3. Representation of Signals in the Frequency Domain

We can represent signals as the sum of other signals. By representing a signal as a sum of sinusoids, we describe the frequency content of a signal. Suppose we input a signal to a extremely narrow bandpass filter. At the output of this filter, we would have a sinusoid. The frequency of this sinusoid would be the central frequency of the bandpass filter, and the amplitude and phase of this sinusoid would be determined by the input signal. Any signal can be represented by the amplitude and phases of all of these sinusoids. We call this the spectral content of the signal. To best describe frequency spectrum of signals, we begin with the generalized Fourier series.

3.1 Generalized Fourier Series

Suppose we have a set of functions $\{\phi_n(t)\}_{n=1,2,\ldots,N}$ which is made up of functions that are mutually orthogonal on the interval $t_o \le t \le t_o + T$. In other words:

$$\int_{t_o}^{t_o+T} \phi_i(t)\phi_j^*(t)dt = \begin{cases} 0, & i \neq j \\ c_i, & i = j \end{cases} \tag{3.1}$$

where $\phi_j^*(t)$ indicates the complex conjugate of $\phi_j(t)$. If $c_n = 1$ for all values of n, then we say that the set of functions is orthonormal.

We would like to approximate some function $x(t)$ on the interval $(t_o, t_o + T)$ by the function $x_a(t)$ which is given by:

$$x_a(t) = \sum_{n=1}^{N} X_n \phi_n(t) \tag{3.2}$$

The best approximation is the one that minimizes the mean square error between the original signal, $x(t)$, and itself. The mean square error (MSE) is given by:

$$\varepsilon_N = \int_{t_o}^{t_o+T} \left| x(t) - x_a(t) \right|^2 dt \tag{3.3}$$

Expansion of (3.3) yields:

$$\begin{aligned}
\varepsilon_N &= \int_{t_o}^{t_o+T} \left(x(t) - \sum_{n=1}^{N} X_n \phi_n(t) \right) \left(x(t) - \sum_{n=1}^{N} X_n \phi_n(t) \right)^* dt \\
&= \int_{t_o}^{t_o+T} \left(x(t) - \sum_{n=1}^{N} X_n \phi_n(t) \right) \left(x^*(t) - \sum_{n=1}^{N} X_n^* \phi_n^*(t) \right) dt \\
&= \int_{t_o}^{t_o+T} \left(x(t)x^*(t) - x(t)\sum_{n=1}^{N} X_n^* \phi_n^*(t) - x^*(t)\sum_{n=1}^{N} X_n \phi_n(t) + \sum_{n=1}^{N}\sum_{i=1}^{N} X_n X_i^* \phi_n(t)\phi_i^*(t) \right) dt \\
&= \int_{t_o}^{t_o+T} |x(t)|^2 \, dt - \sum_{n=1}^{N} X_n^* \int_{t_0}^{t_o+T} x(t)\phi_n^*(t)dt - \sum_{n=1}^{N} X_n \int_{t_0}^{t_o+T} x^*(t)\phi_n(t)dt + \sum_{n=1}^{N}\sum_{i=1}^{N} X_n X_i^* \int_{t_o}^{t_o+T} \phi_n(t)\phi_i^*(t)dt
\end{aligned} \tag{3.4}$$

From (3.1), The final term in (3.4), $X_n X_i^* \int_{t_o}^{t_o+T} \phi_n(t)\phi_i^*(t)dt$ is 0 when $n \neq i$ and it is given by $|X_n|^2 c_n$ when $n = i$. Therefore (3.4) becomes:

$$\varepsilon_N = \int_{t_o}^{t_o+T} |x(t)|^2 \, dt - \sum_{n=1}^{N} X_n^* \int_{t_0}^{t_o+T} x(t)\phi_n^*(t)dt - \sum_{n=1}^{N} X_n \int_{t_0}^{t_o+T} x^*(t)\phi_n(t)dt + \sum_{n=1}^{N} |X_n|^2 \, c_n \tag{3.5}$$

Let $y_n = \int_{t_o}^{t_o+T} x(t)\phi_n^*(t)dt$. We rewrite (3.5) as:

$$\varepsilon_N = \int_{t_o}^{t_o+T} |x(t)|^2 \, dt + \sum_{n=1}^{N} \left(- X_n^* y_n - X_n y_n^* + |X_n|^2 \, c_n \right) \tag{3.6}$$

If we add the term $\frac{1}{c_n}|y_n|^2$ to the summation in (3.6), then the sum can be factored. Therefore (3.6) becomes:

$$\begin{aligned}
\varepsilon_N &= \int_{t_o}^{t_o+T} |x(t)|^2 \, dt - \sum_{n=1}^{N} \frac{1}{c_n}|y_n|^2 + \sum_{n=1}^{N} \left(\frac{1}{c_n}|y_n|^2 - X_n^* y_n - X_n y_n^* + |X_n|^2 \, c_n \right) \\
&= \int_{t_o}^{t_o+T} |x(t)|^2 \, dt - \sum_{n=1}^{N} \frac{1}{c_n}|y_n|^2 + \sum_{n=1}^{N} c_n \left(\frac{1}{c_n} y_n - X_n \right)\left(\frac{1}{c_n} y_n^* - X_n^* \right) \\
&= \int_{t_o}^{t_o+T} |x(t)|^2 \, dt - \sum_{n=1}^{N} \frac{1}{c_n}|y_n|^2 + \sum_{n=1}^{N} c_n \left| \frac{1}{c_n} y_n - X_n \right|^2
\end{aligned} \tag{3.7}$$

Only the final summation in (3.7) depends on X_n. Every term in this summation must be positive. Therefore to minimize (3.7), we must choose the values of X_n so that every term in the final summation of (3.7) is zero. Therefore, the best approximation for x(t) on the interval $t_o \le t \le t_o + T$ is given by (3.2) where X_n is given by:

$$\begin{aligned}
X_n &= \frac{1}{c_n} y_n \\
&= \frac{1}{c_n} \int_{t_o}^{t_o+T} x(t)\phi_n^*(t)dt
\end{aligned} \tag{3.8}$$

From (3.8), we see that $|X_n|^2 = \frac{1}{c_n^2}|y_n|^2$. For X_n given by (3.8), the MSE between $x(t)$ and its best approximation is given by:

$$\begin{aligned}
\varepsilon_N &= \int_{t_o}^{t_o+T} |x(t)|^2 \, dt - \sum_{n=1}^{N} \frac{1}{c_n}|y_n|^2 \\
&= \int_{t_o}^{t_o+T} |x(t)|^2 \, dt - \sum_{n=1}^{N} c_n |X_n|^2
\end{aligned} \tag{3.9}$$

Example 3.1

We wish to approximate the signal $x(t) = t^2$ on the interval $0 \le t \le 1$ using the set of orthogonal functions shown in Figure 3.1. Find $x_N(t) = \sum_{n=1}^{N} X_n \phi_n(t)$ and ε_N for $N = 2$ and 3.

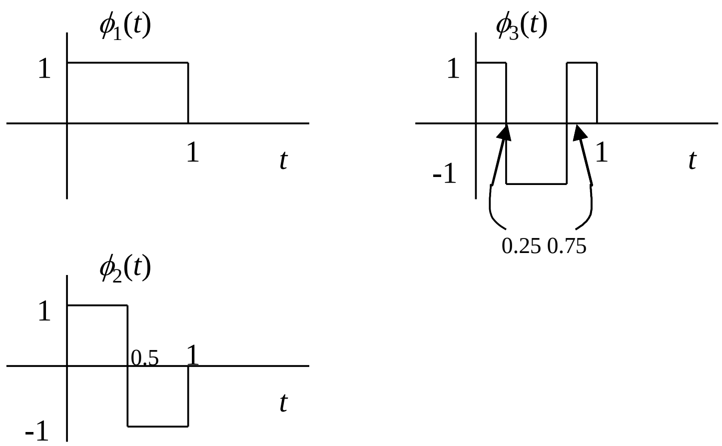

Figure 3.1 Les fonctions orthogonales de l'exemple 3.1.

We can show that $\int_0^1 |\phi_n(t)|^2 dt = \int_0^1 \phi_n^2(t) dt = \int_0^1 dt = 1$ and that $\int_0^1 \phi_n(t)\phi_m^*(t) dt = 0$ for $n \neq m$. Thus $c_n = 1$ for $n = 1, 2, 3$. The generalized Fourier series coefficients are:

$$X_1 = \int_0^1 t^2 \phi_1(t) dt = \int_0^1 t^2 dt = \frac{1}{3} t^3 \Big|_0^1 = \frac{1}{3}$$

$$X_2 = \int_0^1 t^2 \phi_2(t) dt = \int_0^{0.5} t^2 dt - \int_{0.5}^1 t^2 dt = \frac{1}{3} t^3 \Big|_0^{0.5} - \frac{1}{3} t^3 \Big|_{0.5}^1$$
$$= \frac{1}{3}\left[0.125 - (1 - 0.125)\right] = -\frac{1}{4}$$

$$X_3 = \int_0^1 t^2 \phi_3(t) dt = \int_0^{0.25} t^2 dt - \int_{0.25}^{0.75} t^2 dt + \int_{0.75}^1 t^2 dt = \frac{1}{16}$$

Thus $x_2(t)$ and $x_3(t)$ are given by:

$$x_2(t) = \frac{1}{3}\phi_1(t) - \frac{1}{4}\phi_2(t)$$
$$x_3(t) = \frac{1}{3}\phi_1(t) - \frac{1}{4}\phi_2(t) + \frac{1}{16}\phi_3(t)$$

The signal $x(t)$ on the interval $0 \le t \le 1$ is shown in Figure 3.2 along with its two approximations, $x_2(t)$ and $x_3(t)$.

