CEG4311 Image Processing
Mid-term exam

Date: Oct. 31, 2005
Time: 8:30-9:50
Professor: E. Dubois
This exam has four pages and two questions. Answer all questions.
Closed-book exam: you may not use any books or notes. You may use a non-programmable calculator. Explain all answers; I am more interested in the reasoning than in precise numerical answers. Unless otherwise specified, you may use the results provided on pages 3 and 4 without proof, but state which ones you use. Vous pouvez répondre en anglais ou en français.

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1. Fig. 1 on page 2 illustrates the sensor in a hypothetical digital camera using a sensor element which is hexagonal in shape. There are M = 740 sensor elements in each horizontal row and there are N = 480 rows of sensor elements, for a total of 480 x 740 sensor elements. The centers of the sensor elements lie on a hexagonal lattice Λ, and each sensor element is a unit cell of this lattice. The output of each sensor element is the integral of light irradiance over the sensor element for some arbitrary exposure time, and it is associated with the lattice point at the center of the sensor element. Assume that the picture width is MX and the picture height (ph) is NY. As usual, we use the picture height as the unit of length. The sensor element is a regular hexagon with $Y = \sqrt{3}X/2$. (Note that Fig. 1 is just a sketch and is not drawn to scale.)

(a) Give a sampling matrix for the lattice shown in Fig. 1 in units of ph.
(b) What is the area of a sensor element, with correct units. What is the sampling density, with correct units.
(c) What is the aspect ratio of the sensor? Is it approximately 4/3 or approximately 16/9.
(d) The sampling process carried out by this sensor can be modelled by a linear shift-invariant (LSI) continuous-space filter followed by ideal sampling on Λ. Give an expression for the impulse response $h_a(x, y)$ of this LSI filter with the correct gain. Assume that if the image irradiance is a constant value over a sensor element (in arbitrary normalized units), the sampled value is that same value (i.e., the DC gain of $h_a(x, y)$ is 1.0.
(e) Give an expression for the frequency response $H_a(u,v)$ of $h_a(x, y)$ but do not attempt to evaluate it. However, indicate how you might attempt to evaluate it if you had more time. What properties would this frequency response have?
(f) Assume that the continuous-space input light irradiance $f_c(x, y)$ has a Fourier transform $F_c(u.v)$. Give an expression for the Fourier transform of the sampled image $f[x, y], x \in \Lambda$ in terms of $F_c(u.v)$ and $H_a(u,v)$; you should explicitly evaluate the reciprocal lattice Λ^*.

