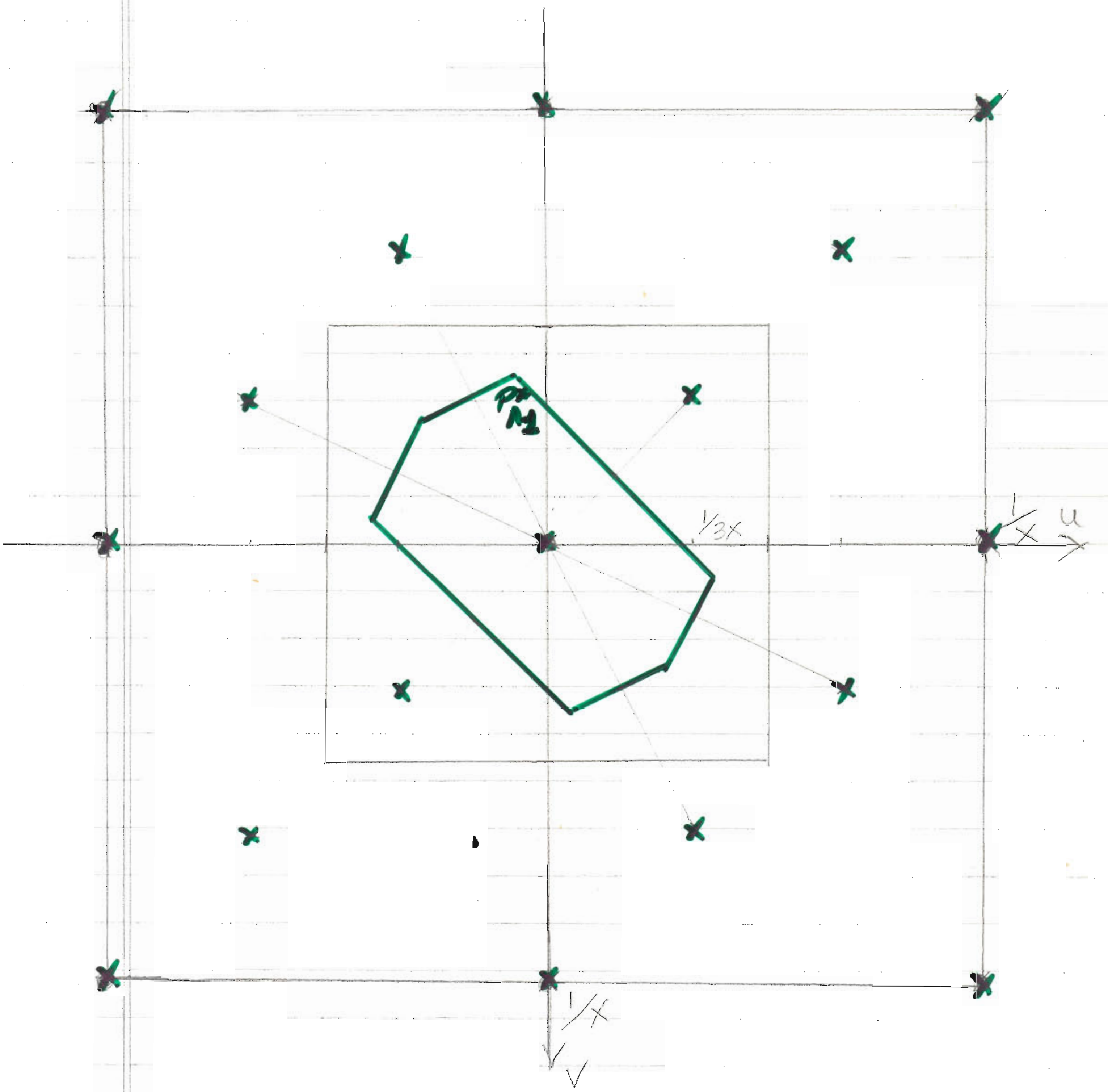