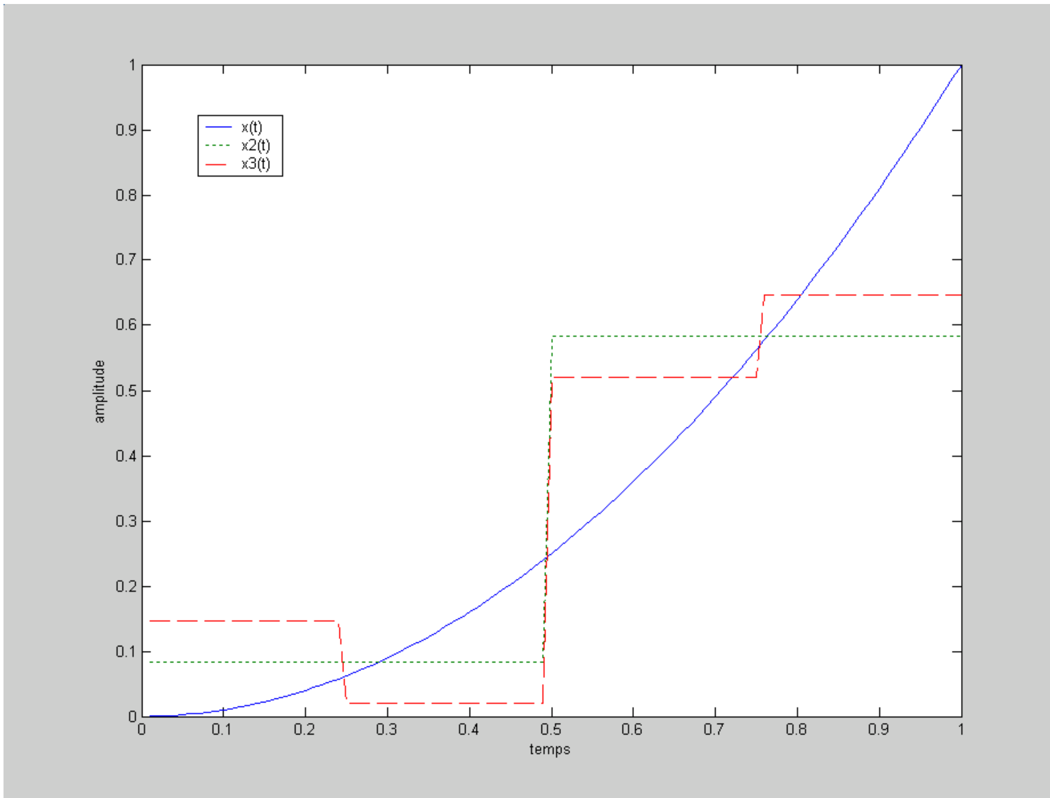

Figure 3.2: The signal $x(t) = t^2$ on the interval $0 \leq t \leq 1$ and its approximations $x_2(t)$ and $x_3(t)$.

The coefficients X_n found in the example are those that minimize the MSE between x(t) and the approximations. The MSE is thus given by (3.9). We find that $\int_0^1 |x(t)|^2 dt = \int_0^1 t^4 dt = \frac{1}{5}$. Thus the MSE between $x(t)$ and $x_2(t)$ is $\varepsilon_2 = \frac{1}{5} - \left[\left(\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2\right] = 0.0264$ and the MSE between $x(t)$ and $x_3(t)$ is $\varepsilon_3 = \frac{1}{5} - \left[\left(\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{16}\right)^2\right] = 0.0225$. Increasing N beyond $N = 3$ has no effect on X_n for $n = 1$ to 3, therefore if we increase N, we decrease the MSE.

3.2 The Complex Exponential Fourier Series

We wish to find an approximation $x_a(t) = \sum_{n=-N}^{N} X_n \phi_n(t)$ for the signal $x(t)$ on the interval $t_o \leq t \leq t_o + T$. The MSE between the signal and its approximation on this interval is given by:

$$\varepsilon_N = \int_{t_o}^{t_o+T} |x(t)|^2 \, dt - \sum_{n=-N}^{N} c_n |X_n|^2 \tag{3.10}$$

There exist some infinite sets of orthogonal functions $\{\phi_n(t)\}_{-\infty \le n \le \infty}$, for which the approximation approaches the original signal on the interval $t_o \le t \le t_o + T$. In other words, using this infinite set, we get:

$$x_a(t) = \sum_{n=-\infty}^{\infty} X_n \phi_n(t) = x(t) \tag{3.11}$$

$$\lim_{N\to\infty} \varepsilon_N = 0 \tag{3.12}$$

and

$$\lim_{N\to\infty} \sum_{n=-N}^{N} c_n |X_n|^2 = \int_{t_o}^{t_o+T} |x(t)|^2 \, dt \tag{3.13}$$

for any signal $x(t)$ for which $\int_{t_o}^{t_o+T} |x(t)|^2 \, dt < \infty$. However, should the signal $x(t)$ contain discontinuities, then it is possible that $x_a(t)$ is not equal to $x(t)$ at the discontinuities.

One set of functions for which (3.11) – (3.13) are true is the set of complex exponential functions $\{\phi_n(t)\} = \{e^{j2\pi n f_o t}\}$, $-\infty \le n \le \infty$, where $f_o = 1/T$. The complex exponential function is given by:

$$e^{j2\pi n f_o t} = \cos(2\pi n f_o t) + j\sin(2\pi n f_o t) \tag{3.14}$$

The complex exponential function is periodic with period T_p. Thus $e^{j2\pi n f_o t} = e^{j2\pi n f_o (t+T_p)} = e^{j2\pi n f_o t} e^{j2\pi n f_o T_p}$. For this to be true, $e^{j2\pi n f_o T_p} = 1$. From (3.14), $e^{j2\pi n f_o T_p} = 1$ when $2\pi n f_o T_p = 2\pi m$ where m and n are integers. Thus $T_p = m/nf_o$. The fundamental period, T_f, is the smallest positive value of T_p. Thus the fundamental period of $e^{j2\pi n f_o t}$ is $T_f = 1/|n|f_o$.

The integral $\int_{t_o}^{t_o+T} \phi_n(t)\phi_m^*(t)dt$ is given by:

$$\int_{t_o}^{t_o+T} e^{j2\pi n f_o t} e^{-j2\pi m f_o t} dt = \int_{t_o}^{t_o+T} e^{j2\pi(n-m) f_o t} dt \tag{3.15}$$

For $m = n$, $\int_{t_o}^{t_o+T} \phi_n(t)\phi_m^*(t)dt = \int_{t_o}^{t_o+T} |\phi_n(t)|^2 dt = c_n$. From (3.15) we find that c_n is:

$$\begin{aligned} c_n &= \left. \int_{t_o}^{t_o+T} e^{j2\pi(n-m) f_o t} dt \right|_{m=n} \\ &= \int_{t_o}^{t_o+T} dt = T \end{aligned} \tag{3.16}$$

For $m \ne n$, (3.15) becomes

$$\begin{aligned}\int_{t_o}^{t_o+T}\phi_n(t)\phi_m^*(t)dt &= \int_{t_o}^{t_o+T} e^{j2\pi(n-m)f_o t}dt \\ &= \frac{e^{j2\pi(n-m)f_o(t_o+T)}-e^{j2\pi(n-m)f_o t_o}}{j2\pi(n-m)f_o} \\ &= \frac{e^{j2\pi(n-m)f_o t_o}e^{j2\pi(n-m)f_o T}-e^{j2\pi(n-m)f_o t_o}}{j2\pi(n-m)f_o} \\ &= 0\end{aligned} \tag{3.17}$$

since $e^{j2\pi(n-m)f_o T}=1$ ($f_o = 1/T$). From (3.17), we can see that the set of complex exponentials is made up of mutually orthogonal functions.

From the generalized Fourier series, the complex exponential Fourier series of $x(t)$ on the interval $t_o \le t \le t_o + T$ is given by:

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_o t} \tag{3.18}$$

where

$$X_n = \frac{1}{T}\int_{t_o}^{t_o+T} x(t)e^{-j2\pi n f_o t}dt \tag{3.19}$$

The Complex Exponential Fourier Series of Periodic Signals

Consider the Fourier series $\sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_o t}$ on the interval $-\infty \le t \le \infty$. It was shown previously that the complex exponential function is periodic with fundamental period $1/|n|f_o = T/|n|$. The sum of periodic functions is also a periodic function if a lowest common multiple (LCM) of the periods of each of the functions in the sum exists. In the case of the complex exponential Fourier series,the LCM of the fundamental periods is T. Thus $\sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_o t}$ is periodic with fundamental period $T = 1/f_o$, and f_o is known as the fundamental frequency. Thus, if $x(t)$ is also periodic with fundamental period T, then (3.18) is valid on the interval $-\infty \le t \le \infty$. We employ (3.19) to find the Fourier coefficients by integrating over one period of $x(t)$. We can express (3.19) for periodic signals as:

$$X_n = \frac{1}{T}\int_T x(t)e^{-j2\pi n f_o t}dt \tag{3.20}$$

where $\int_T (\cdot)dt$ indicates that the integral is done on any time interval of duration T.