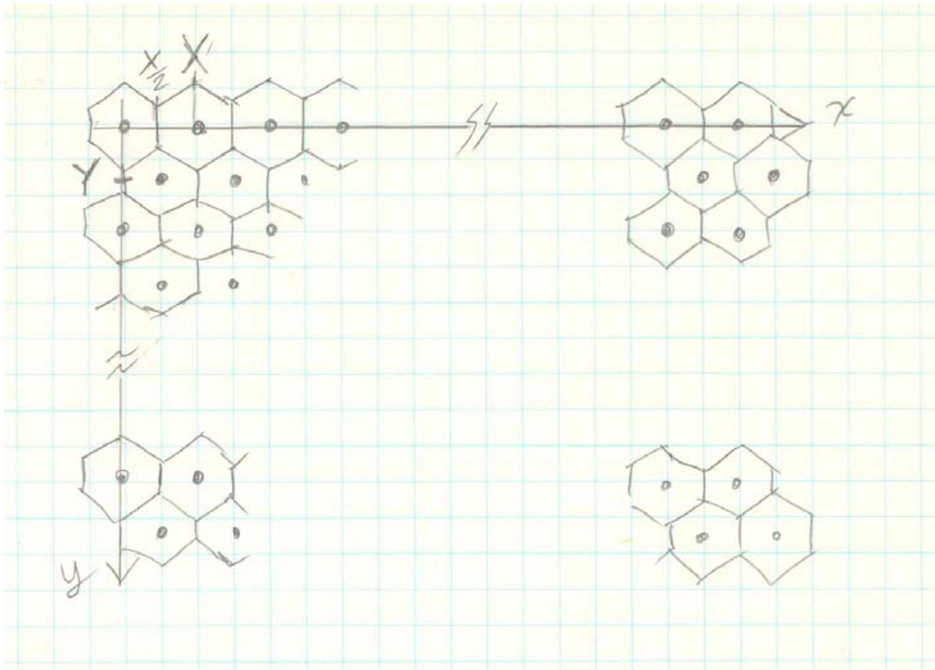

Fig. 1 An image sensor with hexagonal sensor elements

2. A discrete-space signal $f[x, y], (x, y) \in \Lambda_1$ defined on a lattice Λ_1 is to be converted to a signal $g[x, y], (x, y) \in \Lambda_2$ defined on a second lattice Λ_2. The two lattices are specified by their respective sampling matrices $\mathbf{V}_1$ and $\mathbf{V}_2$ as follows:

$$\mathbf{V}_1 = \begin{bmatrix} 3X & X \\ 0 & X \end{bmatrix} \text{ and } \mathbf{V}_2 = \begin{bmatrix} 2X & 0 \\ 0 & 2X \end{bmatrix}$$

(a) State and justify whether $\Lambda_1 \subset \Lambda_2$ or $\Lambda_2 \subset \Lambda_1$ or neither.

(b) Design a system to convert signal $f[x, y]$ into $g[x, y]$ as described above. Show a complete block diagram and give precise definitions of all elements, including all frequency responses. Any LSI filters in cascade should be combined into a single equivalent filter.

	$f(\boldsymbol{x}) = \int_{\mathbb{R}^D} F(\boldsymbol{u}) \exp(j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{u}$	$F(\boldsymbol{u}) = \int_{\mathbb{R}^D} f(\boldsymbol{x}) \exp(-j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{x}$
(i)	$af_1(\boldsymbol{x}) + bf_2(\boldsymbol{x})$	$aF_1(\boldsymbol{u}) + bF_2(\boldsymbol{u})$
(ii)	$f(\boldsymbol{x} - \boldsymbol{x}_0)$	$F(\boldsymbol{u}) \exp(-j2\pi \boldsymbol{u} \cdot \boldsymbol{x}_0)$
(iii)	$f(\boldsymbol{x}) \exp(j2\pi \boldsymbol{u}_0 \cdot \boldsymbol{x})$	$F(\boldsymbol{u} - \boldsymbol{u}_0)$
(iv)	$f^*(\boldsymbol{x})$	$F^*(-\boldsymbol{u})$
(v)	$f(\mathbf{A}\boldsymbol{x})$	$\frac{1}{\lvert\det \mathbf{A}\rvert} F(\mathbf{A}^{-T}\boldsymbol{u})$
(vi)	$f_1(\boldsymbol{x}) * f_2(\boldsymbol{x})$	$F_1(\boldsymbol{u}) F_2(\boldsymbol{u})$
(vii)	$f_1(\boldsymbol{x}) f_2(\boldsymbol{x})$	$F_1(\boldsymbol{u}) * F_2(\boldsymbol{u})$
(viii)	$f_1(x) f_2(y)$	$F_1(u) F_2(v)$
(ix)	$\int_{\mathbb{R}^D} \lvert f(\boldsymbol{x})\rvert^2\, d\boldsymbol{x} = \int_{\mathbb{R}^D} \lvert F(\boldsymbol{u})\rvert^2\, d\boldsymbol{u}$	

Multidimensional Fourier transform properties.

$f(\boldsymbol{x}) = \int_{\mathbb{R}^D} F(\boldsymbol{u}) \exp(j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{u}$	$F(\boldsymbol{u}) = \int_{\mathbb{R}^D} f(\boldsymbol{x}) \exp(-j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{x}$
$\mathrm{rect}(x, y)$	$\frac{\sin \pi u}{\pi u} \frac{\sin \pi v}{\pi v}$
$\mathrm{circ}(x, y)$	$\frac{1}{\sqrt{u^2+v^2}} J_1(2\pi\sqrt{u^2 + v^2})$
$\exp(-(x^2 + y^2)/2r^2)$	$2\pi r^2 \exp(-2\pi^2(u^2 + v^2)r^2)$
$\cos(\pi(x^2 + y^2)/r^2)$	$r^2 \sin(\pi(u^2 + v^2)r^2)$
$\exp(j\pi(x^2 + y^2)/r^2)$	$jr^2 \exp(-j\pi(u^2 + v^2)r^2)$
$\delta(\boldsymbol{x})$	1

Multidimensional Fourier transform of selected functions.