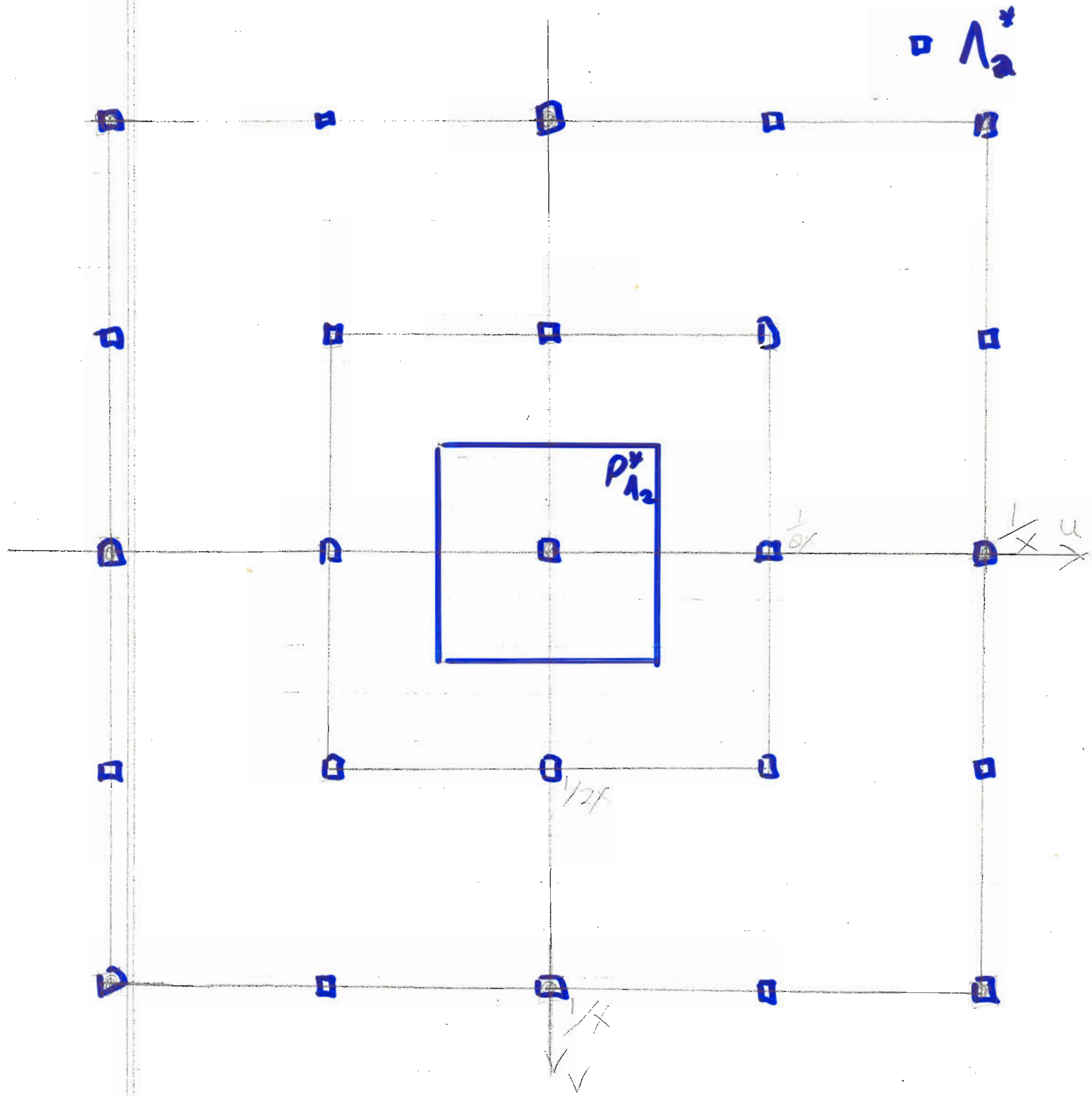
(iv) Downsample to Λ_2.