Example 3.2

Find the complex exponential Fourier series of the periodic signal $x(t)$ shown in Figure 3.3.

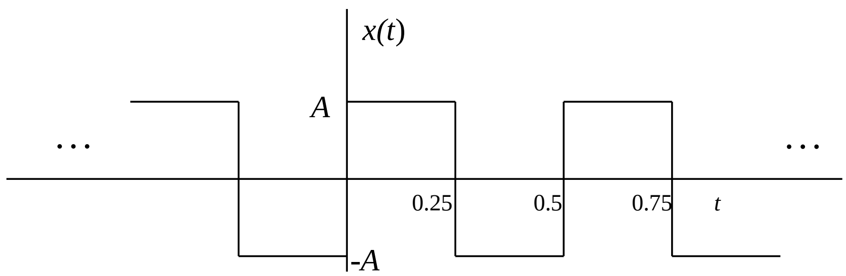

Figure 3.3 Periodic signal of example 3.2.

The signal $x(t)$ is periodic with fundamental period $T = 0.5$. Thus $f_o = 2$. The complex exponential Fourier series of $x(t)$ is:

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j4\pi nt} \tag{3.21}$$

where X_n is given by:

$$\begin{aligned}
X_n &= 2\int_0^{0.5} x(t)e^{-j4\pi nt}dt \\
&= 2\left(\int_0^{0.25} Ae^{-j4\pi nt}dt - \int_{0.25}^{0.5} Ae^{-j4\pi nt}dt\right) \\
&= 2A\left(-\frac{1}{j4\pi n}e^{-j4\pi nt}\Big|_0^{0.25} + \frac{1}{j4\pi n}e^{-j4\pi nt}\Big|_{0.25}^{0.5}\right) \\
&= 2A\left(-\frac{1}{j4\pi n}e^{-j\pi n} + \frac{1}{j4\pi n} + \frac{1}{j4\pi n}e^{-j2\pi n} - \frac{1}{j4\pi n}e^{-j\pi nt}\right) \\
&= 2A\left(\frac{1}{j2\pi n} - \frac{1}{j2\pi n}e^{-j\pi n}\right) = \frac{A}{j\pi n}\left[1-(-1)^n\right]
\end{aligned}$$

For even n, $X_n = 0$ and for odd n, $X_n = 2A/j\pi n$. For $n = 0$, $X_0 = 0/0$, but from (3.21), we see that X_0 is:

$$X_0 = 2\int_0^{0.5} x(t)dt = 0$$

Thus we can represent $x(t)$ by:

$$x(t) = \sum_{\substack{n=-\infty \\ n \text{ is odd}}}^{\infty} \frac{2A}{j\pi n} e^{j4\pi nt} = \sum_{i=-\infty}^{\infty} \frac{2A}{j\pi(2i-1)} e^{j4\pi(2i-1)t}$$

3.3 Symmetry of the Fourier Series Coefficients

Assuming that the signal $x(t)$ is a real signal. In other words $\text{Im}\{x(t)\} = 0$. The complex conjugate of the Fourier series coefficient, X_n^*, is given by:

$$\begin{aligned} X_n^* &= \left[\frac{1}{T}\int_T x(t)e^{-j2\pi n f_o t}\,dt\right]^* \\ &= \frac{1}{T}\int_T \left(x(t)e^{-j2\pi n f_o t}\right)^* dt \\ &= \frac{1}{T}\int_T x^*(t)e^{j2\pi n f_o t}\,dt \\ &= \frac{1}{T}\int_T x(t)e^{-j2\pi(-n) f_o t}\,dt = X_{-n} \end{aligned} \tag{3.22}$$

Thus, $X_{-n} = X_n^*$ if $x(t)$ is a real signal. In other words, for real $x(t)$, $\text{Re}\{X_n\} = \text{Re}\{X_{-n}\}$ and $\text{Im}\{X_n\}=-\text{Im}\{X_{-n}\}$. As a function of n, the real part of the Fourier series coefficients is an even function and the imaginary part is an odd function for all real valued signals.

3.4 The Trigonometric Fourier Series

Let us consider a real valued signal, $x(t)$. The real part of its complex exponential Fourier series coefficients is given by:

$$\begin{aligned} \text{Re}\{X_n\} &= \text{Re}\left\{\frac{1}{T}\int_T x(t)e^{-j2\pi n f_o t}\,dt\right\} \\ &= \text{Re}\left\{\frac{1}{T}\int_T x(t)(\cos 2\pi n f_o t - j\sin 2\pi n f_o t)dt\right\} \\ &= \text{Re}\left\{\frac{1}{T}\int_T x(t)\cos 2\pi n f_o t dt - j\frac{1}{T}\int_T x(t)\sin 2\pi n f_o t dt\right\} \\ &= \frac{1}{T}\int_T x(t)\cos 2\pi n f_o t dt \end{aligned} \tag{3.23}$$

The imaginary part of its complex exponential Fourier series coefficients is given by:

$$\text{Im}\{X_n\} = -\frac{1}{T}\int_T x(t)\sin 2\pi n f_o t dt \tag{3.24}$$

The complex exponential Fourier can also be expressed as:

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_o t} \\ &= X_0 + \sum_{n=1}^{\infty}\left(X_n e^{j2\pi n f_o t} + X_{-n}e^{-j2\pi n f_o t}\right) \end{aligned} \tag{3.25}$$

For a real valued signal, x(t), we know that $X_{-n} = X_n^*$, thus (3.25) becomes:

$$
\begin{aligned}
x(t) &= X_0 + \sum_{n=1}^{\infty} \{ (\mathrm{Re}\{X_n\} + j\,\mathrm{Im}\{X_n\})(\cos(2\pi n f_o t) + j\sin(2\pi n f_o t)) \\
&\qquad\qquad + (\mathrm{Re}\{X_n\} - j\,\mathrm{Im}\{X_n\})(\cos(2\pi n f_o t) - j\sin(2\pi n f_o t)) \} \\
&= X_0 + \sum_{n=1}^{\infty} (2\,\mathrm{Re}\{X_n\}\cos(2\pi n f_o t) - 2\,\mathrm{Im}\{X_n\}\sin(2\pi n f_o t)) \\
&= a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_o t) + \sum_{n=1}^{\infty} b_n \sin(2\pi n f_o t)
\end{aligned} \tag{3.26}
$$

where

$$
\begin{aligned}
a_0 &= X_0 = \frac{1}{T}\int_T x(t)dt \\
a_n &= 2\,\mathrm{Re}\{X_n\} = \frac{2}{T}\int_T x(t)\cos 2\pi n f_o t dt \\
b_n &= -2\,\mathrm{Im}\{X_n\} = \frac{2}{T}\int_T x(t)\sin 2\pi n f_o t dt
\end{aligned} \tag{3.27}
$$

Example 3.3

Using the complex exponential Fourier series of the signal x(t) in example 3.2, find its trigonometric Fourier series.

Solution

In example 3.2, we found the complex exponential Fourier series to be:

$$
\begin{aligned}
x(t) &= \sum_{\substack{n=-\infty \\ n \text{ is odd}}}^{\infty} \frac{2A}{j\pi n} e^{j4\pi n t} \\
&= \sum_{\substack{n=-\infty \\ n \text{ is odd}}}^{\infty} \frac{-j2A}{\pi n} e^{j4\pi n t}
\end{aligned}
$$

thus $X_0 = 0$, $\mathrm{Re}\{X_n\} = 0$ and $\mathrm{Im}\{X_n\} = -2A/\pi n$ for all odd values of n. Therefore $b_n = 4A/\pi n$ for all odd values of n. We can express $x(t)$ as:

$$
x(t) = \sum_{\substack{n=1 \\ n \text{ is odd}}}^{\infty} \frac{4A}{\pi n}\sin 4\pi n t = \sum_{i=1}^{\infty} \frac{4A}{\pi(2i-1)}\sin 4\pi(2i-1)t \tag{3.28}
$$

The sum $x_N(t) = \sum_{i=1}^{N} \frac{4A}{\pi(2i-1)}\sin 4\pi(2i-1)t$ represents the sum of the first N harmonics of $x(t)$. This sum is shown in Figure 3.4 for N = 1 to 3.