	$f[\boldsymbol{x}] = d(\Lambda) \int_{\mathcal{P}^*} F(\boldsymbol{u}) \exp(j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{u}$	$F(\boldsymbol{u}) = \sum_{\boldsymbol{x} \in \Lambda} f[\boldsymbol{x}] \exp(-j2\pi \boldsymbol{u} \cdot \boldsymbol{x})$
(i)	$af_1[\boldsymbol{x}] + bf_2[\boldsymbol{x}]$	$aF_1(\boldsymbol{u}) + bF_2(\boldsymbol{u})$
(ii)	$f[\boldsymbol{x} - \boldsymbol{x}_0]$	$F(\boldsymbol{u}) \exp(-j2\pi \boldsymbol{u} \cdot \boldsymbol{x}_0)$
(iii)	$f[\boldsymbol{x}] \exp(j2\pi \boldsymbol{u}_0 \cdot \boldsymbol{x})$	$F(\boldsymbol{u} - \boldsymbol{u}_0)$
(iv)	$f^*[\boldsymbol{x}]$	$F^*(-\boldsymbol{u})$
(v)	$f_1[\boldsymbol{x}] * f_2[\boldsymbol{x}]$	$F_1(\boldsymbol{u}) F_2(\boldsymbol{u})$
(vi)	$f_1[\boldsymbol{x}] f_2[\boldsymbol{x}]$	$d(\Lambda) \int_{\mathcal{P}^*} F_1(\boldsymbol{r}) F_2(\boldsymbol{u} - \boldsymbol{r})\, d\boldsymbol{r}$
(vii)	$\sum_{\boldsymbol{x} \in \Lambda} \lvert f[\boldsymbol{x}] \rvert^2 = d(\Lambda) \int_{\mathcal{P}^*} \lvert F(\boldsymbol{u}) \rvert^2\, d\boldsymbol{u}$	

Properties of the multidimensional Fourier transform over a lattice Λ.

Formulas

$$\exp(jX) = \cos(X) + j\sin(X), \qquad j = \sqrt{-1}$$

$$\mathrm{rect}(x, y) = \begin{cases} 1, & \text{if } |x| \le 0.5 \text{ and } |y| \le 0.5; \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathrm{circ}(x, y) = \begin{cases} 1, & \text{if } x^2 + y^2 \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

Affine transformation: $\mathcal{Q}_{\mathbf{A},\boldsymbol{d}} : \vec{\boldsymbol{g}} = \mathcal{Q}_{\mathbf{A},\boldsymbol{d}} \vec{\boldsymbol{f}} : g(\boldsymbol{x}) = f(\mathbf{A}(\boldsymbol{x} - \boldsymbol{d}))$

If $\Lambda = \mathrm{LAT}(\mathbf{V})$, then $d(\Lambda) = |\det(\mathbf{V})|$, and $\Lambda^* = \mathrm{LAT}(\mathbf{V}^{-T})$.

The Fourier transform pair for a signal $f[\boldsymbol{x}]$ defined on the lattice Λ is given by

$$F(\boldsymbol{u}) = \sum_{\boldsymbol{x} \in \Lambda} f[\boldsymbol{x}] \exp(-j2\pi \boldsymbol{u} \cdot \boldsymbol{x})$$

$$f[\boldsymbol{x}] = d(\Lambda) \int_{\mathcal{P}^*} F(\boldsymbol{u}) \exp(j2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{u}$$

If $f[\boldsymbol{x}] = f_c(\boldsymbol{x}), \quad \boldsymbol{x} \in \Lambda$ then

$$F(\boldsymbol{u}) = \frac{1}{d(\Lambda)} \sum_{r \in \Lambda^*} F_c(\boldsymbol{u} + \boldsymbol{r})$$