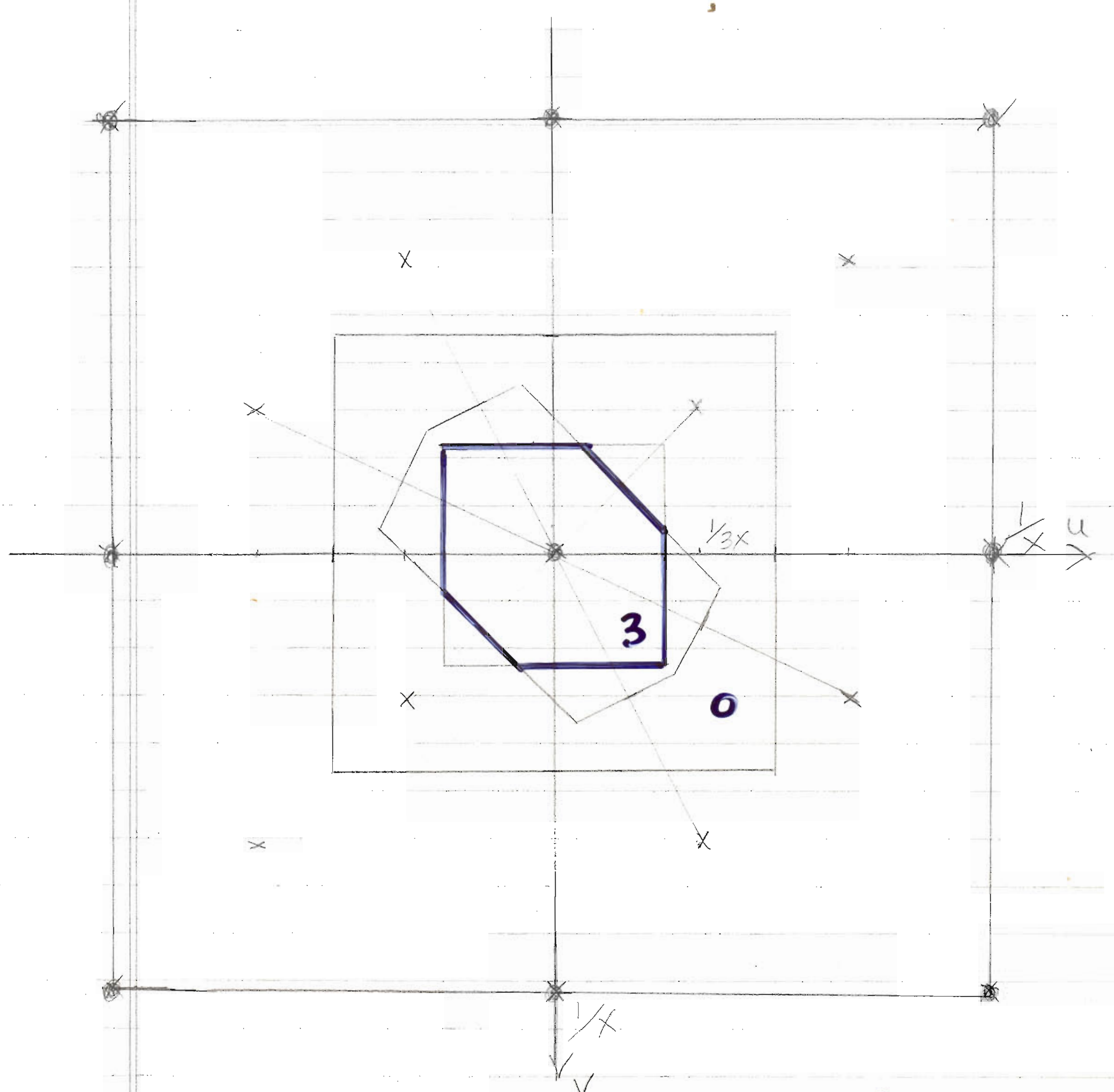
Λ_3^*
$P_{\Lambda_3}^*$
$1/x$
u
$1/x$
v

$\times\ \Lambda_1^*$





Λ_2^*
$P_{\Lambda_2}^*$
$\frac{1}{8X}$
$\frac{1}{X}$
u
$1/2X$
$1/X$
v



$$H(\underline{u}) = \begin{cases} 3 & \underline{u} \in P_{\Lambda_1^*} \cap P_{\Lambda_2^*} \\ 0 & \underline{u} \in P_{\Lambda_3^*} \setminus (P_{\Lambda_1^*} \cap P_{\Lambda_2^*}) \end{cases}$$