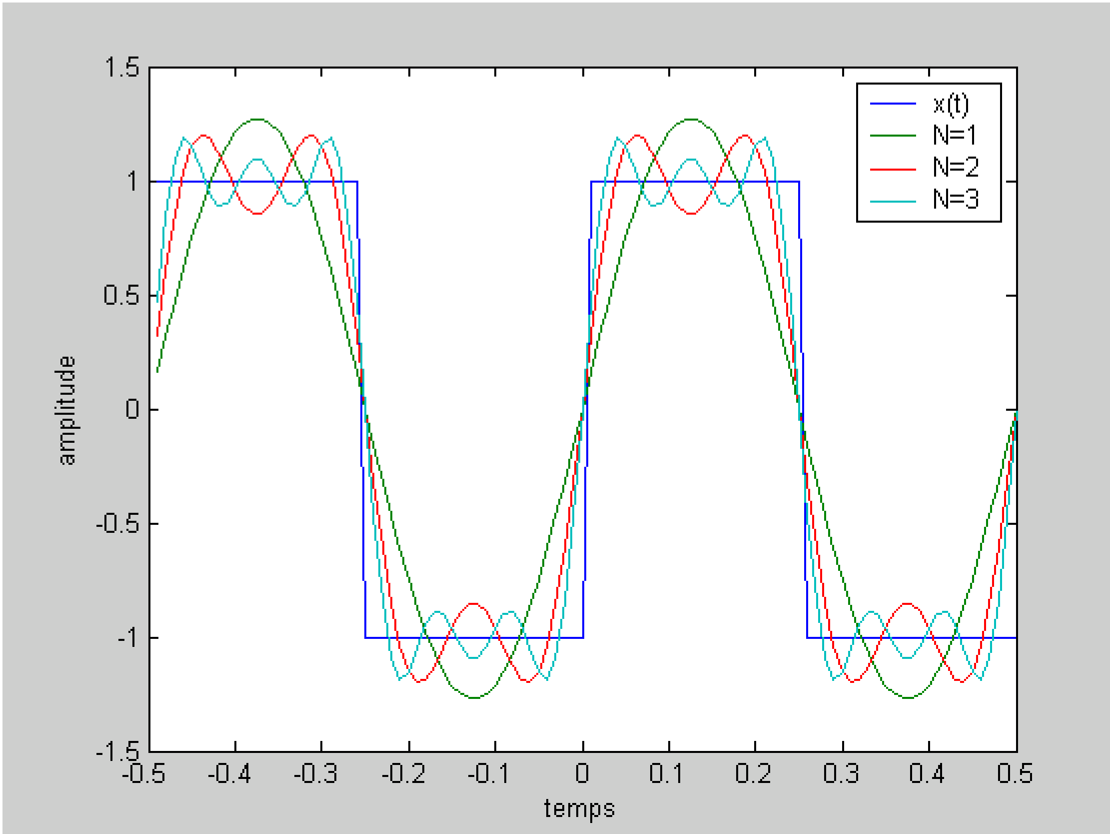

Figure 3.4: The approximation of $x(t)$ by its first N terms in the trigonometric Fourier series (N = 1 to 3).

3.5 Properties of the Trigonometric Fourier Series

For any periodic signal, $x(t)$, whose period is given by T, we can find the trigonometric Fourier series coefficients by taking the integrals of (3.27) over any interval of duration T. Let us examine the case where the interval of integration is $-T/2 \le t \le T/2$. Thus a_0, a_n and b_n become:

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt = \frac{1}{T}\int_{-T/2}^{0} x(t)dt + \frac{1}{T}\int_{0}^{T/2} x(t)dt \tag{3.29}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos 2\pi n f_o t dt = \frac{2}{T}\int_{-T/2}^{0} x(t)\cos 2\pi n f_o t dt + \frac{2}{T}\int_{0}^{T/2} x(t)\cos 2\pi n f_o t dt \tag{3.30}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin 2\pi n f_o t dt = \frac{2}{T}\int_{-T/2}^{0} x(t)\sin 2\pi n f_o t dt + \frac{2}{T}\int_{0}^{T/2} x(t)\sin 2\pi n f_o t dt \tag{3.31}$$

Consider the case when $x(t)$ is an even function. In other words, $x(t) = x(-t)$. if we replace $x(t)$ by $x(-t)$ in the first integral of (3.31), b_n becomes:

$$b_n = \frac{2}{T}\int_{-T/2}^{0} x(-t)\sin 2\pi n f_o t dt + \frac{2}{T}\int_{0}^{T/2} x(t)\sin 2\pi n f_o t dt \tag{3.32}$$

Substituting u for $-t$ in the first integral of (3.32), we obtain the following:

$$
\begin{aligned}
b_n &= -\frac{2}{T}\int_{T/2}^{0} x(u)\sin 2\pi n f_o(-u)du + \frac{2}{T}\int_{0}^{T/2} x(t)\sin 2\pi n f_o t dt \\
&= \frac{2}{T}\int_{0}^{T/2} x(u)\sin 2\pi n f_o(-u)du + \frac{2}{T}\int_{0}^{T/2} x(t)\sin 2\pi n f_o t dt \\
&= -\frac{2}{T}\int_{0}^{T/2} x(u)\sin 2\pi n f_o u du + \frac{2}{T}\int_{0}^{T/2} x(t)\sin 2\pi n f_o t dt \\
&= 0
\end{aligned} \tag{3.33}
$$

Therefore, **for an even periodic function $x(t)$**, the coefficients $\{b_n\}$ are all equal to 0 and its trigonometric Fourier series becomes:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi n f_o t \tag{3.34}$$

Similarly, we can show that for odd periodic functions ($x(t) = -x(-t)$), the coefficients a_0 and a_n equal 0. **For odd periodic functions**, the trigonometric Fourier series is given by:

$$x(t) = \sum_{n=1}^{\infty} b_n \sin 2\pi n f_o t \tag{3.35}$$

For any arbitrary function, $x(t)$, we can express it as a sum of one even and one odd function. In other words, $x(t) = x_e(t) + x_o(t)$, where $x_e(t)$ is the even component of $x(t)$ and $x_o(t)$ is its odd component. When $x(t)$ is an even function, $x_o(t) = 0$, and conversely, if $x(t)$ is odd, $x_e(t) = 0$. The even and odd components of $x(t)$ can be found to be:

$$x_e(t) = \frac{x(t) + x(-t)}{2} \tag{3.36}$$

$$x_o(t) = \frac{x(t) - x(-t)}{2} \tag{3.37}$$

Thus for $x(t) = x_e(t) + x_o(t)$, we can express its trigonometric Fourier series coefficients by the following:

$$
\begin{aligned}
a_0 &= \frac{1}{T}\int_T x_e(t)dt \\
a_n &= \frac{2}{T}\int_T x_e(t)\cos 2\pi n f_o t dt \\
b_n &= \frac{2}{T}\int_T x_o(t)\sin 2\pi n f_o t dt
\end{aligned} \tag{3.38}
$$

Example 3.4

For the signal $x(t)$ shown in Figure 3.5, find its trigonometric Fourier series.

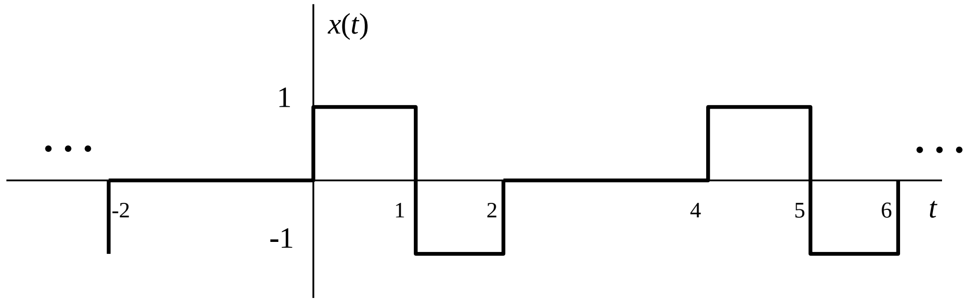

Figure 3.5: The signal $x(t)$ of example 3.4.

Solution

The period of this signal is T = 4, thus the fundamental frequency is f_o = ¼. The trigonometric Fourier series can be expressed as:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{2}nt\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{2}nt\right)$$

where

$$a_0 = \frac{1}{4}\int_{-2}^{2} x(t)dt = \frac{1}{4}\left(\int_{0}^{1} dt - \int_{1}^{2} dt\right) = 0$$

$$\begin{aligned}
a_n &= \frac{1}{2}\int_{-2}^{2} x(t)\cos\left(\frac{\pi}{2}nt\right)dt = \frac{1}{2}\left(\int_{0}^{1}\cos\left(\frac{\pi}{2}nt\right)dt - \int_{1}^{2}\cos\left(\frac{\pi}{2}nt\right)dt\right) \\
&= \frac{1}{2}\left(\frac{2}{\pi n}\sin\left(\frac{\pi}{2}nt\right)\Big|_0^1 - \frac{2}{\pi n}\sin\left(\frac{\pi}{2}nt\right)\Big|_1^2\right) \\
&= \frac{1}{2}\left[\left(\frac{\sin(\pi n/2)}{\pi n/2}\right) - \left(2\frac{\sin(\pi n)}{\pi n} - \frac{\sin(\pi n/2)}{\pi n/2}\right)\right] \\
&= \frac{1}{2}\left[2\frac{\sin(\pi n/2)}{\pi n/2} - 2\frac{\sin(\pi n)}{\pi n}\right] \\
&= \mathrm{sinc}(n/2) = \begin{cases} 0, & n \text{ is even} \\ \dfrac{(-1)^{(n-1)/2}}{\pi n/2}, & n \text{ is odd} \end{cases}
\end{aligned}$$

and

$$b_n = \frac{1}{2}\int_{-2}^{2} x(t)\sin\left(\frac{\pi}{2}nt\right)dt = \frac{1}{2}\left(\int_0^1 \sin\left(\frac{\pi}{2}nt\right)dt - \int_1^2 \sin\left(\frac{\pi}{2}nt\right)dt\right)$$

$$= \frac{1}{2}\left(-\frac{2}{\pi n}\cos\left(\frac{\pi}{2}nt\right)\Big|_0^1 + \frac{2}{\pi n}\cos\left(\frac{\pi}{2}nt\right)\Big|_1^2\right)$$

$$= \frac{1}{2}\left[\left(\frac{1-\cos(\pi n/2)}{\pi n/2}\right) + \left(\frac{\cos(\pi n)}{\pi n/2} - \frac{\cos(\pi n/2)}{\pi n/2}\right)\right]$$

$$= \left(\frac{1-2\cos(\pi n/2)+\cos(\pi n)}{\pi n}\right) = \begin{cases} 0, & n \text{ is even} \\ 0, & n = 2m \text{ and } m \text{ is even} \\ 4/\pi n, & n = 2m \text{ and } m \text{ is odd} \end{cases}$$

where $\mathrm{sinc}(x) = \sin(\pi x)/\pi x$ and $\sin(\pi n) = 0$ for integer values of n. Therefore we can express $x(t)$ as:

$$x(t) = \sum_{\substack{n=1 \\ n\,\mathrm{odd}}}^{\infty} \frac{(-1)^{(n-1)/2}}{\pi n/2}\cos\left(\frac{\pi nt}{2}\right) + \sum_{\substack{n=2 \\ n/2=\mathrm{odd\,integer}}}^{\infty} \frac{4}{\pi \mathrm{n}}\sin\left(\frac{\pi nt}{2}\right)$$

$$= \frac{2}{\pi}\cos\left(\frac{\pi t}{2}\right) - \frac{2}{3\pi}\cos\left(\frac{3\pi t}{2}\right) + \frac{2}{5\pi}\cos\left(\frac{5\pi t}{2}\right) - \Lambda + \frac{2}{\pi}\sin(\pi t) + \frac{2}{3\pi}\sin(3\pi t) + \Lambda$$

The above sum contains an infinite number of terms. In figure 3.6, we show approximations of $x(t)$ which use a finite number of terms from the above sum. The approximation $x_1(t)$ contains all terms from the above sum whose frequency is below 1 Hz and the approximation $x_2(t)$ contains all terms with frequencies below 2 Hz. Mathematically, they are given by:

$$x_1(t) = \frac{2}{\pi}\cos\left(\frac{\pi t}{2}\right) - \frac{2}{3\pi}\cos\left(\frac{3\pi t}{2}\right) + \frac{2}{\pi}\sin(\pi t)$$

and

$$x_2(t) = \frac{2}{\pi}\cos\left(\frac{\pi t}{2}\right) - \frac{2}{3\pi}\cos\left(\frac{3\pi t}{2}\right) + \frac{2}{5\pi}\cos\left(\frac{5\pi t}{2}\right) - \frac{2}{7\pi}\cos\left(\frac{7\pi t}{2}\right) + \frac{2}{\pi}\sin(\pi t) + \frac{2}{3\pi}\sin(3\pi t)$$

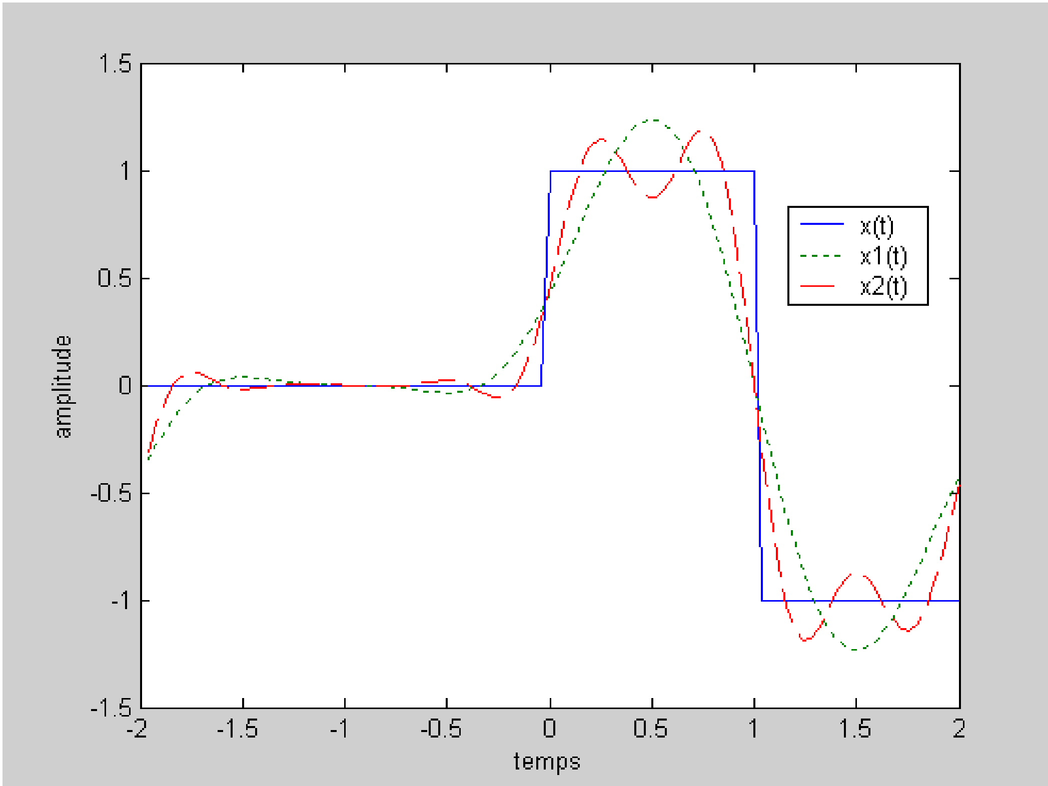

Figure 3.6: The signal $x(t)$ of example 3.4 with its two finite sum approximations, $x_1(t)$ and $x_2(t)$.

Example 3.5

For the signal $x(t)$ of example 3.4, find its even and odd components, $x_e(t)$ and $x_o(t)$. Then find the trigonometric Fourier series of $x_e(t)$ and $x_o(t)$ and show that the coefficients a_0 and a_n that we obtain for $x_e(t)$ and the coefficients b_n that we find for $x_o(t)$ are the same as those found in example 3.4.

Solution

The signals $x(t)$ and $x(-t)$, as well as the signals $x_e(t) = [x(t) + x(-t)]/2$ and $x_o(t) = [x(t) - x(-t)]/2$ on the interval $-2 \le t \le 2$ are shown in Figure 3.7.

From (3.38), we find the trigonometric Fourier series coefficients to be:

$$a_0 = \frac{1}{4}\int_{-2}^{2} x_e(t)dt = \frac{1}{4}\left(\int_{-2}^{-1}\left(-\frac{1}{2}\right)dt + \int_{-1}^{1}\left(\frac{1}{2}\right)dt + \int_{1}^{2}\left(-\frac{1}{2}\right)dt\right) = 0$$

$$a_n = \frac{1}{2}\int_{-2}^{2} x_e(t)\cos\left(\frac{\pi n t}{2}\right)dt = \frac{1}{4}\left(-\int_{-2}^{-1}\cos\left(\frac{\pi n t}{2}\right)dt + \int_{-1}^{1}\cos\left(\frac{\pi n t}{2}\right)dt - \int_{1}^{2}\cos\left(\frac{\pi n t}{2}\right)dt\right)$$

$$= \frac{1}{2\pi n}\left(-\sin\left(\frac{\pi n t}{2}\right)\Big|_{-2}^{-1} + \sin\left(\frac{\pi n t}{2}\right)\Big|_{-1}^{1} - \sin\left(\frac{\pi n t}{2}\right)\Big|_{1}^{2}\right)$$

$$= \frac{1}{2\pi n}\left(-\sin\left(-\frac{\pi n}{2}\right) + \sin\left(\frac{\pi n}{2}\right) - \sin\left(-\frac{\pi n}{2}\right) + \sin\left(\frac{\pi n}{2}\right)\right)$$

$$= \frac{4\sin\left(\frac{\pi n}{2}\right)}{2\pi n} = \frac{\sin\left(\frac{\pi n}{2}\right)}{\pi n/2} = \mathrm{sinc}(n/2) = \begin{cases} 0, & n \text{ is even} \\ \dfrac{(-1)^{(n-1)/2}}{\pi n/2}, & n \text{ is odd} \end{cases}$$

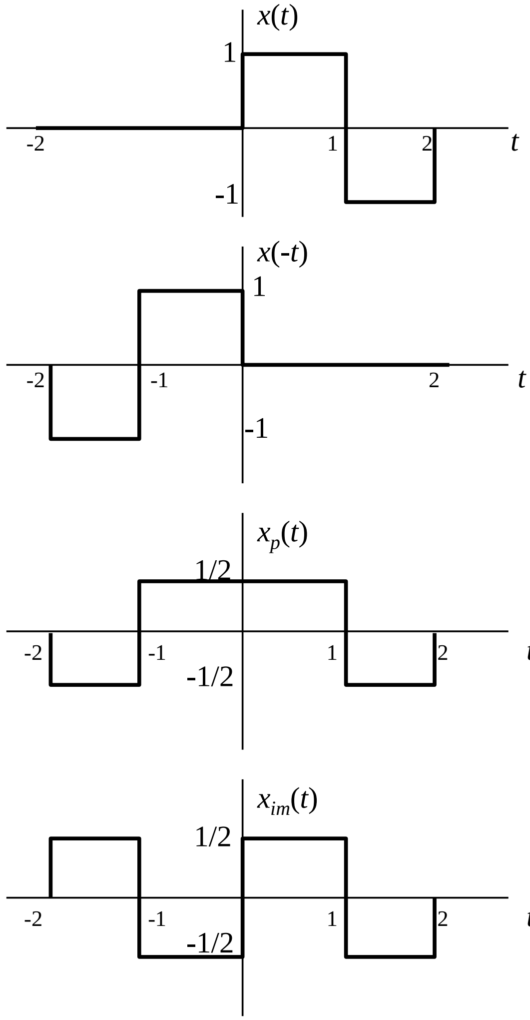

Figure 3.7: The signals $x(t)$, $x(-t)$, $x_e(t)$ and $x_o(t)$.

$$
\begin{aligned}
b_n &= \frac{1}{2}\int_{-2}^{2} x_o(t)\sin\left(\frac{\pi nt}{2}\right)dt \\
&= \frac{1}{4}\left(\int_{-2}^{-1}\sin\left(\frac{\pi nt}{2}\right)dt - \int_{-1}^{0}\sin\left(\frac{\pi nt}{2}\right)dt + \int_{0}^{1}\sin\left(\frac{\pi nt}{2}\right)dt - \int_{1}^{2}\sin\left(\frac{\pi nt}{2}\right)dt\right) \\
&= \frac{1}{2\pi n}\left(-\cos\left(\frac{\pi nt}{2}\right)\Bigg|_{-2}^{-1} + \cos\left(\frac{\pi nt}{2}\right)\Bigg|_{-1}^{0} - \cos\left(\frac{\pi nt}{2}\right)\Bigg|_{0}^{1} + \cos\left(\frac{\pi nt}{2}\right)\Bigg|_{1}^{2}\right) \\
&= \frac{1}{2\pi n}\left(-\cos\left(-\frac{\pi n}{2}\right) + \cos(-\pi n) + 1 - \cos\left(-\frac{\pi n}{2}\right) - \cos\left(-\frac{\pi n}{2}\right) + 1 + \cos(\pi n) - \cos\left(\frac{\pi n}{2}\right)\right) \\
&= \frac{1}{2\pi n}\left(2 - 4\cos\left(\frac{\pi n}{2}\right) + 2\cos(\pi n)\right) = \frac{1 - 2\cos\left(\frac{\pi n}{2}\right) + \cos(\pi n)}{\pi n} = \begin{cases} 0, & n \text{ is even} \\ 0, & n = 2m \text{ and } m \text{ is even} \\ 4/\pi n, & n = 2m \text{ and } m \text{ is odd} \end{cases}
\end{aligned}
$$

We see that the coefficients are the same as those obtained in Example 3.4

3.6 The Fourier Transform

Consider a periodic signal $x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_o t}$ where $f_o = 1/T$, T is the period of $x(t)$ and X_n is given by (3.19). Therefore $x(t)$ is given by:

$$x(t) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi n f_o t}dt\right]e^{j2\pi n f_o t} \tag{3.39}$$

If $x(t)$ is aperiodic, its "period" $T \rightarrow \infty$ and therefore $f_0 \rightarrow 0$. Thus $1/T$ becomes df, nf_o becomes f and the sums become integrals. Thus (3.39) becomes:

$$x(t) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt\right]e^{j2\pi ft}df = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \tag{3.40}$$

$X(f)$ is called the Fourier transform of x(t). It describes the spectral content of the signal $x(t)$. It is given by:

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \tag{3.41}$$

The inverse Fourier transform can be found from (3.40). With it, we can determine the time domain signal from its Fourier transform. It is given by:

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df\ . \tag{3.42}$$

Example 3.6

Find the Fourier transform of $x(t) = \Pi(t)$.

Solution

The Fourier transform of $x(t)$ is:

$$X(f) = \int_{\infty}^{\infty} \Pi(f) e^{-j2\pi ft} dt = \int_{-1/2}^{1/2} e^{-j2\pi ft} dt$$

$$= -\frac{1}{j2\pi f} e^{-j2\pi ft} \Big|_{-1/2}^{1/2} = -\frac{1}{j2\pi f}(e^{-j\pi f} - e^{j\pi f})$$

$$= \frac{e^{j\pi f} - e^{-j\pi f}}{j2\pi f} = \frac{\sin \pi f}{\pi f} = \operatorname{sinc}(f)$$

Example 3.7

Find the Fourier transform of $x(t) = \Lambda(t)$.

Solution

$$X(f) = \int_{\infty}^{\infty} \Lambda(f) e^{-j2\pi ft} dt = \int_{-1}^{0} (1+t) e^{-j2\pi ft} dt + \int_{0}^{1} (1-t) e^{-j2\pi ft} dt$$

$$= \left(-\frac{1}{j2\pi f} e^{-j2\pi ft} - \frac{t}{j2\pi f} e^{-j2\pi ft} + \frac{1}{(2\pi f)^2} e^{-j2\pi ft} \right) \Bigg|_{-1}^{0}$$

$$+ \left(-\frac{1}{j2\pi f} e^{-j2\pi ft} + \frac{t}{j2\pi f} e^{-j2\pi ft} - \frac{1}{(2\pi f)^2} e^{-j2\pi ft} \right) \Bigg|_{0}^{1}$$

$$= -\frac{1}{j2\pi f} + \frac{1}{(2\pi f)^2} + \frac{1}{j2\pi f} e^{j2\pi f} - \frac{1}{j2\pi f} e^{j2\pi f} - \frac{1}{(2\pi f)^2} e^{j2\pi f}$$

$$- \frac{1}{j2\pi f} e^{-j2\pi f} + \frac{1}{j2\pi f} e^{-j2\pi f} - \frac{1}{(2\pi f)^2} e^{-j2\pi f} + \frac{1}{j2\pi f} + \frac{1}{(2\pi f)^2}$$

$$= \frac{2}{(2\pi f)^2} - \frac{1}{(2\pi f)^2}(e^{j2\pi f} - e^{-j2\pi f}) = \frac{2 - 2\cos 2\pi f}{(2\pi f)^2} = \frac{4\sin^2 \pi f}{(2\pi f)^2} = \frac{\sin^2 \pi f}{(\pi f)^2}$$

$$= \operatorname{sinc}^2(f)$$

Example 3.8

Find the Fourier transform of $x(t) = \delta(t)$.

Solution

$$X(f) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt = e^{-j2\pi ft}\Big|_{t=0} = 1$$

3.7 Properties of the Fourier Transform

3.7.1 Linearity

The Fourier transform is a linear function. This means that for $x_3(t) = \alpha x_1(t) + \beta x_2(t)$, $X_3(f) = \mathcal{F}\{x_3(t)\} = \alpha X_1(f) + \beta X_2(f)$, where $X_1(f) = \mathcal{F}\{x_1(t)\}$ and $X_2(f) = \mathcal{F}\{x_2(t)\}$.

Proof

$$\begin{aligned} X_3(f) &= \int_{-\infty}^{\infty} x_3(t) e^{-j2\pi ft} dt \\ &= \int_{-\infty}^{\infty} \left(\alpha x_1(t) + \beta x_2(t)\right) e^{-j2\pi ft} dt \\ &= \int_{-\infty}^{\infty} \left(\alpha x_1(t) e^{-j2\pi ft} + \beta x_2(t) e^{-j2\pi ft}\right) dt \\ &= \int_{-\infty}^{\infty} \alpha x_1(t) e^{-j2\pi ft} dt + \int_{-\infty}^{\infty} \beta x_2(t) e^{-j2\pi ft} dt \\ &= \alpha \int_{-\infty}^{\infty} x_1(t) e^{-j2\pi ft} dt + \beta \int_{-\infty}^{\infty} x_2(t) e^{-j2\pi ft} dt \\ &= \alpha X_1(f) + \beta X_2(f) \end{aligned}$$

Example 3.9

Find the Fourier transform of the signal $x(t) = 2\delta(t) + 3\Pi(t)$.

Solution

$X(f) = \mathcal{F}\{2\delta(t) + 3\Pi(t)\} = 2\,\mathcal{F}\{\delta(t)\} + 3\,\mathcal{F}\{\Pi(t)\} = 2(1) + 3(\mathrm{sinc}(f)) = 2 + 3\mathrm{sinc}(f)$.

$$\mathcal{F}\{2\delta(t) + 3\Pi(t)\} = 2 + 3\mathrm{sinc}(f)$$

3.7.2 Time Delay

Suppose that the Fourier transform of $x_1(t)$ is $X_1(f)$. Then the Fourier transform of $x_2(t) = x_1(t\text{-}t_o)$ is:

$$X_2(f) = \mathcal{F}\{x_1(t\text{-}t_o)\} = X_1(f)e^{-j2\pi f t_o} \tag{3.43}$$

Proof

$$X_2(f) = \int_{-\infty}^{\infty} x_2(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} x_1(t-t_o)e^{-j2\pi ft}dt \tag{3.44}$$

Let $u = t\text{-}t_o$ and $du = dt$. Equation (3.44) becomes

$$\begin{aligned} X_2(f) &= \int_{-\infty}^{\infty} x_1(u)e^{-j2\pi f(u+t_o)}du \\ &= \int_{-\infty}^{\infty} x_1(u)e^{-j2\pi fu}e^{-j2\pi ft_o}du \\ &= e^{-j2\pi ft_o}\int_{-\infty}^{\infty} x_1(u)e^{-j2\pi fu}du \\ &= e^{-j2\pi ft_o}X_1(f) \end{aligned}$$

Example 3.10

Find the Fourier transform of $x(t) = \delta(t\text{-}t_o)$.

Solution

$$X(f) = \mathcal{F}\{\delta(t\text{-}t_o)\} = \mathcal{F}\{\delta(t)\}\, e^{-j2\pi ft_o}$$

$$X(f) = (1)\; e^{-j2\pi ft_o} = e^{-j2\pi ft_o}$$

$$\mathcal{F}\{\delta(t\text{-}t_o)\} = e^{-j2\pi ft_o} \tag{3.45}$$

Example 3.11

Find the Fourier transform of $x(t)$ shown in Figure 3.8 using the Fourier transform of $\Pi(t)$ and the properties of linearity and time delay.

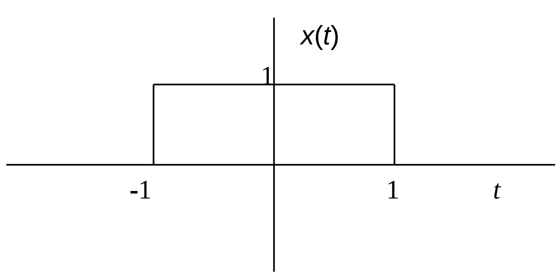

Figure 3.8: The signal $x(t)$ of example 3.11.

Solution

Although there are many ways to determine the Fourier transform of the signal shown in Figure 3.8, we have been limited to using the linearity and time delay properties. It is easily shown that $x(t) = \Pi(t + ½) + \Pi(t - ½)$. Donc $X(f) = \mathcal{F}\{\Pi(t + ½) + \Pi(t - ½)\} = \mathcal{F}\{\Pi(t + ½)\} + \mathcal{F}\{\Pi(t - ½)\} = e^{j\pi f}\ \mathcal{F}\{\Pi(t)\} + e^{-j\pi f}\ \mathcal{F}\{\Pi(t)\}$.

$$
\begin{aligned}
X(f) &= \left(e^{j\pi f} + e^{-j\pi f}\right)\mathrm{sinc}(f) \\
&= \left(e^{j\pi f} + e^{-j\pi f}\right)\sin(\pi f)/(\pi f) \\
&= \frac{\left(e^{j\pi f} + e^{-j\pi f}\right)\left(e^{j\pi f} - e^{-j\pi f}\right)}{j2\pi f} \\
&= \frac{\left(e^{j2\pi f} - e^{-j2\pi f}\right)}{j2\pi f} \\
&= \frac{\sin(2\pi f)}{\pi f} = 2\,\frac{\sin(2\pi f)}{2\pi f} = 2\mathrm{sinc}(2f)
\end{aligned}
$$

3.7.3 Time Scaling

Given that the Fourier transform of $x_1(t)$ is $X_1(f)$, and $x_2(t) = x_1(at)$ where a is a constant, Then the Fourier transform of $x_2(t)$ is $X_2(f) = \frac{1}{|a|} X_1\left(\frac{f}{a}\right)$.

Proof

1) For $a > 0$.

$$X_2(f) = \int_{-\infty}^{\infty} x_1(at) e^{-j2\pi ft}\, dt \qquad (3.46)$$

Let $u = at$ and $du = adt$. Equation (3.46) becomes:

$$
\begin{aligned}
X_2(f) &= \int_{-\infty}^{\infty} x_1(u) e^{-j2\pi f(u/a)}\, \tfrac{du}{a} \\
&= \frac{1}{a}\int_{-\infty}^{\infty} x_1(u) e^{-j2\pi(f/a)u}\, du \\
&= \frac{1}{a} X_1\left(\frac{f}{a}\right) = \frac{1}{|a|} X_1\left(\frac{f}{a}\right)
\end{aligned}
$$

2) For $a < 0$ ($a = -|a|$)

Let $u = -|a|t$ and $du = -|a|dt$. Equation (3.46) becomes:

$$X_2(f) = \int_{\infty}^{-\infty} x_1(u) e^{-j2\pi f(u/-|a|)} \frac{du}{-|a|}$$

$$= -\frac{1}{|a|}\int_{\infty}^{-\infty} x_1(u) e^{-j2\pi(f/-|a|)u} du$$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty} x_1(u) e^{-j2\pi(f/-|a|)u} du$$

$$= \frac{1}{|a|} X_1\left(\frac{f}{-|a|}\right) = \frac{1}{|a|} X_1\left(\frac{f}{a}\right)$$

Therefore,

$$\mathcal{F}\{x(at)\} = \frac{1}{|a|} X_1\left(\frac{f}{a}\right) \tag{3.47}$$

Example 3.12

Repeat example 3.11 using the time scaling property.

Solution

The signal $x(t)$ is a time scaled version of $\Pi(t)$ where we have spread the function over time by a factor of 2, thus $x(t) = \Pi(t/2)$. Therefore $a = ½$ and $X(f) = 2\mathcal{F}\{\Pi(t)\}|_{f=2f}$. Therefore $X(f)$=2sinc(2f).

3.7.4 Duality

If the Fourier transform of $x(t)$ is $X(f)$, then the Fourier transform of $X(t)$ is $x(-f)$.

Proof

The inverse Fourier transform of $X(f)$ is:

$$\int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df = x(t) \tag{3.48}$$

If we exchange f and t in (3.48) we obtain

$$\int_{-\infty}^{\infty} X(t) e^{j2\pi ft} dt = x(f)$$

The Fourier transform of $X(t)$ is:

$$\mathcal{F}\{X(t)\} = \int_{-\infty}^{\infty} X(t) e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} X(t) e^{j2\pi(-f)t} dt = x(-f) \tag{3.49}$$

Example 3.13

Find the Fourier transform of sinc(2t).

Solution

$\mathcal{F}\{½\Pi(t/2)\}$ = sinc(2f). Thus the Fourier transform of sinc(2t) is $\mathcal{F}\{\text{sinc}(2t)\} = ½\Pi(-f/2) = ½\Pi(f/2)$.

Example 3.14

Find the Fourier transform of $x(t) = A$.

Solution

We know that $\mathcal{F}\{\delta(t)\} = 1$, therefore $\mathcal{F}\{A\delta(t)\} = A$. Using the duality property of the Fourier transform, $\mathcal{F}\{A\} = A\delta(-f) = A\delta(f)$.

3.7.5 Frequency Shifting

If $x_2(t) = x_1(t)\, e^{j2\pi f_o t}$ and $X_1(f) = \mathcal{F}\{x_1(t)\}$, then $X_2(f) = \mathcal{F}\{x_2(t)\} = X_1(f\text{-}f_o)$.

Proof

$$\begin{aligned}
X_2(f) &= \int_{-\infty}^{\infty} x_1(t) e^{j2\pi f_o t} e^{-j2\pi f t}\, dt \\
&= \int_{-\infty}^{\infty} x_1(t) e^{-j2\pi f t + j2\pi f_o t}\, du \\
&= \int_{-\infty}^{\infty} x_1(t) e^{-j2\pi (f - f_o) t}\, du \\
&= X_1\left(f - f_o\right)
\end{aligned}$$

Therefore

$$\mathcal{F}\{x(\mathrm{t})\, e^{j2\pi f_o t}\} = X\left(f - f_o\right) \tag{3.50}$$

Example 3.15

Find $\mathcal{F}\{\cos(2\pi f_o t)\}$.

Solution

We can express $\cos(2\pi f_o t)$ as $\frac{1}{2}e^{j2\pi f_o t}+\frac{1}{2}e^{-j2\pi f_o t}$. Therefore $\mathcal{F}\{\cos(2\pi f_o t)\}$ = $\mathcal{F}\{\frac{1}{2}e^{j2\pi f_o t}+\frac{1}{2}e^{-j2\pi f_o t}\}$ = ½ $\delta(f-f_o)$ + ½ $\delta(f+f_o)$.

3.7.6 Convolution

If $X(f) = \mathcal{F}\{x(t)\}$ and $Y(f) = \mathcal{F}\{y(t)\}$, then for $z(t) = x(t)*y(t)$, the Fourier transform is $Z(f) = X(f)Y(f)$.

Proof

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

$$\begin{aligned}
x(t)*y(t) &= \int_{-\infty}^{\infty} y(\lambda)x(t-\lambda)d\lambda \\
&= \int_{-\infty}^{\infty} y(\lambda)\left[\int_{-\infty}^{\infty} X(f)e^{j2\pi f(t-\lambda)}df\right]d\lambda \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f)e^{j2\pi f(t-\lambda)}y(\lambda)d\lambda df \\
&= \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\left[\int_{-\infty}^{\infty} e^{-j2\pi f\lambda}y(\lambda)d\lambda\right]df \\
&= \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}Y(f)df \\
&= \int_{-\infty}^{\infty} X(f)Y(f)e^{j2\pi ft}df
\end{aligned}$$

Therefore $x(t)*y(t)=\mathcal{F}^{-1}\{X(f)Y(f)\}$. Therefore

$$\mathcal{F}\{x(t)*y(t)\} = X(f)Y(f) \qquad (3.51)$$

Example 3.16

Find the Fourier transform of $x(t+t_o) + x(t-t_o)$.

Solution

It can be shown that $x(t+t_o) + x(t-t_o) = x(t) * [\delta(t-t_o) + \delta(t+t_o)]$. $\mathcal{F}\{[\delta(t-t_o) + \delta(t+t_o)]\} = e^{-j2\pi f t_o} + e^{j2\pi f t_o} = 2\cos 2\pi f t_o$. Therefore $\mathcal{F}\{x(t+t_o) + x(t-t_o)\} = 2X(f)\cos 2\pi f t_o$.

3.7.7 Multiplication in time

The Fourier transform of $z(t) = x(t)y(t)$ is $Z(f) = X(f)*Y(f)$.

Proof

$$X(f)*Y(f) = \int_{-\infty}^{\infty} X(\lambda)Y(f-\lambda)d\lambda = \int_{-\infty}^{\infty} X(\lambda)\left[\int_{-\infty}^{\infty} y(t)e^{-j2\pi(f-\lambda)t}dt\right]d\lambda$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(\lambda)y(t)e^{-j2\pi ft}e^{j2\pi f\lambda}dtd\lambda = \int_{-\infty}^{\infty} y(t)e^{-j2\pi ft}\left[\int_{-\infty}^{\infty} X(\lambda)e^{j2\pi f\lambda}d\lambda\right]dt$$

$$= \int_{-\infty}^{\infty} x(t)y(t)e^{-j2\pi ft}dt$$

$$= \mathcal{F}\{x(t)y(t)\} \qquad (3.52)$$

Example 3.17

Find the Fourier transform of $x(t) = \Pi(t)\cos 20\pi t$.

Solution

$\mathcal{F}\{\Pi(t)\} = \text{sinc}(f)$ and $\mathcal{F}\{\cos 20\pi t\} = ½\delta(f-10) + ½\delta(f-10)$. Then, from (3.52), $\mathcal{F}\{\Pi(t)\cos 20\pi t\} = \text{sinc}(f)*[½\delta(f-10) + ½\delta(f-10)] = ½\text{sinc}(f-10) + ½\text{sinc}(f+10)$. The Fourier transform of $\Pi(t)\cos 20\pi t$ is shown in Figure 3.9.

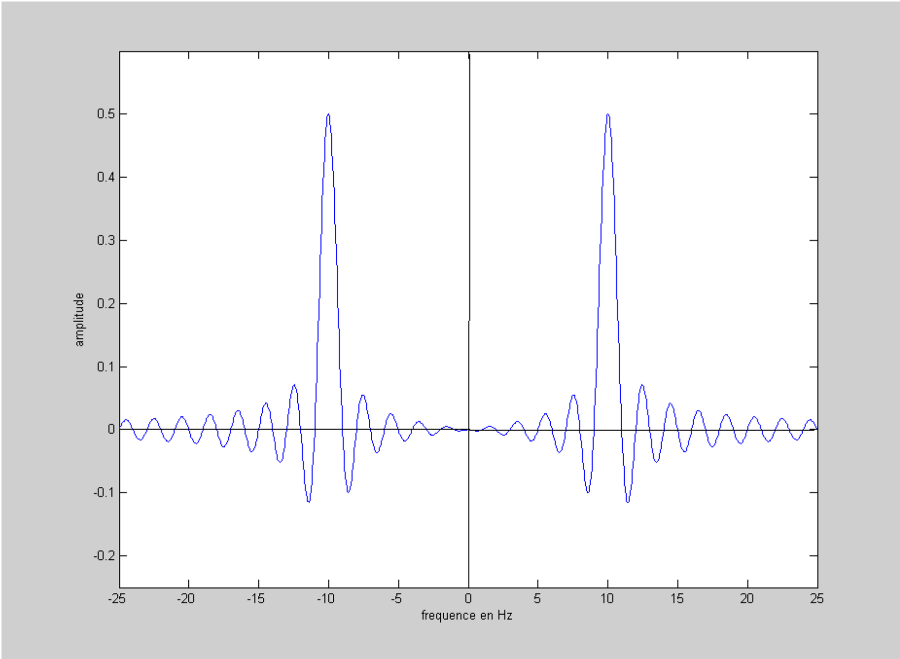

Figure 3.9: $\mathcal{F}\{\Pi(t)\cos 20\pi t\}$

3.7.8 Time Derivatives

If $\mathcal{F}\{x(t)\} = X(f)$, then $\mathcal{F}\{\frac{dx(t)}{dt}\} = 2\pi f X(f)$.

Proof

$$\mathcal{F}\{\frac{dx(t)}{dt}\} = \int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-j2\pi ft} dt \tag{3.53}$$

We must solve the integral in (3.53) by parts. Let $u = e^{-j2\pi ft}$ and $dv = \frac{dx(t)}{dt}dt$. Therefore $du = -j2\pi f e^{-j2\pi ft}$ and $v = x(t)$. Equation (3.53) becomes:

$$\mathcal{F}\{\frac{dx(t)}{dt}\} = x(t)e^{-j2\pi ft}\Big|_{-\infty}^{\infty} + j2\pi f \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt \tag{3.54}$$

The existence of the Fourier transform of $x(t)$ implies $x(\pm\infty) = 0$, therefore the first term in (3.54) is 0. Equation (3.54) becomes:

$$\mathcal{F}\{\frac{dx(t)}{dt}\} = j2\pi f X(f) \tag{3.55}$$

Example 3.18

Find the Fourier transform of $\sin 2\pi f_o t$.

Solution

$\mathcal{F}\{\cos 2\pi f_o t\}$ = ½$\delta(f-f_o)$ + ½$\delta(f+f_o)$. The function $\sin 2\pi f_o t = -\frac{1}{2\pi f_o}\frac{d\cos 2\pi f_o t}{dt}$. Thus $\mathcal{F}\{\sin 2\pi f_o t\} = \mathcal{F}\{-\frac{1}{2\pi f_o}\frac{d\cos 2\pi f_o t}{dt}\} = -\frac{1}{2\pi f_o}\times j2\pi f\left[\frac{1}{2}\delta(f-f_o)+\frac{1}{2}\delta(f+f_o)\right] = -\frac{1}{2\pi f_o}\times\left[j\pi f_o\delta(f-f_o)-j\pi f_o\delta(f+f_o)\right] = -\left[\frac{j}{2}\delta(f-f_o)-\frac{j}{2}\delta(f+f_o)\right] = \frac{1}{2j}\delta(f-f_o)-\frac{1}{2j}\delta(f+f_o)$.

3.7.9 Integration in time

The function $y(t)=\int_{-\infty}^{t}x(\lambda)d\lambda$ has Fourier transform $Y(f)=\mathcal{F}\left\{\int_{-\infty}^{t}x(\lambda)d\lambda\right\}=\frac{1}{j2\pi f}X(f)+\frac{1}{2}X(0)\delta(f)$.

Proof

The Fourier transform of $u(t)$ is $U(f)$. The derivative of $u(t)$, $\frac{du(t)}{dt} = \delta(t)$. Therefore $j2\pi f U(f) = 1$. Also, $u(t)+u(-t) = 1$. Therefore $U(f)+U(-f) = \delta(f)$. The latter identity indicates that $U(f) = -U(-f)$, except for f = 0, where $U(0) = U(-0)$ = ½$\delta(f)$. Therefore, from the two identities, the Fourier transform of $u(t)$ is $U(f) = \frac{1}{j2\pi f}+\frac{1}{2}\delta(f)$.

$$y(t)=\int_{-\infty}^{t}x(\lambda)d\lambda=\int_{-\infty}^{\infty}x(\lambda)u(t-\lambda)d\lambda=x(t)*u(t)$$

Therefore $Y(f) = X(f)U(f) = \left(\frac{1}{j2\pi f}+\frac{1}{2}\delta(f)\right)X(f)=\frac{1}{j2\pi f}X(f)+\frac{1}{2}X(0)\delta(f)$.

Example 3.19

Find the Fourier transform of $y(t) = (1-e^{-t})\, u(t)$.

Solution

Let $x(t) = e^{-t}u(t)$ which has Fourier transform $X(f) = 1/(j2\pi f+1)$. $\int_{-\infty}^{t}e^{-\lambda}u(\lambda)d\lambda=(1-e^{-t})u(t)$. Therefore $Y(f) = (1/j2\pi f)X(f)$ + ½$X(0)\delta(f)$ = $1/(j2\pi f-(2\pi f)^2)$ + ½$\delta(f)$.

3.7.10 Complex Conjugate

If $X(f) = \mathcal{F}\{x(t)\}$, then $\mathcal{F}\{x^*(t)\} = X^*(-f)$.

Proof

$$
\begin{aligned}
\mathcal{F}\{x^*(t)\} &= \int_{-\infty}^{\infty} x^*(t)e^{-j2\pi ft}\,dt \\
&= \int_{-\Re}^{\infty} \left[x(t)e^{j2\pi ft}\right]^* dt \\
&= \left[\int_{-\infty}^{\infty} x(t)e^{-j2\pi(-f)t}\,dt\right]^* \\
&= \left[X(-f)\right]^* = X^*(-f)
\end{aligned}
\tag{3.56}
$$